POINT OPTIMAL TESTING WITH ROOTS THAT ARE FUNCTIONALLY LOCAL TO UNITY

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POINT OPTIMAL TESTING WITH ROOTS THAT ARE FUNCTIONALLY LOCAL TO UNITY

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Abstract. Limit theory for regressions involving local to unit roots (LURs) is now used extensively in time series econometric work, establishing power properties for unit root and cointegration tests, assisting the construction of uniform confidence intervals for autoregressive coefficients, and enabling the development of methods robust to departures from unit roots. The present paper shows how to generalize LUR asymptotics to cases where the localized departure from unity is a time varying function rather than a constant. Such a functional local unit root (FLUR) model has much greater generality and encompasses many cases of additional interest, including structural break formulations that admit subperiods of unit root, local stationary and local explosive behavior within a given sample. Point optimal FLUR tests are constructed in the paper to accommodate such cases. It is shown that against FLUR alternatives, conventional constant point optimal tests can have extremely low power, particularly when the departure from unity occurs early in the sample period. Simulation results are reported and some implications for empirical practice are examined.

Key words and phrases: Functional local unit root; Local to unity; Uniform confidence interval, Unit root model.

JEL Classifications: C22, C65

1. Introduction and Motivation

Local to unit root (LUR) limit theory has played a significant role in the development of econometric methods for nonstationary time series. The primary need for this development came from the desire to assess the effect of local departures on the functional limit theory for unit root processes and its many applications to regression and unit root testing (Chan and Wei (1987), Phillips (1987), Phillips (1988)). The methodology assisted asymptotic power analysis and the construction of point optimal unit root tests (Elliott et al. (1996) and for a recent overview King and Sriananthakumar (2016)). Most recently, the methods have been used to study the uniform properties

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of various methods of confidence interval construction for autoregressive coefficients (Mikusheva (2007), Mikusheva (2012), Phillips (2014)). In all these implementations, local departures from unity have been measured in terms of constant localized coefficient departures from unity of the form $\theta_n = e^{c/n} \sim 1 + \frac{c}{n}$ in terms of the sample size n. This type of constant coefficient LUR specification is extremely convenient because of its parsimony and its standard Pitman form, given the $O(n)$ rate of convergence of an autoregressive estimate of a unit autoregressive coefficient.

In spite of their mathematical convenience, there is nothing particularly relevant in such constant LUR formulations for modeling economic time series in which departures from unity may be expected to take a variety of different forms, including periods of increasing or decreasing persistence, transitions to and from unity, and break points that shift from a unit root to stationary or even explosive roots. Complex departures from unity of this type require greater flexibility in formulation than a constant coefficient. They may be captured using a time varying coefficient function in which the autoregressive coefficient is time varying of the form $\theta_{tn} = \theta_n\left(\frac{t}{n}\right) = e^{c\left(\frac{t}{n}\right)/n} \sim 1 + c\left(\frac{t}{n}\right)/n$. We call such a formulation a functional local unit root (FLUR). Models with FLUR coefficients have already been used in empirical work on modeling bubble contagion (Phillips and Yu (2011), Greenaway-McGrevy and Phillips (2016)) and in some related recent theoretical developments dealing with stochastic unit root models (Lieberman and Phillips (2014), Lieberman and Phillips (2016)) and random coefficient autoregressions (Banerjee et al. (2015)).

The primary purpose of the present paper is to analyze such models and generalize LUR asymptotics to cases where the localized departure from unity is a general time varying function rather than a constant. FLUR models of this type provide a new mechanism for assessing the power properties of UR tests against more complex alternatives. A second contribution of the paper is to develop functional point optimal UR tests constructed to achieve point optimality against a specific functional alternative. With FLUR alternatives, conventional constant point optimal tests can have extremely low power, particularly when departures from unity occur early in the sample period. A third contribution of the paper is to reveal conditions under which such weaknesses typically arise. In the light of this analysis, it is apparent that point optimal tests based on a constant alternative are by no means a universally satisfactory solution to improving power in unit root testing. Indeed, the power envelope itself can be very different under a functional alternative to that which is obtained under the strict condition of a constant local alternative. Simulation results on the proximity of the power function of point optimal tests to a power envelope constructed for constant Pitman-type alternatives can therefore be a misleading indicator of discriminatory power of such tests against more general cases.

More specifically, the paper provides new limit theory for autoregressive models with time varying coefficients that are close to unity. This limit theory enables the development of functional point optimal tests of a unit root, which extend earlier work on constant point optimal tests analysis (Elliott et al. (1996)). The limit theory provides analytic power comparisons between FLUR and standard point optimal tests, showing how the latter tests can have power that is well below the power envelope under certain conditions. The findings are confirmed in simulations that explore the power differences in particular cases, concentrating on empirically relevant situations where time varying coefficients induce structural breaks in the generating mechanism between unit root and local unit root behavior in the data.

The paper is organized as follows. Section 2 describes the general setup, FLUR asymptotics, and the implications of these asymptotics for unit root testing, including analytic power comparisons. This Section also provides numerical simulations that compare the power of standard point-optimal tests based on constant local departures from the null with the actual power envelope under a functional local alternative. Section 3 develops limit theory approximations that explain the power differences. Special focus is given to cases where local power is low and divergence is largest, notably on departures from a unit root that occur briefly early in the sample. Section 4 concludes and proofs are given in the Appendix.

2. Model, Testing, and Asymptotics

2.1. **Setup and first asymptotic results.** We consider a time series generated by the following model

$$X_t = \theta_{tn} X_{t-1} + u_t,\ t = 1, \dots, n,\ X_0 = u_0{}^1 \tag{1}$$

where $u_t \sim i.i.d.\ \mathbb{N}(0, \sigma^2)$ and

$$\theta_{tn} = \exp\left(\frac{c(t/n)}{n}\right) \approx 1 + \frac{c(t/n)}{n}. \tag{2}$$

The coefficient θ_{tn} in the autoregression varies with time and can be arbitrarily close to unity as the function $c(\cdot)$ moves towards zero. It is convenient and involves no loss of generality to ignore in the notation X_t the array nature implied by θ_{tn}. This framework allows for unit root testing against local unit roots that include functional alternatives (2). Such alternatives accommodate structural breaks and smooth transitions in the process, as well as the usual Pitman drift formulations where $c(\cdot)$ is a constant function.

[1] $X_0 = u_0$ can be viewed as starting from $X_{-1} = 0$.

Our goal is to examine the performance characteristics of tests of a unit root, where $c \equiv 0$, against functional local unit root alternatives such as (2).

Many results in the paper remain valid, including the asymptotic behavior of X_t, if $X_0 = O_p(1)$ is replaced by $X_0 = o_p(\sqrt{n})$. However, specific details of the asymptotic results given in Theorems 1 - 3 in Section 3 will change and the results given here are for the simpler case. Additionally, one may weaken the independence and normality conditions to suitable weak dependence and moment conditions with some further adjustments to the limit theory.

Define $X_n(r) = \frac{1}{\sqrt{n}\sigma} S_{\lfloor nr \rfloor}$, where $S_t = u_1 + \cdots + u_t$. Our first result details the asymptotic behavior of the scaled process $n^{-\frac{1}{2}} X_{\lfloor nr \rfloor}$, which is given by

$$n^{-\frac{1}{2}} X_{\lfloor nr \rfloor} \xrightarrow[n \to \infty]{} \sigma K_c(r) = \sigma \int_0^r e^{\int_s^r c(k)dk} dW(s). \tag{3}$$

where the process $K_c(r)$ satisfies the stochastic differential equation $dK_c(r) = c(r)K_c(r)dr + dW(r)$, as shown in Lemma 8 in the Appendix.

A primary focus of the paper is testing the null function $c(\cdot) = 0$, under which X_t is a standard unit root process. The alternative hypothesis involves function space possibilities as any non-zero function $c(\cdot)$ represents a local departure from the null. Composite alternatives inevitably complicate hypothesis testing but are more subtle in the present case because the alternative is a function that may induce subperiods in which the null of a unit root actually holds. An obvious simplifying procedure when faced with such composite functionally-infinite space of alternatives is to assume some fixed function $c(r)$ as a proxy for the alternative hypothesis. Doing so enables application of the the Neyman-Pearson lemma to deliver the best (point-optimal under functional alternatives) test. We proceed to implement that approach in testing the hypothesis of a unit root. The analysis provides a basis of comparison with point optimal tests of a unit root that are based on a constant local alternative.

We need to compare likelihood functions under the null and alternative hypothesis. The sequence X_t involves interdependent random variables. To construct the likelihood, we transform X_t so that the likelihood is a simple product of independent densities. To do this, consider

$$d_t^c := \frac{1}{\sqrt{n}\sigma} \left(X_t - e^{c(t/n)/n} X_{t-1} \right) = \frac{1}{\sqrt{n}\sigma} u_t, \tag{4}$$

where the functions d_t^c are independent for different values of t. Moreover, under Gaussianity $d_t^c \sim N(0, 1/n)$, so that we are left with normal densities $\frac{1}{\sqrt{2\pi/n}} e^{-\frac{d^2}{2/n}}$.

Thus, we can construct a likelihood ratio test of the form

$$\text{Reject if } e^{-\sum_t \frac{\left(d_t^0\right)^2}{2/n}} / e^{-\sum_t \frac{(d_t^c)^2}{2/n}} \leq \eta,$$

where d^0 corresponds to the value of function $d(\cdot)$ under null hypothesis of unit root, so that $d_t^0 = \frac{1}{\sqrt{n}\sigma}(X_t - X_{t-1})$, and d^c stands for the value under alternative of hypothesis of some function $c(\cdot)$, so that $d_t^c = \frac{1}{\sqrt{n}\sigma}\left(X_t - e^{c(t/n)/n}X_{t-1}\right)$. Equivalently,

$$S := \frac{1}{2\sigma^2}\left(\sum_t (X_t - X_{t-1})^2 - \sum_t \left(X_t - e^{c\left(\frac{t}{n}\right)/n}X_{t-1}\right)^2\right) \geq \alpha. \tag{5}$$

To properly choose the value for α we need to calculate the values of S under H_0 and H_1.

Lemma 1. *The test statistic S has the following limit behavior under H_0 and H_1*

$$S \xrightarrow[n\to\infty]{H_0} \int_0^1 c(s)W(s)dW(s) - \frac{1}{2}\int_0^1 c^2(s)W^2(s)ds,$$

$$S \xrightarrow[n\to\infty]{H_1} \int_0^1 c(s)K_c(s)dW(s) + \frac{1}{2}\int_0^1 c^2(s)K_c^2(s)ds.$$

Since the test statistic S in Eq. (5) depends on the proxy alternative function, it may happen that the true function governing the behavior of the process X_t is a different function. In such a case the asymptotic behavior of the test S is described by the next Lemma. To clarify notation, in what follows the proxy (pseudo) alternative function $c(\cdot)$ (used in the definition of the point-optimal test statistic S) is denoted $c^*(\cdot)$. The true value of the function $c(\cdot)$ under H_1 is denoted $\bar{c}(\cdot)$. To examine such misspecifications, we refer to the test in such a case as a pseudo-point-optimal test.

Lemma 2. *Under $H_1 : c(\cdot) = \bar{c}(\cdot)$, the pseudo-point-optimal test S based on $c(\cdot) = c^*(\cdot)$ has the following limit behavior*

$$S \overset{n\to\infty}{\longrightarrow} \int_0^1 K_{\bar{c}}(s)c^*(s)dW(s) + \int_0^1 K_{\bar{c}}^2(s)\bar{c}(s)c^*(s)ds - \frac{1}{2}\int_0^1 K_{\bar{c}}(s)c^{*2}(s)ds.$$

Combining Lemmas 1 and 2, we can write the asymptotic size and power of the tests, constructed based on true and on misspecified alternatives, in terms of a probability involving certain limiting stochastic integrals. For simplicity in the following discussion, we focus on 5% asymptotic critical values but all results remain valid under an arbitrary critical value α. We determine the 5% asymptotic critical values A and A^c through the equations

$$\mathbb{P}\left(\int_0^1 \bar{c}(t)W(t)dW(t) - \frac{1}{2}\int_0^1 \bar{c}^2(t)W^2(t)dt > A\right) = 0.05,$$

$$\mathbb{P}\left(\int_0^1 c^*(t)W(t)dW(t) - \frac{1}{2}\int_0^1 c^{*2}(t)W^2(t)dt > A^c\right) = 0.05.$$

Then the maximal power (corresponding to the correctly specified alternative $c = \bar{c}$, rather than the proxy alternative) is

$$P^{\max} = \mathbb{P}\left(\int_0^1 \bar{c}(t)K_{\bar{c}}(t)dW(t) + \frac{1}{2}\int_0^1 \bar{c}^2(t)K_{\bar{c}}^2(t)dt > A\right), \tag{6}$$

and the power corresponding to a possibly misspecified proxy alternative $c = c^*$ is

$$P^{c^*} = \mathbb{P}\left(\int_0^1 K_{\bar{c}}(t)c^*(t)dW(t) + \int_0^1 K_{\bar{c}}^2(t)\bar{c}(t)c^*(t)dt - \frac{1}{2}\int_0^1 K_{\bar{c}}(t)c^{*2}(t)dt > A^c\right). \tag{7}$$

Formulae such as (6) and (7) are typically intractable analytically even in the simplest case where $c^*(\cdot) = c^* = const$. As shown later, some useful asymptotic expansions of these power functions can be obtained in certain important cases. To analyze the behavior of these power functions directly we first provide some numerical simulations. We focus on the case where c^* is selected to be constant, while $\bar{c}$ is some function representing plausible time changes or evolution in the AR coefficient over a sample period. Low power in testing against a fixed alternative in such cases indicates the importance of taking into account the possibility of a functional alternative. The power envelope in such cases is itself a space of functions, rather than a simple function as it is in the case where only constant c alternatives are considered.

2.2. **Simulations.** The following subsections provide numerical simulations to illustrate power performance from point-optimal tests that are based on some (possibly misspecified) proxy alternative against maximal attainable power. In the first subsection we investigate the performance of a conventional constant alternative $c^*(x) = const$ test when the data generating process corresponds to non-constant function $\bar{c}(x)$. As will be seen, the power of such misspecified point optimal tests can be far below maximal power. Asymptotic theory to explain this phenomenon is given in Section 3.

The next subsection compares power performance that is based on misspecified triangular proxy functions of differing heights when the true function is piecewise constant. The true localizing function $\bar{c}(x)$ is assumed to be zero for $x \leq 0.25$ and $x \geq 0.75$ and to take some negative value $-\lambda$ when $0.25 < x < 0.75$. We find the optimal height of the proxy triangular alternative, i.e. the function that minimizes the area between misspecified power and maximal power as a function of λ.

Finally, we compare the power of the standard Dickey-Fuller (DF) t test with the maximal power and with the power achieved by using constant proxy alternatives in the construction of pseudo-point optimal tests. We find that the DF test also performs poorly compared to maximal power; and DF power is approximately the same as the power of a pseudo-point-optimal test constructed with $c^* = const$ when the constant is chosen optimally for this class of constant alternatives.

2.2.1. *Point-optimal text based on the constant proxy alternative.* We consider divergence from a unit root over subsamples of data corresponding to simple structural breaks. Specifically, we assume that the true function $\bar{c}(\cdot)$ is piece-wise constant and equals to zero on the intervals $[0, r_1)$ and $(r_2, 1]$ (this corresponds to two unit root subperiods in the model) and to some number $\bar{C}$ on the interval $[r_1, r_2]$. We allow $r_1 = 0$ or $r_2 = 1$, so that instead of three segments there may only be two. Numerical simulations are given for six specific examples which differ from each other either by the sign of the constant or by the left/right/middle location of the non-zero segment, which is determined by the value of a parameter λ. The functions are as follows:

- $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x < \lambda\}$;
- $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{x < \lambda\}$;
- $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x > 1 - \lambda\}$;
- $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{x > 1 - \lambda\}$;
- $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{|x - 0.5| < \lambda\}$;
- $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{|x - 0.5| < \lambda\}$.

Assuming $\sigma = 1$, maximal power (when the alternative FLUR hypothesis is specified correctly) and power under the misspecified LUR alternative $c^* = const$ under 5% significance level are shown in Figures 1 - 6. Green lines represent maximal power and blue lines represent power under a misspecified constant alternative, for which we assume $c^* = 1$ in near-explosive cases and $c^* = -5$ in near-stationary cases. These values were selected for the constant proxy alternative because in the simulations they turned out to deliver the best power. To compare powers under different alternatives, see Figures 7 and 8, which are drawn for the first two cases: $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x < \lambda\}$ and $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{x < \lambda\}$.

Evidently from Figures 1 - 6, we see that in all six examples the power of the test constructed by a misspecified alternative function with $c^* = const$ is lower, and often substantially lower, than the maximal power. In particular, it is apparent that for small values of λ, i.e., when the time series model has only a minor difference from a unit root model, the green (maximal power test) and blue (test based on the misspecified alternative) curves have different slopes. In all examples, the slope around zero is smaller for the blue curve. In Section 3, we explain this pattern of the power curves by calculating analytically a power function expansion that allows for such small departures from unity.

2.2.2. *Comparison of different proxy functions.* In this section we consider a double-break point data generating function in which the localizing function has the form

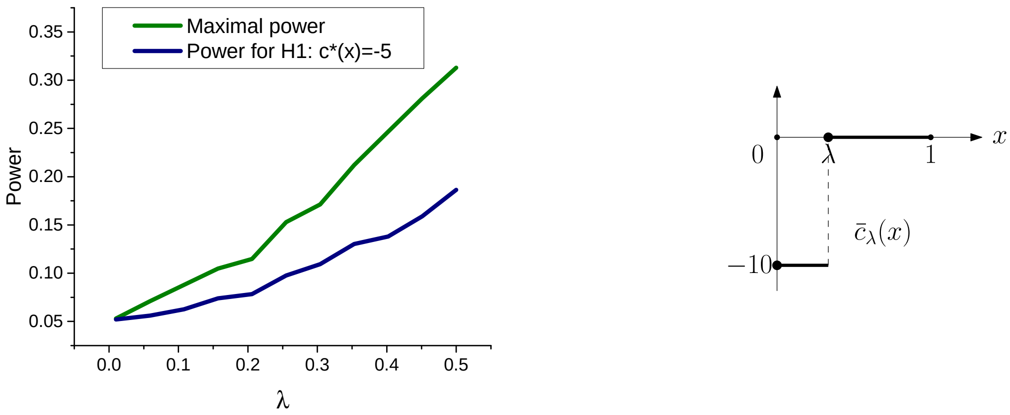

FIGURE 1. Power envelopes for $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x < \lambda\}$.

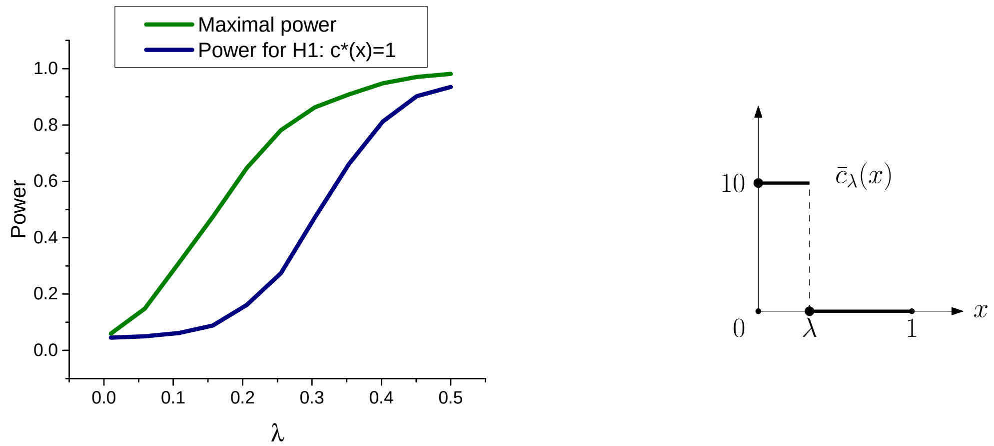

FIGURE 2. Power envelopes for $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{x < \lambda\}$.

$\bar{c}_\lambda(x) = -\lambda \mathbf{1}\{0.25 < x < 0.75\}$. This function gives a period of unit root behavior followed by a (near) stationary period which switches back to unit root behavior in the final period. As a proxy function we take a triangular function of the form ${c^*}_l(x) = -2l\,(x\mathbf{1}\{x <= 0.5\} + (1-x)\mathbf{1}\{x > 0.5\})$, in which the stationary wedge has height l, and the end points of the interval $[0, 1]$ are the intitiating and terminating points of the wedge. As such, the proxy function does not use any information about the break points but is structured in a way that acknowledges the possible presence of stationary behavior within the sample.

We seek to discover how different values of the height l of the wedge function affect the power of this proxy-function point-optimal test based on ${c^*}_l(x)$, when the true localizing coefficient function is the double break point function $\bar{c}_\lambda(x)$. We are also interested in the 'optimal' choice of l, l^*, that minimizes the distance (measured in some sense) between maximal power and power based on the proxy ${c^*}_l$ function. This exercise might be regarded as a functional point-optimal extended version of the 'optimal' selection of the constant in conventional point-optimal testing (Elliott et al.

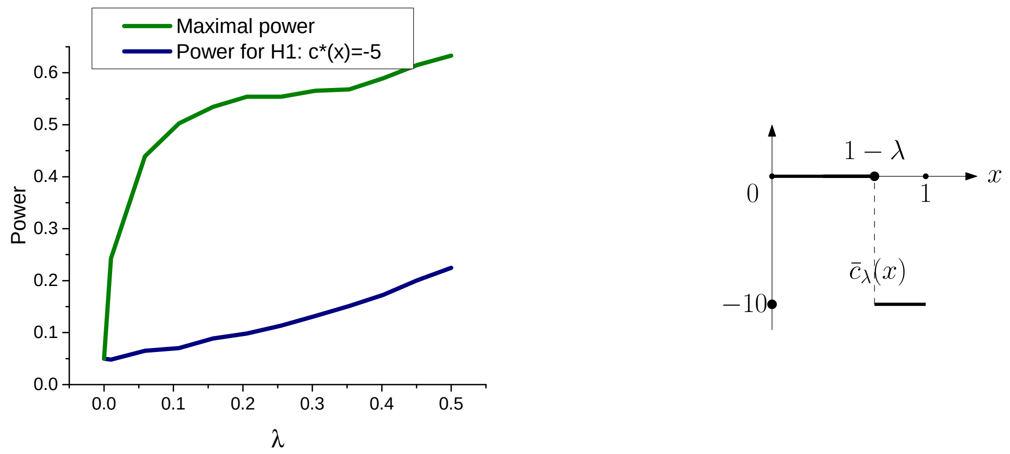

FIGURE 3. Power envelopes for $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x > 1-\lambda\}$.

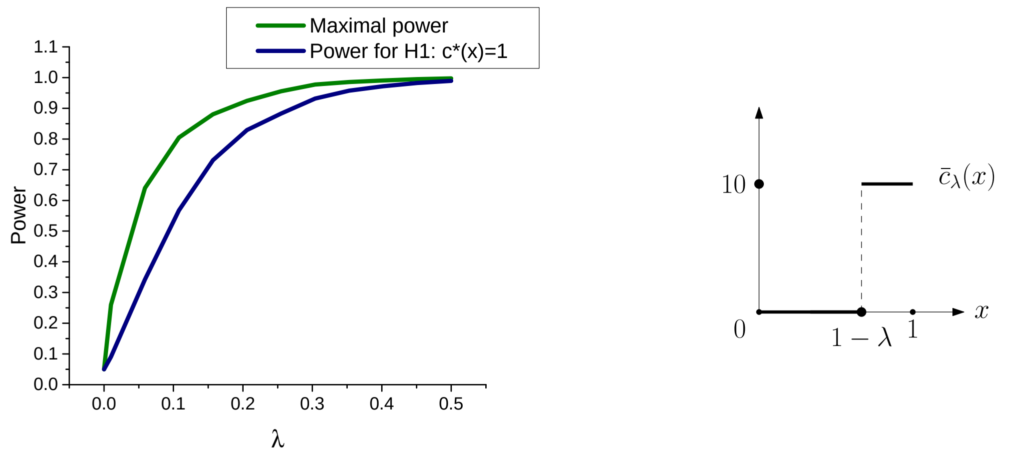

FIGURE 4. Power envelopes for $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{x > 1-\lambda\}$.

(1996)) with the proviso that in the FLUR case the function itself may, as here, be chosen incorrectly. Figure 9 shows the power of the point-optimal test based on the proxy alternatives for different values of l as a function of λ against optimal power for the same value of λ. Evidently, when actual λ is itself large (representing a large functional departure from a unit root) then the pseudo-point optimal test with $l = 20$ (the pink curve) has power closest to maximal power (the green curve). However, the choice $l = 12.5$ minimizes the area between the power curve under the proxy alternative and the maximal power curve. This calculation is based on computations of the difference in the area over a grid of possible choices for l, which showed that when λ is bounded by 40, then the choice $l^* = 12.5$ provides the closest fit to the power envelope over λ (as traced out by the green curve in Figure 9). This choice of l is shown by the blue curve. Apparently, when λ is not too large the blue curve is the closest to the power envelope over λ in the range $[0, 40]$. However, as is apparent from the behavior of the pink curve for $\lambda \geq 30$, if much larger values of λ are countenanced, then correspondingly larger values of l perform better relative to the power envelope.

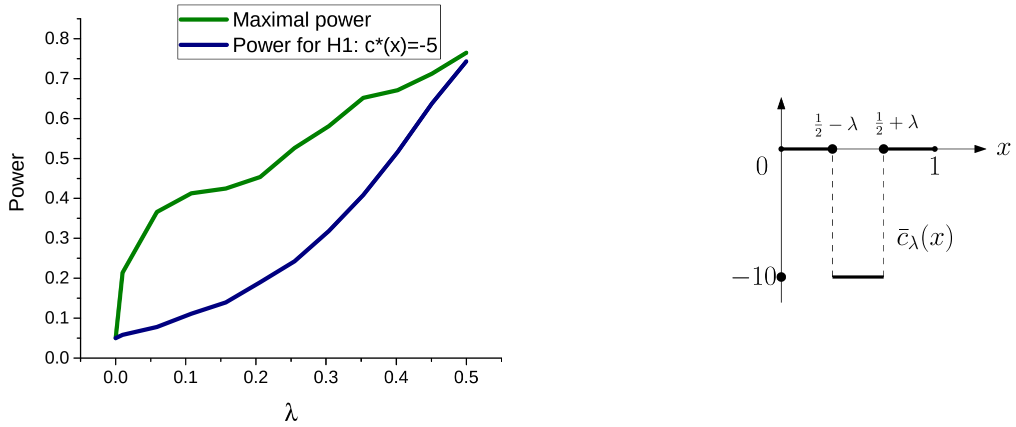

FIGURE 5. Power envelopes for $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{|x - 0.5| < \lambda\}$.

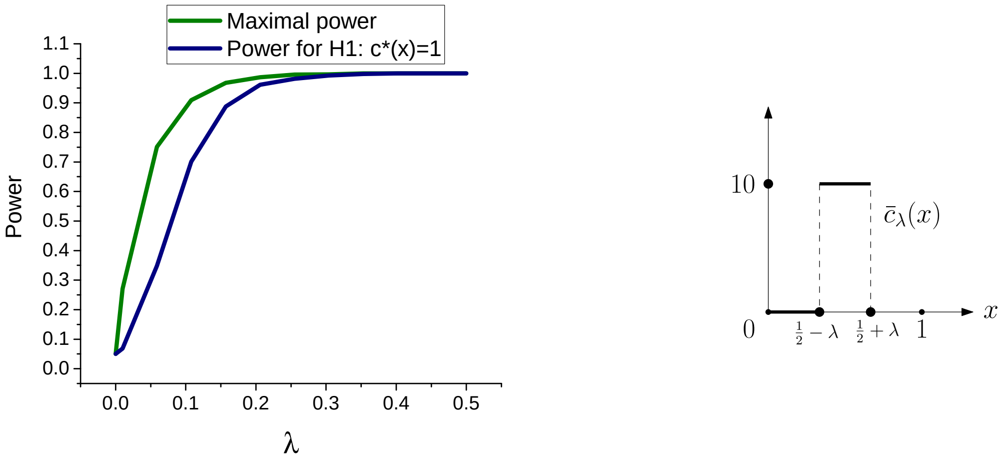

FIGURE 6. Power envelopes for $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{|x - 0.5| < \lambda\}$.

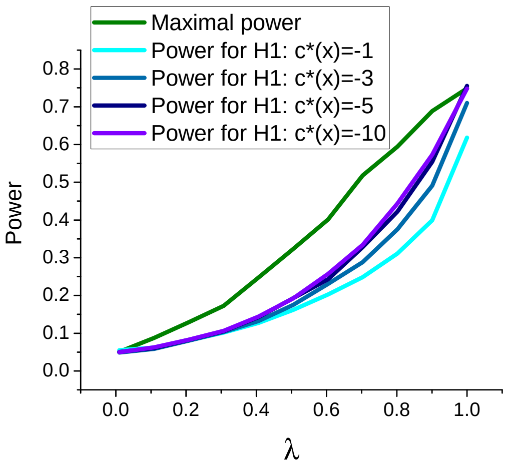

FIGURE 7. Power comparison for $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x < \lambda\}$.

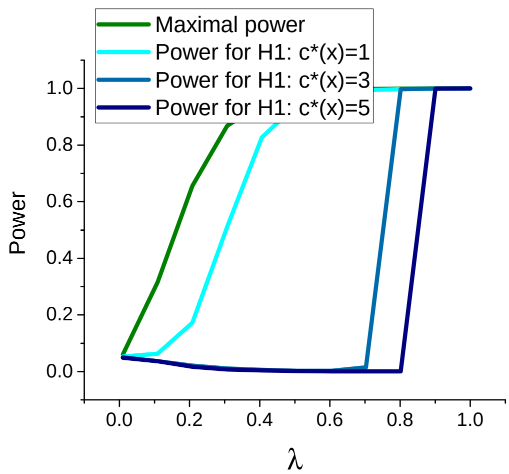

FIGURE 8. Power comparison for $\bar{c}_\lambda(x) = 10 \times \mathbf{1}\{x < \lambda\}$.

2.2.3. *Comparison with the Dickey–Fuller test.* To calibrate against a standard unit root test, we also show the power performance of the Dickey–Fuller (DF) t test. We

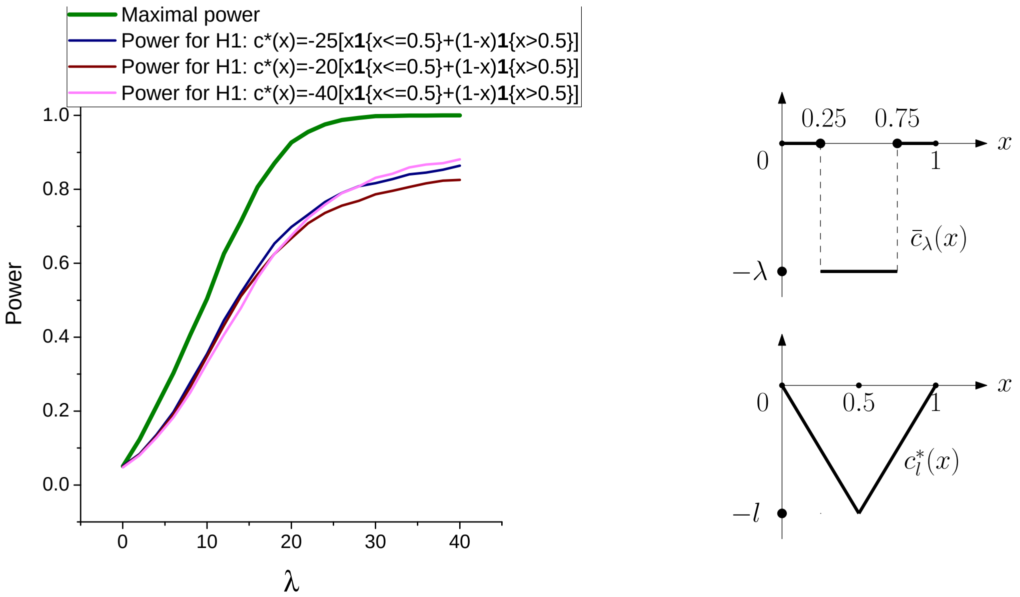

FIGURE 9. Power comparison for
$\bar{c}_\lambda(x) = -\lambda \mathbf{1}\{0.25 < x < 0.75\}$, ${c^*}_l(x) = -2l\,(x\mathbf{1}\{x <= 0.5\} + (1-x)\mathbf{1}\{x > 0.5\})$.

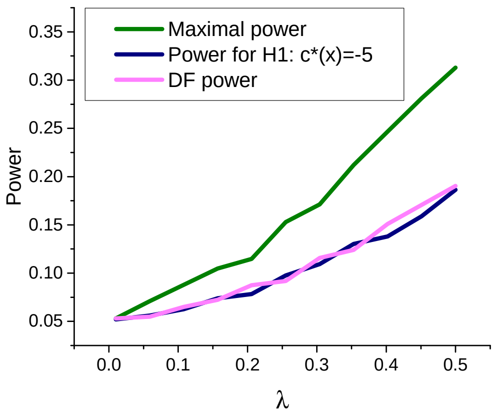

FIGURE 10. DF for $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x < \lambda\}$.

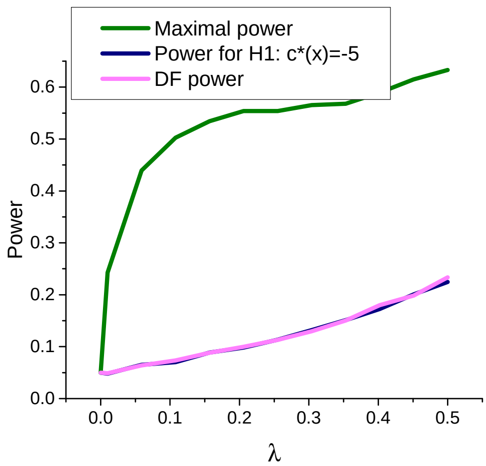

FIGURE 11. DF for $\bar{c}_\lambda(x) = -10 \times \mathbf{1}\{x > 1 - \lambda\}$.

examine the stationary case, where left side testing is the predominant application in practice. Figures 10 and 11 show the power of the Dickey–Fuller test along with the maximal power and power of the pseudo-point-optimal test constructed with $c^*(\cdot) = -5$. The DF test is represented by the pink line, which evidently closely matches the blue line and thereby the power of the pseudo-point-optimal test. So, the DF test does not increase power. In contrast to the pseudo-point-optimal test, the Dickey–Fuller test does not specify an alternate value of the autoregressive coefficient $\theta < 1$, which might appear prima facie to be an advantage when the actual alternative is more complex, as in the present case. But this flexibility does not appear to be useful even when testing against an alternative that differs from a fixed alternative. Thus, the additional flexibility of the DF test does not raise power even in this misspecified case.

3. Failure of Point Optimal Testing under a Constant Alternative

The above numerical examples show that assuming a constant alternative c^* produces very poor power when the true data-generating process corresponds to a functional LUR of the form $\bar{c}(x) = \bar{C}\mathbf{1}\{|x-\ell| \leq \lambda\}$ for $\ell \in [0,1]$ and λ is small. To provide some theoretical foundation for this finding we focus on the following explicit process:

$$\begin{aligned}
\bar{c}(x) &= \bar{C} \times \mathbf{1}\left\{x \leq \frac{1}{n}\right\}, \; \theta_{tn} = 1 + \frac{\bar{c}(t/n)}{n}, \; u_t \sim_{iid} N(0,\sigma^2), \\
X_{-1} &= 0, \; X_0 = \theta_{0n}X_{-1} + u_0 = u_0, \; X_1 = \theta_{1n}X_0 + u_1 = \left(1 + \frac{\bar{C}}{n}\right)u_0 + u_1, \\
X_t &= \theta_{tn}X_{t-1} + u_t = X_{t-1} + u_t = \left(1 + \frac{\bar{C}}{n}\right)u_0 + u_1 + \cdots + u_t, t > 1. \qquad (8)
\end{aligned}$$

In this example $\ell = 0$ and the process differs from a simple unit root model only at the first observation $t = 1$. We show that in such a case the maximal power is $0.05 + \frac{\bar{C}}{n} \times \text{const} + o(n^{-1})$, where const > 0. In contrast, if we construct a test based on a misspecified constant alternative with $c^*(x) = C^*$, we get significantly smaller power: power in this case is bounded from above by $0.05 + \text{const}_\varepsilon\, n^{-\frac{3}{2}+\varepsilon}$ for arbitrary $\varepsilon > 0$. Here and in what follows the notation 'const_ε' signifies a positive constant that depends on ε and may change from line to line.

Similar results hold if instead we assume $\bar{c}(x) = \bar{C} \times \mathbf{1}\{x = \alpha\}$, so that divergence from a unit root process occurs in the middle of the sample, at the point $t = \lfloor \alpha n \rfloor$. Note that for $\alpha = 1$ a difference occurs at the last observation. Then maximal power is $0.05 + \text{const}\, n^{-0.5} + o(n^{-0.5})$, whereas power of a test based on $c^*(x) = C^*$ is at most $0.05 + \text{const}_\varepsilon n^{-1+\varepsilon}$, where $\varepsilon > 0$ is arbitrary.

Moreover, the above results on single-point structural changes in the generating mechanism can be generalized to small infinity regions of originating or terminating data. In such cases the true data-generating process differs from a UR specification not only at a single point but over an interval of L points, where the parameter L is allowed to pass to infinity but at a slower rate than n. This extension is considered later.

The next series of lemmas and theorems show that, in a model based on the above structural break specification, maximal power under a functional point-optimal test is asymptotically higher than power of tests based on a misspecified constant local alternative with $H_1 : c(x) = C^*$. These results make use of the following lemma, which is proved in the Appendix.

Lemma 3. *Suppose $A \in \mathbb{R}$, $k \in \mathbb{R}_+$, (Y, Z) are two random variables with finite moment generating function, and the density of Z is locally bounded at A. Then*

$$\mathbb{P}\left\{Z + \frac{1}{n^k} Y > A\right\} \leq \mathbb{P}\{Z > A\} + O\left(\frac{1}{n^k}\log n\right) \leq \mathbb{P}\{Z > A\} + const_\varepsilon n^{-k+\varepsilon},$$

where $\varepsilon > 0$ is arbitrary.

3.1. **Unit Root Break at $t = 1$.**

Lemma 4. *Under $H_0 : c(x) \equiv 0$, the test statistic for the point-optimal test with maximal power takes the form*

$$S|_{H_0} = \frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1 - \frac{\bar{C}}{n}u_0\right).$$

Under $H_1 : c(x) \equiv \bar{c}(x)$, the test statistic for the point-optimal test with maximal power has the form

$$S|_{H_1} = \frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1 + \frac{\bar{C}}{n}u_0\right).$$

Theorem 1. *In the setting of model (8), the maximal power for detecting a unit root break at $t = 1$ is $P^m = 0.05 + \frac{\bar{C}}{n} \cdot const + o(n^{-1})$, $const > 0$.*

Lemma 5. *Under $H_0 : c(x) \equiv 0$, the test statistic for the pseudo-point-optimal test based on the function $c^*(x) \equiv C^*$ has the form*

$$S|_{H_0} = \frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right).$$

Under $H_1 : c(x) \equiv \bar{c}(x)$, the test statistic for the pseudo-point-optimal test based on the constant function $c^*(x) \equiv C^*$ has the form

$$\begin{aligned} S|_{H_1} = {} & \frac{1}{2\sigma^2}\sum_{t=1}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right) + \frac{C^*\bar{C}}{\sigma^2 n^2}u_0^2 \\ & + \frac{C^*\bar{C}}{2\sigma^2 n^2}u_0\sum_{t=2}^{n}\left(2u_t - \frac{C^*\bar{C}}{n^2}u_0\right) - \frac{C^{*2}\bar{C}}{\sigma^2 n^3}u_0\sum_{t=2}^{n}\sum_{\tau=0}^{t-1}u_\tau. \end{aligned}$$

Theorem 2. *In the setting of model (8), the pseudo-point-optimal test based on the constant function $c^*(x) \equiv C^*$ for detection of a unit root break at $t = 1$ has power $P^c \leq 0.05 + const_\varepsilon n^{-\frac{3}{2}+\varepsilon}$ for arbitrary $\varepsilon > 0$.*

Remark. *The above specification assumes C^* to be a constant. An alternative specification is a decreasing sequence so that as n goes to infinity the proxy alternative* *becomes less distinguishable from a unit root model. In such a case, when the pseudo-point-optimal test is based on a decreasing sequence C_n^* instead of C^*, the above results still hold. To see this, note that the maximal power does not depend on C_n^*, so we only need to consider an appropriately modified version of the result, as is done in Theorem 3 below.*

Theorem 3. *In the setting (8), the pseudo-point-optimal test based on the function $c^*(x) \equiv C_n^*$, where C_n^* is a positive decreasing sequence of n, has the power $P^c \le 0.05 + const_\varepsilon\, n^{-\frac{3}{2}+\varepsilon}$.*

3.2. **Unit root break at $t = \lfloor \alpha n \rfloor$.** This subsection focuses on the special case where the constant alternative is $c^*(x) \equiv C^*$ and the true function $\bar{c}(x)$ differs from zero only at a single fractional point $\alpha > 0$ in the sample. That is, $\bar{c}(x) = \bar{C}\mathbf{1}\{x = \alpha\}$ so that $\theta_{tn} = 1 + \frac{\bar{C}}{n}\mathbf{1}\{t = \lfloor \alpha n \rfloor\}$.

Lemma 6. *Under $H_0 : c(x) \equiv 0$, the point-optimal test with maximal power has the form*

$$S|_{H_0} = \frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t \left(2u_{\lfloor \alpha n \rfloor} - \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t\right).$$

Under $H_1 : c(x) \equiv \bar{c}(x)$, the point-optimal test with maximal power has the form

$$S|_{H_1} = \frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t \left(2u_{\lfloor \alpha n \rfloor} + \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t\right).$$

Theorem 4. *Maximal power for detection of a unit root break at $t = \lfloor \alpha n \rfloor$ is $P^m = 0.05 + \frac{1}{\sqrt{n}} \times const + o(n^{-0.5})$, for some $const > 0$.*

Lemma 7. *Under $H_0 : c(x) \equiv 0$, the pseudo-point-optimal test based on the function $c^*(x) \equiv C^*$ has the form*

$$S|_{H_0} = \frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau \left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right).$$

Under $H_1 : c(x) \equiv \bar{c}(x)$, the pseudo-point-optimal test based on the function $c^*(x) \equiv C^*$ has the form

$$\begin{aligned} S|_{H_1} = \frac{1}{2\sigma^2}\Bigg[&\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau \left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right) + 2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t\right)^2 \\ &+ 2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t\right)\sum_{t > \lfloor \alpha n \rfloor} u_t - 2\frac{\bar{C}C^{*2}}{n^3}\left(\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t\right)\sum_{t > \lfloor \alpha n \rfloor}\sum_{\tau=0}^{t-1} u_\tau \\ &- \frac{\bar{C}^2 C^{*2}}{n^4}\left(\sum_{t=0}^{\lfloor \alpha n \rfloor - 1} u_t\right)^2 (n - \lfloor \alpha n \rfloor)\Bigg]. \end{aligned}$$

Theorem 5. *The pseudo-point-optimal test based on the function* $c^*(x) \equiv C^*$ *for detection of a unit root break at* $t = \lfloor \alpha n \rfloor$ *has power* $P^c \leq 0.05 + const_\varepsilon\, n^{-1+\varepsilon}$*, for arbitrary* $\varepsilon > 0$.

The differences in power magnitude between early and later shifts in the generating mechanism may be explained as follows. The generating mechanism changes according to the presence or absence of the term $\frac{c}{n} X_{t-1}$. For an early shift with $\alpha = 0$ and $t = 1$, we have $\frac{c}{n} X_{t-1} = o_p(n^{-1/2})$ when $X_0 = o_p(n^{1/2})$, whereas for a later shift with $\alpha > 0$, we have $\frac{c}{n} X_{t-1} = O_p(n^{-1/2})$. Hence, the differential in the generating mechanism has a greater order of magnitude for a later shift, thereby enhancing discriminatory power in both the point optimal and pseudo-point optimal tests when $\alpha > 0$. The difference is greater also when $X_0 = O_p(1)$. So the order of magnitude of the initial condition impacts discriminatory power in the presence of an early shift in the generating mechanism. This explanation is relevant to earlier studies on unit root testing where the effects of initial conditions have been observed in simulation outcomes.

3.3. **Extension to Small Infinity Regions of originating data.** To extend Theorems 1 and 2, we consider a structural break FLUR model where the LUR coefficient $1+\frac{\bar{C}}{n}$, holds for $\{t = 1, ..., L\}$ and shifts to a UR coefficient for $\{t = L + 1, ..., n\}$, where L satisfies $\frac{1}{L} + \frac{L^2}{n} \to 0$ as $n \to \infty$. That is, we work with the model

$$X_t = \left(1 + \frac{\bar{C}}{n}\mathbf{1}\{t \leq L\}\right) X_{t-1} + u_t, \quad u_t \sim_{iid} N(0, \sigma^2), \tag{9}$$

so that $X_t = \theta_{tn} X_{t-1} + u_t$ with $\theta_{tn} = 1 + \frac{\bar{C}}{n}\mathbf{1}\{t \leq L\}$ for $t = 0, ..., n$, with originating data

$$X_{-1} = 0,\ X_0 = \theta_{0n} X_{-1} + u_0 = u_0, X_t = \left(1 + \frac{\bar{C}}{n}\right) X_{t-1} + u_t,\ t = 1, ..., L.$$

In this FLUR model, the effects of departures from a UR are confined to a small infinity ($L \to \infty$) of the originating data.

We proceed to examine the behavior of the functional point optimal test based on this specification and a pseudo-point optimal test based on the use of a constant LUR specification $\theta_{tn} = \left(1 + \frac{C^*}{n}\right)$ for all $t = 0, ..., n$.

Conjecture 1. *Suppose that* $L, n \to \infty$ *with* $\frac{L^2}{n} \to 0$ *as* $n \to \infty$*. In the setting of model* (9)*, maximal power is bounded from below by* $0.05 + const_\varepsilon(\frac{L}{n})^{1+\varepsilon}$ *and power of the pseudo-point-optimal test based on the function* $c^*(x) \equiv C^*$ *is bounded from above by* $0.05 + const_\varepsilon \left(\frac{L}{n}\right)^{3/2-\varepsilon}$*, where* $\varepsilon > 0$ *is arbitrary and* $const_\varepsilon$ *is a positive constant depending on* ε.

The Appendix presents an outline proof of this conjecture that is supported by numerical simulations in the unproven part of the argument where a required bound has not been obtained analytically. The authors have not yet been able to obtain a fully rigorous proof. So the result is stated as a conjecture.

4. Conclusion and Further Research

While there is substantial empirical evidence for the presence of unit root autoregressive roots in many economic and financial time series, insistence on a strict unit root model specification is known to be restrictive. In relaxing this specification, more realistic formulations will often allow for shifts and transitions in the generating mechanism for which there may be empirical or institutional evidence. In such cases, functional specifications of local to unity behavior in the autoregressive coefficient allow greater flexibility than constant departures of the Pitman drift form. As this paper shows, functional specifications have three main effects: (i) the limit form of the standardized time series now embodies nonlinearities that reflect the functional departures from unity leading to a nonlinear diffusion limit process; (ii) the discriminatory power of standard unit root tests and point optimal tests is diminished, in many cases considerably; and (iii) the power envelope is defined by a space of functions and is no longer given by a simple single curve derived from point optimal tests against constant local to unity departures.

The present contribution has focused on studying these effects. Obvious extensions to models with drifts are possible in which there will be corresponding changes to the so-called GLS detrending procedures. These procedures actually rely on quasi-differencing the data rather than full GLS transforms. In a FLUR model with drift, it is necessary to take account of functional departures from unity, which in turn implies more complex quasi-differencing methods to achieve more efficient detrending. These may be investigated along the lines of the present study. Further extensions of function space alternatives to mildly integrated and mildly explosive processes (Phillips and Magdalinos (2007)) are possible and accommodate subperiod departures of this type from unit root models that accord with recent empirical investigations of bubble phenomena (Phillips et al. (2015)). A major additional area of interest in inference concerns the construction of confidence intervals. The validity of such intervals and the possibility of uniform inference about autoregressive behavior in the presence of more complex departures from unity is clearly of importance in practical work. These matters are currently under investigation and will be reported in later work.

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5. Appendix

Lemma 8. *For data X_t generated by model* (1) *and* (2)

$$n^{-\frac{1}{2}}X_{\lfloor nr\rfloor} \xrightarrow[n\to\infty]{} \sigma K_c(r) := \sigma\int_0^r e^{\int_s^r c(k)dk}dW(s),$$

where $K_c(r)$ satisfies the stochastic differential equation:

$$dK_c(r) = c(r)K_c(r)dr + dW(r).$$

Proof. First, note that

$$\begin{aligned} X_t &= \theta_{tn}X_{t-1} + u_t = \theta_{tn}\theta_{nt-1}X_{t-2} + \theta_{nt}u_{t-1} + u_t \\ &= \sum_{j=1}^t u_j e^{\frac{1}{n}\sum_{k=1}^{t-j} c\left(\frac{t-k+1}{n}\right)} + e^{\frac{1}{n}\sum_{j=1}^t c\left(\frac{j}{n}\right)}X_0. \end{aligned} \tag{10}$$

Then, as $e^{\frac{1}{n}\sum_{j=1}^t c\left(\frac{j}{n}\right)}X_0 = O_p(1)$ (or $o_p(n^{0.5})$ if $X_0 = o_p(n^{0.5})$), using (10) we get

$$\begin{aligned} n^{-\frac{1}{2}}X_{\lfloor nr\rfloor} &= \sigma\sum_{j=1}^{\lfloor nr\rfloor} e^{\frac{1}{n}\sum_{k=1}^{\lfloor nr\rfloor-j} c\left(\frac{\lfloor nr\rfloor-k+1}{n}\right)}\int_{\frac{j-1}{n}}^{\frac{j}{n}} dX_n(s) + o_p(1)) \\ &= \sigma\sum_{j=1}^{\lfloor nr\rfloor}\int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{\int_s^r c(k)dk}dX_n(s) + o_p(1)) \\ &= \sigma\int_0^r e^{\int_s^r c(k)dk}dX_n(s) + o_p(1)). \end{aligned} \tag{11}$$

Using integration by parts

$$\int_0^r e^{\int_s^r c(k)dk}dX_n(s) = X_n(r) + \int_0^r e^{\int_s^r c(k)dk}c(s)X_n(s)ds,$$

taking limits as $n\to\infty$ in (11), and using continuous mapping, we get

$$n^{-\frac{1}{2}}X_{\lfloor nr\rfloor} \to \sigma\left(W(r) + \int_0^r e^{\int_s^r c(k)dk}c(s)W(s)ds\right) = \sigma\int_0^r e^{\int_s^r c(k)dk}dW(s),$$

giving the required result. Finally, denoting $K_c(r) = \int_0^r e^{\int_s^r c(k)dk}dW(s)$, we get

$$K_c(r) = W(r) + \int_0^r e^{\int_s^r c(k)dk}c(s)W(s)ds,$$

and

$$\begin{aligned} dK_c(r) &= dW(r) + c(r)W(r)dr + c(r)\int_0^r e^{\int_s^r c(k)dk}c(s)W(s)dsdr \\ &= dW(r) + c(r)K_c(r)dr, \end{aligned}$$

as required. □

Proof of Lemma 1.

Proof. Define $r = \frac{t-1}{n}$ and $\Delta r = \frac{1}{n}$. Under H_0 we have that $K_c(r) = \int_0^r dW(s) = W(r)$, so by Eq. (3),

$$\begin{aligned} d_t^0 &= \frac{1}{\sqrt{n}\sigma}\left(X_t - X_{t-1}\right) \approx K_c(r+\Delta r) - K_c(r) = \int_r^{r+\Delta r} dW(s) \\ &= W(r+\Delta r) - W(r), \end{aligned}$$

$$\begin{aligned} d_t^c &= \frac{1}{\sqrt{n}\sigma}\left(X_t - e^{c(t/n)/n}X_{t-1}\right) \approx K_c(r+\Delta r) - e^{c(r)\Delta r}K_c(r) \\ &= W(r+\Delta r) - e^{c(r)\Delta r}W(r), \end{aligned}$$

so that

$$\begin{aligned} \left(d_t^c\right)^2 &\approx \left(W(r+\Delta r) - e^{c(r)\Delta r}W(r)\right)^2 \\ &= \left(W(r+\Delta r) - W(r) + W(r)\left(1 - e^{c(r)\Delta r}\right)\right)^2 \\ &= \left(d_t^0\right)^2 + 2d_t^0 W(r)\left(1 - e^{c(r)\Delta r}\right) + W^2(r)\left(1 - e^{c(r)\Delta r}\right)^2. \end{aligned} \tag{12}$$

Plugging the expression for $\left(d_t^c\right)^2$ from Eq. (12) into Eq. (5), we get

$$\begin{aligned} \sum_t \left(d_t^0\right)^2 - \sum_t \left(d_t^c\right)^2 &= -2\sum d_t^0 W(r)\left(1 - e^{c(r)\Delta r}\right) \\ &\quad - \sum W^2(r)\left(1 - e^{c(r)\Delta r}\right)^2 \\ &\approx 2\Delta r \sum c(r)\left(W(r+\Delta r) - W(r)\right)W(r) \\ &\quad - (\Delta r)^2 \sum W^2(r)c^2(r) \\ &\approx 2\Delta r \int_0^1 c(s)W(s)dW(s) - \Delta r \int_0^1 c^2(s)W^2(s)ds. \end{aligned} \tag{13}$$

Using (13), we see that $S \xrightarrow{H_0} \int_0^1 c(s)W(s)dW(s) - \frac{1}{2}\int_0^1 c^2(s)W^2(s)ds$. Similarly, we can calculate the limit under H_1. By Eq. (3)

$$d_t^0 = \frac{1}{\sqrt{n}\sigma}\left(X_t - X_{t-1}\right) \approx K_c(r+\Delta r) - K_c(r),$$

$$d_t^c = \frac{1}{\sqrt{n}\sigma}\left(X_t - e^{c(t/n)/n}X_{t-1}\right) \approx K_c(r+\Delta r) - e^{c(r)\Delta r}K_c(r),$$

so that

$$\sum_t \left(d_t^0\right)^2 - \sum_t \left(d_t^c\right)^2 \approx \sum \left(K_c(r+\Delta r) - e^{c(r)\Delta r}K_c(r) + K_c(r)\left(e^{c(r)\Delta r}-1\right)\right)^2$$
$$-\sum \left(K_c(r+\Delta r) - e^{c(r)\Delta r}K_c(r)\right)^2$$
$$= 2\sum \left(K_c(r+\Delta r) - e^{c(r)\Delta r}K_c(r)\right)K_c(r)\left(c(r)\Delta r\right) + \sum K_c^2(r)c^2(r)(\Delta r)^2$$
$$\approx \Delta r\int_0^1 K_c^2(s)c^2(s)ds + 2\sum e^{c(r)\Delta r}\left(W(r+\Delta r)-W(r)\right)K_c(r)\left(c(r)\Delta r\right)$$
$$\approx \Delta r\int_0^1 K_c^2(s)c^2(s)ds + 2\Delta r\int_0^1 K_c(s)c(s)dW(s).$$

Thus, $S \xrightarrow{H_1} \int_0^1 c(s)K_c(s)dW(s) + \frac{1}{2}\int_0^1 c^2(s)K_c^2(s)ds.$ □

Proof of Lemma 2.

Proof. We decompose the test statistic into two summations:

$$\sum_t \left(d_t^0\right)^2 - \sum_t \left(d_t^{c^*}\right)^2 = \left(\sum_i \left(d_i^0\right)^2 - \sum_t \left(d_t^{\bar{c}}\right)^2\right) + \left(\sum_t \left(d_t^{\bar{c}}\right)^2 - \sum_t \left(d_t^{c^*}\right)^2\right).$$

From Lemma 1 we know that

$$\frac{1}{2\Delta r}\left(\sum_t \left(d_t^0\right)^2 - \sum_t \left(d_t^{\bar{c}}\right)^2\right) \xrightarrow{H_1} \int_0^1 \bar{c}(s)K_{\bar{c}}(s)dW(s) + \frac{1}{2}\int_0^1 \bar{c}^2(s)K_{\bar{c}}^2(s)ds. \tag{14}$$

So we are left with the second term $\sum_t \left(d_t^{\bar{c}}\right)^2 - \sum_t \left(d_t^{c^*}\right)^2$. We will apply the same technique as in Lemma 1 to approximate that term. In particular

$$\sum_t \left(d_t^{\bar{c}}\right)^2 - \sum_t \left(d_t^{c^*}\right)^2 \approx \sum \left(K_{\bar{c}}(r+\Delta r) - e^{\bar{c}(r)\Delta r}K_{\bar{c}}(r)\right)^2$$
$$-\sum \left(K_{\bar{c}}(r+\Delta r) - e^{\bar{c}(r)\Delta r}K_{\bar{c}}(r) + K_{\bar{c}}(r)\left(e^{\bar{c}(r)\Delta r} - e^{c^*(r)\Delta r}\right)\right)^2$$
$$= -2\sum \left(K_{\bar{c}}(r+\Delta r) - e^{\bar{c}(r)\Delta r}K_{\bar{c}}(r)\right)K_{\bar{c}}(r)\left(e^{\bar{c}(r)\Delta r} - e^{c^*(r)\Delta r}\right)$$
$$-\sum K_{\bar{c}}^2(r)\left(e^{\bar{c}(r)\Delta r} - e^{c^*(r)\Delta r}\right)^2 \tag{15}$$
$$\approx -2\sum e^{\bar{c}(r)\Delta r}\left(W(r+\Delta r)-W(r)\right)K_{\bar{c}}(r)\Delta r\left(\bar{c}(r)-c^*(r)\right)$$
$$-\sum K_{\bar{c}}^2(r)(\Delta r)^2\left(\bar{c}(r)-c^*(r)\right)^2$$
$$\approx -2\Delta r\int_0^1 K_{\bar{c}}(s)(\bar{c}(s)-c^*(s))dW(s) - \Delta r\int_0^1 K_{\bar{c}}^2(s)(\bar{c}(s)-c^*(s))^2ds.$$

Combining Eq. 14 and 15 we get that

$$S = \frac{1}{2\Delta r}\left(\sum_t \left(d_t^0\right)^2 - \sum_t \left(d_t^{c^*}\right)^2\right) \xrightarrow{H_1} \int_0^1 \bar{c}(s)K_{\bar{c}}(s)dW(s) + \frac{1}{2}\int_0^1 \bar{c}^2(s)K_{\bar{c}}^2(s)$$
$$- \int_0^1 K_{\bar{c}}(s)(\bar{c}(s) - c^*(s))dW(s) - \frac{1}{2}\int_0^1 K_{\bar{c}}^2(s)(\bar{c}(s) - c^*(s))^2 ds$$
$$= \int_0^1 K_{\bar{c}}(s)c^*(s)dW(s) + \int_0^1 K_{\bar{c}}^2(s)\bar{c}(s)c^*(s)ds - \frac{1}{2}\int_0^1 K_{\bar{c}}(s)c^{*2}(s)ds.$$

□

Proof of Lemma 3.

Proof. Define the joint density of (Z, Y) as $f(Z, Y)$ and the marginal densities as $f_Z(Z)$ and $f_Y(Y)$. Then

$$\begin{aligned}
\mathbb{P}\left\{Z + \frac{1}{n^k}Y > A\right\} &= \int_{\mathbb{R}}\int_{n^k(A-z)}^{\infty} f(z,y)dydz \\
&= \left(\int_{-\infty}^{A} + \int_{A}^{\infty}\right)\int_{n^k(A-z)}^{\infty} f(z,y)dydz \\
&\leq \left(\int_{-\infty}^{A-\varepsilon} + \int_{A-\varepsilon}^{A}\right)\int_{n^k(A-z)}^{\infty} f(z,y)dydz + \int_{A}^{\infty} f_Z(z)dz \\
&= \left(\int_{-\infty}^{A-\varepsilon} + \int_{A-\varepsilon}^{A}\right)\int_{n^k(A-z)}^{\infty} f(z,y)dydz + \mathbb{P}\left\{Z > A\right\},
\end{aligned} \tag{16}$$

where ε is arbitrary positive number.

Note that

$$\begin{aligned}
\int_{-\infty}^{A-\varepsilon}\int_{n^k(A-z)}^{\infty} f(z,y)dydz &\leq \int_{-\infty}^{A-\varepsilon}\int_{\varepsilon n^k}^{\infty} f(z,y)dydz \leq \int_{\varepsilon n^k}^{\infty} f_Y(y)dy \\
&= \mathbb{P}\left\{Y \geq \varepsilon n^k\right\} \leq \mathbb{P}\left\{|Y| \geq n^k\varepsilon\right\} \\
&= \mathbb{P}\left\{e^{\alpha k|Y|} \geq e^{\alpha k n^k \varepsilon}\right\} \leq \frac{\mathbb{E}\left(e^{\alpha k|Y|}\right)}{e^{n^k \alpha k \varepsilon}};
\end{aligned} \tag{17}$$

$$\int_{A-\varepsilon}^{A}\int_{n^k(A-z)}^{\infty} f(z,y)dydz \leq \int_{A-\varepsilon}^{A} f_Z(z)dz \leq B\varepsilon \tag{18}$$

where B is the upper bound of density of Z, $f(z)$, around point $z = A$.

Let $\varepsilon = \frac{\log n}{n^k}$, so that $\varepsilon \xrightarrow[n\to\infty]{} 0$. Then plugging Eq. 17 and 18 into Eq. (16), we get

$$\begin{aligned}
\mathbb{P}\left\{Z + \frac{1}{n^k}Y > A\right\} &\leq \mathbb{P}\left\{Z > A\right\} + B\varepsilon + \frac{\mathbb{E}\left(e^{\alpha k|Y|}\right)}{e^{n^k k\alpha\varepsilon}} \\
&= \mathbb{P}\left\{Z > A\right\} + Bn^{-k}\log n + \frac{\mathbb{E}\left(e^{\alpha k|Y|}\right)}{e^{\alpha k \log n}} \\
&= \mathbb{P}\left\{Z > A\right\} + O\left(\frac{1}{n^{k\alpha}} + n^{-k}\log n\right),
\end{aligned}$$

so that

$$\mathbb{P}\left\{Z+\frac{1}{n^k}Y>A\right\}\leq\mathbb{P}\left\{Z>A\right\}+O\left(n^{-k}\log n\right)\leq\mathbb{P}\left\{Z>A\right\}+\mathrm{const}_\varepsilon n^{-k+\varepsilon},$$

where $\varepsilon>0$ is arbitrary. □

Proof of Lemma 4.

Proof. Under H_0, $X_t=X_{t-1}+u_t=\sum_{\tau=0}^{t}u_\tau$.

$$\frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t-X_{t-1})^2\overset{H_0}{=}\frac{1}{2\sigma^2}\sum_{t=0}^{n}u_t^2. \tag{19}$$

$$\begin{aligned}\frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t-\left(1+\frac{\bar{c}(t/n)}{n}\right)X_{t-1}\right)^2&\overset{H_0}{=}\frac{1}{2\sigma^2}u_0^2+\frac{1}{2\sigma^2}\left(u_1-\frac{\bar{C}}{n}u_0\right)^2+\frac{1}{2\sigma^2}\sum_{t=2}^{n}u_t^2\\&=\frac{1}{2\sigma^2}\left(\sum_{t=0}^{n}u_t^2+\frac{\bar{C}}{n}u_0\left(\frac{\bar{C}}{n}u_0-2u_1\right)\right).\end{aligned} \tag{20}$$

Combining 19 and 20, we get

$$\begin{aligned}S&\overset{H_0}{=}\frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t-X_{t-1})^2-\frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t-\left(1+\frac{\bar{c}(t/n)}{n}\right)X_{t-1}\right)^2\\&=\frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1-\frac{\bar{C}}{n}u_0\right).\end{aligned} \tag{21}$$

Under H_1, $X_t=\left(1+\frac{\bar{c}(t/n)}{n}\right)X_{t-1}+u_t$.

$$\begin{aligned}\frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t-X_{t-1})^2&\overset{H_1}{=}\frac{1}{2\sigma^2}u_0^2+\frac{1}{2\sigma^2}\left(u_1+\frac{\bar{C}}{n}u_0\right)^2+\frac{1}{2\sigma^2}\sum_{t=2}^{n}u_t^2\\&=\frac{1}{2\sigma^2}\left(\sum_{t=0}^{n}u_t^2+\frac{\bar{C}}{n}u_0\left(\frac{\bar{C}}{n}u_0+2u_1\right)\right).\end{aligned} \tag{22}$$

$$\frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t-\left(1+\frac{\bar{c}(t/n)}{n}\right)X_{t-1}\right)^2\overset{H_1}{=}\frac{1}{2\sigma^2}\sum_{t=0}^{n}u_t^2. \tag{23}$$

Combining 22 and 23 we get

$$\begin{aligned}S&\overset{H_1}{=}\frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t-X_{t-1})^2-\frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t-\left(1+\frac{\bar{c}(t/n)}{n}\right)X_{t-1}\right)^2\\&=\frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1+\frac{\bar{C}}{n}u_0\right).\end{aligned} \tag{24}$$

□

Proof of Theorem 1.

Proof. We reject the null hypothesis at the 5% significance level when the test statistic is greater than the 5% critical value cv. Using the formula for $S|_{H_0}$ from Lemma 4, the constant cv is determined by

$$\mathbb{P}\left(\frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1-\frac{\bar{C}}{n}u_0\right)>cv\right)=0.05.$$

Note also that $\frac{1}{2\sigma^2}\bar{C}u_0\left(2u_1-\frac{\bar{C}}{n}u_0\right)\approx\frac{1}{\sigma^2}\bar{C}u_0u_1$ when n is large, so that $A^m:=n\cdot cv$ is approximately constant.

Plugging in $S|_{H_1}$ from Lemma 4, we get the following expression for maximal power, P^m:

$$\begin{aligned}
P^m &= \mathbb{P}\left(\frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1+\frac{\bar{C}}{n}u_0\right)>cv\right)\\
&= \mathbb{P}\left(\frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1-\frac{\bar{C}}{n}u_0\right)+\frac{1}{\sigma^2}\left(\frac{\bar{C}}{n}u_0\right)^2>cv\right)\\
&= 0.05+\mathbb{P}\left(\frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1-\frac{\bar{C}}{n}u_0\right)\le cv,\frac{1}{2\sigma^2}\frac{\bar{C}}{n}u_0\left(2u_1+\frac{\bar{C}}{n}u_0\right)>cv\right)\\
&= 0.05+\mathbb{P}\left(u_0>0,u_1\in\left(\frac{\sigma^2 cv}{\frac{\bar{C}}{n}u_0}-\frac{\bar{C}}{2n}u_0,\frac{\sigma^2 cv}{\frac{\bar{C}}{n}u_0}+\frac{\bar{C}}{2n}u_0\right]\right)\\
&+\mathbb{P}\left(u_0<0,u_1\in\left(\frac{\sigma^2 cv}{\frac{\bar{C}}{n}u_0}+\frac{\bar{C}}{2n}u_0,\frac{\sigma^2 cv}{\frac{\bar{C}}{n}u_0}-\frac{\bar{C}}{2n}u_0\right]\right)
\end{aligned}$$

$$\begin{aligned}
&= 0.05+\mathbb{P}\left(u_0>0,u_1\in\left(\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0,\frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0\right]\right)\\
&+\mathbb{P}\left(u_0<0,u_1\in\left(\frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0,\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0\right]\right)\\
&= 0.05+\frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}e^{-u^2/2\sigma^2}\frac{\bar{C}}{n}ue^{-\tilde{u}(u)^2/2\sigma^2}du\\
&= 0.05+\frac{\bar{C}}{n}\frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}e^{-u^2/2\sigma^2}u\left(e^{-\frac{\sigma^2(A^m)^2}{2\bar{C}^2u^2}}+o(1)\right)du\\
&= 0.05+\frac{\bar{C}}{n}\cdot\text{const}+o(n^{-1}),
\end{aligned}\tag{25}$$

where $\tilde{u}(u)$ lies in the neighbourhood of $\dfrac{\sigma^2A^m}{\bar{C}u}$

To justify the last expression, we use the following truncation argument. We split the domain of u_0 into the two regions $\left\{|u_0|\le n^\delta\right\}$ and $\left\{|u_0|>n^\delta\right\}$ for some $\delta\in(0,1)$.

Then, we have

$$\begin{aligned}
&\mathbb{P}\left(u_0>0, u_1\in\left(\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0, \frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0\right]\right)\\
=&\ \frac{1}{2\pi\sigma^2}\int_0^\infty e^{-\frac{1}{2}u_0^2/\sigma^2}\int_{\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0}^{\frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0} e^{-\frac{1}{2}u_1^2/\sigma^2}du_1du_0\\
=&\ \frac{1}{2\pi\sigma^2}\left(\int_0^{n^\delta}+\int_{n^\delta}^\infty\right)e^{-\frac{1}{2}u_0^2/\sigma^2}\int_{\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0}^{\frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0} e^{-\frac{1}{2}u_1^2/\sigma^2}du_1du_0.
\end{aligned}$$

First

$$\begin{aligned}
&\frac{1}{2\pi\sigma^2}\int_0^{n^\delta} e^{-\frac{1}{2}u_0^2/\sigma^2}\int_{\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0}^{\frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0} e^{-\frac{1}{2}u_1^2/\sigma^2}du_1du_0\\
&=\frac{1}{2\pi\sigma^2}\int_0^{n^\delta} e^{-\frac{1}{2}u_0^2/\sigma^2}\frac{\bar{C}}{n}u_0e^{-\frac{1}{2}\tilde{u}_1(u_0)^2/\sigma^2}du_0,\\
&\text{with } \tilde{u}_1(u_0)\in\left(\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0, \frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0\right) \qquad (26)\\
&=\frac{\bar{C}}{n2\pi\sigma^2}\int_0^\infty e^{-\frac{1}{2}u_0^2/\sigma^2}u_0e^{-\frac{1}{2}\tilde{u}_1(u_0)^2/\sigma^2}du_0\\
&=\frac{\bar{C}}{n}\times const,
\end{aligned}$$

by integrability of $u_0e^{-\frac{1}{2}u_0^2/\sigma^2}$ and boundedness of $e^{-\frac{1}{2}\tilde{u}_1(u_0)^2/\sigma^2}$ since $\frac{u_0}{n}=o(1)$ throughout $\left\{|u_0|\le n^\delta\right\}$, which makes $\left(\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0, \frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0\right)$ a shrinking interval of $\frac{\sigma^2A^m}{\bar{C}u_0}$, justifying (26). Second,

$$\begin{aligned}
&\left|\frac{1}{2\pi\sigma^2}\int_{n^\delta}^\infty e^{-\frac{1}{2}u_0^2/\sigma^2}\int_{\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0}^{\frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0} e^{-\frac{1}{2}u_1^2/\sigma^2}du_1du_0\right|\\
\le&\ \frac{1}{\sqrt{2\pi}\sigma}\int_{n^\delta}^\infty e^{-\frac{1}{2}u_0^2/\sigma^2}du_0=O\left(n^{-\delta}e^{-\frac{1}{2\sigma^2}n^{2\delta}}\right), \text{ by Mills ratio,}
\end{aligned}$$

which gives

$$\mathbb{P}\left(u_0>0, u_1\in\left(\frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0, \frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0\right]\right)=\frac{\bar{C}}{n}\times const.\ +\ o\left(n^{-1}\right).$$

A similar argument shows that

$$\mathbb{P}\left(u_0<0, u_1\in\left(\frac{\sigma^2A^m}{\bar{C}u_0}+\frac{\bar{C}}{2n}u_0, \frac{\sigma^2A^m}{\bar{C}u_0}-\frac{\bar{C}}{2n}u_0\right]\right)=\frac{\bar{C}}{n}\times const.\ +\ o\left(n^{-1}\right)$$

giving the required result (25). □

Proof of Lemma 5.

Proof. Under H_0, $X_t = X_{t-1} + u_t = \sum_{\tau=0}^{t} u_\tau$.

$$\frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 \stackrel{H_0}{=} \frac{1}{2\sigma^2}\sum_{t=0}^{n} u_t^2. \tag{27}$$

$$\begin{aligned}
&\frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t - \left(1+\frac{C^*}{n}\right)X_{t-1}\right)^2 \stackrel{H_0}{=} \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(u_t - \frac{C^*}{n}X_{t-1}\right)^2 \\
&= \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right)^2 \\
&= \frac{1}{2\sigma^2}\left[\sum_{t=0}^{n}\left(u_t^2 + \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau - 2u_t\right)\right)\right].
\end{aligned} \tag{28}$$

Combining 27 and 28 we get

$$\begin{aligned}
S &\stackrel{H_0}{=} \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t - \left(1+\frac{C^*}{n}\right)X_{t-1}\right)^2 \\
&= \frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right).
\end{aligned} \tag{29}$$

Under H_1, $X_t = \left(1+\frac{\bar{c}(t/n)}{n}\right)X_{t-1} + u_t = \frac{\bar{C}}{n}u_0 + \sum_{\tau=0}^{t} u_\tau, t \geq 1,\ X_0 = u_0$.

$$\begin{aligned}
\frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 &\stackrel{H_1}{=} \frac{1}{2\sigma^2}u_0^2 + \frac{1}{2\sigma^2}\left(u_1 + \frac{\bar{C}}{n}u_0\right)^2 + \frac{1}{2\sigma^2}\sum_{t=2}^{n}u_t^2 \\
&= \frac{1}{2\sigma^2}\left(\sum_{t=0}^{n}u_t^2 + \frac{\bar{C}}{n}u_0\left(\frac{\bar{C}}{n}u_0 + 2u_1\right)\right).
\end{aligned} \tag{30}$$

(31)

$$\begin{aligned}
&\frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t - \left(1+\frac{C^*}{n}\right)X_{t-1}\right)^2 \\
&\stackrel{H_1}{=} \frac{1}{2\sigma^2}\left(u_0^2 + \left(u_1 + u_0\left(\frac{\bar{C}}{n} - \frac{C^*}{n}\right)\right)^2 + \sum_{t=2}^{n}\left(u_t - \frac{C^*}{n}X_{t-1}\right)^2\right) \\
&= \frac{1}{2\sigma^2}\left(\sum_{t=0}^{n}u_t^2 + u_0\frac{\bar{C}-C^*}{n}\left(2u_1 + u_0\frac{\bar{C}-C^*}{n}\right) + \sum_{t=2}^{n}\frac{C^*}{n}X_{t-1}\left(\frac{C^*}{n}X_{t-1} - 2u_t\right)\right) \\
&= \frac{1}{2\sigma^2}\left[\sum_{t=0}^{n}u_t^2 + u_0\frac{\bar{C}-C^*}{n}\left(2u_1 + u_0\frac{\bar{C}-C^*}{n}\right)\right. \\
&\left. + \sum_{t=2}^{n}\frac{C^*}{n}\left(\frac{\bar{C}}{n}u_0 + \sum_{\tau=0}^{t-1}u_\tau\right)\left(\frac{C^*}{n}\left(\frac{\bar{C}}{n}u_0 + \sum_{\tau=0}^{t-1}u_\tau\right) - 2u_t\right)\right].
\end{aligned}$$

Combining 30 and 31 we get

(32)

$$
\begin{aligned}
S &\overset{H_1}{=} \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t - \left(1+\frac{C^*}{n}\right)X_{t-1}\right)^2 \\
&= \frac{1}{2\sigma^2}\left[\frac{\bar{C}C^*}{n^2}u_0^2 + \frac{C^*}{n}u_0\left(2u_1 + \frac{\bar{C}-C^*}{n}u_0\right)\right. \\
&\left. - \sum_{t=2}^{n}\frac{C^*}{n}\left(\frac{\bar{C}}{n}u_0 + \sum_{\tau=0}^{t-1}u_\tau\right)\left(\frac{C^*}{n}\left(\frac{\bar{C}}{n}u_0 + \sum_{\tau=0}^{t-1}u_\tau\right) - 2u_t\right)\right] \\
&= \frac{C^*\bar{C}}{\sigma^2 n^2}u_0^2 + \frac{1}{2\sigma^2}\sum_{t=1}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right) \\
&+ \frac{1}{2\sigma^2}\sum_{t=2}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(-\frac{C^*\bar{C}}{n^2}u_0\right) + \frac{C^*\bar{C}}{2\sigma^2 n^2}u_0\sum_{t=2}^{n}\left(2u_t - \frac{C^*}{n}\left(\frac{\bar{C}}{n}u_0 + \sum_{\tau=0}^{t-1}u_\tau\right)\right) \\
&= \frac{1}{2\sigma^2}\sum_{t=1}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right) + \frac{C^*\bar{C}}{\sigma^2 n^2}u_0^2 \\
&+ \frac{C^*\bar{C}}{2\sigma^2 n^2}u_0\sum_{t=2}^{n}\left(2u_t - \frac{C^*\bar{C}}{n^2}u_0\right) - \frac{C^{*2}\bar{C}}{\sigma^2 n^3}u_0\sum_{t=2}^{n}\sum_{\tau=0}^{t-1}u_\tau.
\end{aligned}
$$

□

Proof of Theorem 2.

Proof. The proof is similar to that of Theorem 1. Using the formula for $S|_{H_0}$ from Lemma 5, the 5% critical value cv is determined by

$$
\mathbb{P}\left(\frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right) > cv\right) = 0.05.
$$

Note that

$$
\begin{aligned}
\frac{2}{n}\sum_{t=0}^{n}u_t\sum_{\tau=0}^{t-1}u_\tau &= \frac{1}{n}\left((u_0+\cdots+u_n)^2 - \sum_{0}^{n}u_t^2\right) \\
&= \left(\frac{u_0+\cdots+u_n}{\sqrt{n}}\right)^2 - \frac{1}{n}\sum_{0}^{n}u_t^2 = \sigma^2\chi_1^2 - \sigma^2 + o(1).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{1}{n^2}\sum_{t=0}^{n}\left(\sum_{\tau=0}^{t-1}u_\tau\right)\left(\sum_{\tau=0}^{t-1}u_\tau\right) &= \frac{1}{n}\sum_{0}^{n}\left(\frac{u_0+\cdots+u_{t-1}}{\sqrt{n}}\right)^2 \\
&= \frac{1}{n}\sum_{0}^{n}\sigma^2 W^2\left(\frac{t-1}{n}\right) = \sigma^2\int_0^1 W^2(s)ds + o(1),
\end{aligned}
$$

so that

$$\frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t-\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right)=\xi+o(1), \tag{33}$$

and cv is approximately constant with respect to n.

Thus, plugging in $S|_{H_1}$ from Lemma 5, P^c becomes

$$P^c=\mathbb{P}\Bigg(\frac{1}{2\sigma^2}\sum_{t=1}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t-\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right)$$
$$+\frac{C^*\bar{C}}{\sigma^2n^2}u_0^2+\frac{C^*\bar{C}}{2\sigma^2n^2}u_0\sum_{t=2}^{n}\left(2u_t-\frac{C^*\bar{C}}{n^2}u_0\right)-\frac{C^{*2}\bar{C}}{\sigma^2n^3}u_0\sum_{t=2}^{n}\sum_{\tau=0}^{t-1}u_\tau>cv\Bigg).$$

Note that

$$\frac{C^*\bar{C}}{\sigma^2n^2}u_0^2=O(n^{-2}),$$

$$\frac{1}{n^2}u_0\sum_{t=2}^{n}\left(2u_t-\frac{C^*\bar{C}}{n^2}u_0\right)=n^{-3/2}u_0\frac{u_2+\cdots+u_n}{\sqrt{n}}-\frac{C^*\bar{C}}{n^3}u_0^2$$
$$=n^{-3/2}u_0\cdot(N(0,\sigma^2)+o(1))+O(n^{-3})=O(n^{-3/2}).$$

Finally,

$$\frac{1}{\sigma^2n^3}\sum_{t=2}^{n}\sum_{\tau=0}^{t-1}u_\tau=\sigma^{-2}n^{-5/2}\left(\frac{u_0+u_1}{\sqrt{n}}+\cdots+\frac{u_0+u_1+\cdots+u_{n-1}}{\sqrt{n}}\right)$$
$$=n^{-5/2}\left(W(1/n)+W(2/n)+\cdots+W(1)\right)$$
$$=n^{-3/2}\left(\int_0^1W(s)ds+o(1)\right)=O(n^{-3/2}),$$

so that we have

$$P^c=\mathbb{P}\left(\xi+n^{-3/2}\eta>cv\right)=\mathbb{P}(\xi+O(n^{-3/2})>cv)$$
$$\le\mathbb{P}(\xi>cv)+\text{const}_\varepsilon\,n^{-\frac{3}{2}+\varepsilon}=0.05+\text{const}_\varepsilon\,n^{-\frac{3}{2}+\varepsilon},$$

where the inequality follows from Lemma 3, ξ was defined in Eq. (33), and $n^{-3/2}\eta=\frac{C^*\bar{C}}{\sigma^2n^2}u_0^2+\frac{C^*\bar{C}}{2\sigma^2n^2}u_0\sum_{t=2}^{n}\left(2u_t-\frac{C^*\bar{C}}{n^2}u_0\right)-\frac{C^{*2}\bar{C}}{\sigma^2n^3}u_0\sum_{t=2}^{n}\sum_{\tau=0}^{t-1}u_\tau$. □

Proof of Theorem 3.

Proof. We can use formulae for the test statistics from Theorem 2 replacing C^* by C^*_n. Only asymptotic behavior may change. The critical value cv is such that

$$\mathbb{P}\left(\frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C^*_n}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t-\frac{C^*_n}{n}\sum_{\tau=0}^{t-1}u_\tau\right)>cv\right)=0.05.$$

As before, we are interested in behavior as n goes to infinity. We can write:

$$\begin{aligned}\frac{2}{n}\sum_{t=0}^{n} u_t \sum_{\tau=0}^{t-1} u_\tau &= \frac{1}{n}\left((u_0+\cdots+u_n)^2 - \sum_0^n u_t^2\right)\\ &= \left(\frac{u_0+\cdots+u_n}{\sqrt{n}}\right)^2 - \frac{1}{n}\sum_0^n u_t^2 = \sigma^2\chi_1^2 - \sigma^2 + o(1).\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{1}{n^2}\sum_{t=0}^{n}\left(\sum_{\tau=0}^{t-1} u_\tau\right)\left(\sum_{\tau=0}^{t-1} u_\tau\right) &= \frac{1}{n}\sum_0^n \left(\frac{u_0+\cdots+u_{t-1}}{\sqrt{n}}\right)^2\\ &= \frac{1}{n}\sum_0^n \sigma^2 W^2\left(\frac{t-1}{n}\right) = \sigma^2\int_0^1 W^2(s)ds + o(1),\end{aligned}$$

so that

$$\frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{1}{n}\sum_{\tau=0}^{t-1} u_\tau\left(2u_t - \frac{C_n^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right) = \frac{1}{2}\left(\chi_1^2+1\right) + \frac{\lim\limits_{n\to\infty} C_n^*}{2}\int_0^1 W^2(s)ds + o(1),$$

and $\frac{cv}{C_n^*}$ is also approximately constant with respect to n.

Similarly,

$$\frac{\bar{C}}{\sigma^2 n^2}u_0^2 = O(n^{-2}),$$

$$\begin{aligned}\frac{1}{n^2}u_0\sum_{t=2}^{n}\left(2u_t - \frac{C_n^*\bar{C}}{n^2}u_0\right) &= n^{-3/2}u_0\frac{u_2+\cdots+u_n}{\sqrt{n}} - \frac{C_n^*\bar{C}}{n^3}u_0^2\\ &= n^{-3/2}u_0\cdot\left(N(0,\sigma^2)+o(1)\right) + O(n^{-3}) = O(n^{-3/2}),\end{aligned}$$

and

$$\begin{aligned}\frac{C_n^*}{\sigma^2 n^3}\sum_{t=2}^{n}\sum_{\tau=0}^{t-1} u_\tau &= C_n^*\sigma^{-2}n^{-5/2}\left(\frac{u_0+u_1}{\sqrt{n}} + \cdots + \frac{u_0+u_1+\cdots+u_{n-1}}{\sqrt{n}}\right)\\ &= C_n^* n^{-5/2}\left(W(1/n)+W(2/n)+\cdots+W(1)\right)\\ &= C_n^* n^{-3/2}\left(\int_0^1 W(s)ds + o(1)\right) = C_n^* O(n^{-3/2}).\end{aligned}$$

Thus, we can rewrite power as

$$\begin{aligned}
P^c &= \mathbb{P}\Bigg[\frac{1}{2\sigma^2}\sum_{t=1}^{n}\frac{1}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C_n^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right) \\
&\quad + \frac{\bar{C}}{\sigma^2 n^2}u_0^2 + \frac{\bar{C}}{2\sigma^2 n^2}u_0\sum_{t=2}^{n}\left(2u_t - \frac{C_n^*\bar{C}}{n^2}u_0\right) - \frac{C_n^*\bar{C}}{\sigma^2 n^3}u_0\sum_{t=2}^{n}\sum_{\tau=0}^{t-1}u_\tau > \frac{cv}{C_n^*}\Bigg] \\
&= \mathbb{P}\left[\frac{1}{2}\left(\chi_1^2+1\right) + \frac{\lim_{n\to\infty} C_n^*}{2}\int_0^1 W^2(s)ds + O(n^{-3/2}) > \frac{cv}{C_n^*}\right] \\
&\le \mathbb{P}\left[\frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C_n^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t - \frac{C_n^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right) > cv\right] + \mathrm{const}_\varepsilon\, n^{-\frac{3}{2}+\varepsilon} \\
&= 0.05 + \mathrm{const}_\varepsilon\, n^{-\frac{3}{2}+\varepsilon},
\end{aligned}$$

giving the required result. □

Proof of Lemma 6.

Proof. Under H_0, $X_t = X_{t-1} + u_t = \sum_{\tau=0}^{t} u_\tau$, so

$$\sum_{t=0}^{n}(X_t - X_{t-1})^2 = \sum_{t=0}^{n}u_t^2; \tag{34}$$

(35)

$$\begin{aligned}
&\sum_{t=0}^{n}\left(X_t - \left(1 + \frac{\bar{C}}{n}\mathbf{1}\{t = \lfloor \alpha n \rfloor\}\right)X_{t-1}\right)^2 = \sum_{t\neq\lfloor \alpha n \rfloor}u_t^2 + \left(u_{\lfloor \alpha n \rfloor} - \frac{\bar{C}}{n}\sum_{\tau=0}^{\lfloor \alpha n \rfloor - 1}u_\tau\right)^2 \\
&= \sum_{t=0}^{n}u_t^2 + \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1}u_t\left(\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1}u_t - 2u_{\lfloor \alpha n \rfloor}\right).
\end{aligned}$$

Combining equations (34) and (35), we get

$$S|_{H_0} := \frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1}u_t\left(2u_{\lfloor \alpha n \rfloor} - \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor \alpha n \rfloor - 1}u_t\right).$$

Under H_1,

$$X_t = \begin{cases} X_{t-1} + u_t = \sum_{\tau=0}^{t}u_\tau, & t < \lfloor \alpha n \rfloor; \\ \left(1 + \frac{\bar{C}}{n}\right)X_{t-1} + u_t = \sum_{\tau=0}^{t}u_\tau + \frac{\bar{C}}{n}\sum_{\tau=0}^{t-1}u_\tau, & t = \lfloor \alpha n \rfloor; \\ X_{t-1} + u_t = \sum_{\tau=0}^{t}u_\tau + \frac{\bar{C}}{n}\sum_{\tau=0}^{\lfloor \alpha n \rfloor - 1}u_\tau, & t > \lfloor \alpha n \rfloor. \end{cases}$$

So, we have

$$\sum_{t=0}^{n}(X_t - X_{t-1})^2 = \sum_{t\neq\lfloor\alpha n\rfloor}^{n} u_t^2 + \left(u_{\lfloor\alpha n\rfloor} + \frac{\bar{C}}{n}\sum_{\tau=0}^{\lfloor\alpha n\rfloor-1} u_\tau\right)^2$$
$$= \sum_{t=0}^{n} u_t^2 + \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor} + \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right), \tag{36}$$

and

$$\sum_{t=0}^{n}\left(X_t - \left(1 + \frac{\bar{C}}{n}\mathbf{1}\left\{t = \lfloor\alpha n\rfloor\right\}\right)X_{t-1}\right)^2 = \sum_{t=0}^{n} u_t^2. \tag{37}$$

Combining equations (36) and (37) gives

$$S|_{H_0} := \frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor} + \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right).$$

□

Proof of Theorem 4.

Proof. We reject the null hypothesis at the 5% significance level when the test statistic is greater than the 5% critical value cv. The constant acv is determined by

$$\mathbb{P}\left(\frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor} - \frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right) > cv\right) = 0.05,$$

where we use Lemma 6 for $S|_{H_0}$.

Because $\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t = \frac{\bar{C}\sqrt{\lfloor\alpha n\rfloor}}{n}\frac{1}{\sqrt{\lfloor\alpha n\rfloor}}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t \sim \sqrt{\frac{\alpha}{n}}N(0,\bar{C}^2\sigma^2) = O_p(n^{-0.5})^2$, we get $S_{H_0} = O_p(n^{-0.5})$. Thus, we also have that $cv = O_p(n^{-0.5})$ or $\sqrt{n}\cdot cv = O_p(1)$.

[2]Here $\alpha > 0$ matters, as otherwise we can not use CLT for the partial sum $\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t$.

Plugging $S|_{H_1}$ from Lemma 6, we obtain the following expression for maximal power

(38)

$$\begin{aligned}
P^m &= \mathbb{P}\left[\frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor}+\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)>cv\right]\\
&= \mathbb{P}\left[\frac{1}{2\sigma^2}\left(\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor}-\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)+2\left(\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)^2\right)>cv\right]\\
&= \mathbb{P}\left[\frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor}-\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)>cv\right]\\
&\quad+\mathbb{P}\left[\frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor}-\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)\le cv,\right.\\
&\qquad\left.\frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor}+\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)>cv\right]\\
&= 0.05+\mathbb{P}\left[\frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor}-\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)\le cv,\right.\\
&\qquad\left.\frac{1}{2\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\left(2u_{\lfloor\alpha n\rfloor}+\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\right)>cv\right]\\
&=: 0.05+X.
\end{aligned}$$

Denote $v=\frac{1}{\sigma}\sum_{t=0}^{\lfloor\alpha n\rfloor-1} u_t\sim N(0,\lfloor\alpha n\rfloor)$, $u=\frac{1}{\sigma}u_{\lfloor\alpha n\rfloor}\sim N(0,1)$. Then $v\perp u$ and we are left with calculating

(39)

$$\begin{aligned}
X &= \mathbb{P}\left(\frac{\bar{C}}{n}uv-\frac{\bar{C}^2}{2n^2}v^2\le cv,\ \frac{\bar{C}}{n}uv+\frac{\bar{C}^2}{2n^2}v^2>cv\right)\\
&= \mathbb{P}\left(u\frac{v}{\sqrt{n}}\in\left(\frac{cv\sqrt{n}}{\bar{C}}-\frac{\bar{C}}{2n^{3/2}}v^2,\frac{cv\sqrt{n}}{\bar{C}}+\frac{\bar{C}}{2n^{3/2}}v^2\right]\right)\\
&= \int_0^\infty\frac{1}{\sqrt{2\pi\bar{C}^2(\lfloor\alpha n\rfloor/n)}}e^{-\frac{y^2}{2\bar{C}^2(\lfloor\alpha n\rfloor/n)}}\int_{\frac{\sqrt{n}cv}{y}-\frac{y}{2\sqrt{n}}}^{\frac{\sqrt{n}cv}{y}+\frac{y}{2\sqrt{n}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dxdy\\
&+\int_{-\infty}^0\frac{1}{\sqrt{2\pi\bar{C}^2(\lfloor\alpha n\rfloor/n)}}e^{-\frac{y^2}{2\bar{C}^2(\lfloor\alpha n\rfloor/n)}}\int_{\frac{\sqrt{n}cv}{y}+\frac{y}{2\sqrt{n}}}^{\frac{\sqrt{n}cv}{y}-\frac{y}{2\sqrt{n}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dxdy
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\infty} \frac{e^{-\frac{y^2}{2\bar{C}^2(\lfloor \alpha n \rfloor/n)}}}{\sqrt{2\pi \bar{C}^2(\lfloor \alpha n \rfloor/n)}} \left[\Phi\left(\frac{\sqrt{n}\cdot cv}{y} + \frac{y}{2\sqrt{n}}\right) - \Phi\left(\frac{\sqrt{n}\cdot cv}{y} - \frac{y}{2\sqrt{n}}\right)\right] dy \\
&+ \int_{-\infty}^{0} \frac{e^{-\frac{y^2}{2\bar{C}^2(\lfloor \alpha n \rfloor/n)}}}{\sqrt{2\pi \bar{C}^2(\lfloor \alpha n \rfloor/n)}} \left[\Phi\left(\frac{\sqrt{n}\cdot cv}{y} - \frac{y}{2\sqrt{n}}\right) - \Phi\left(\frac{\sqrt{n}\cdot cv}{y} + \frac{y}{2\sqrt{n}}\right)\right] dy \\
&= \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \frac{|y| e^{-\frac{y^2}{2\bar{C}^2(\lfloor \alpha n \rfloor/n)}}}{\sqrt{2\pi \bar{C}^2(\lfloor \alpha n \rfloor/n)}} \phi\left(\frac{\sqrt{n}\cdot cv}{y} + \varepsilon\right) dy + o\left(n^{-0.5}\right) \\
&= \frac{1}{\sqrt{n}} \times \text{const} + o(n^{-0.5}),
\end{aligned}
$$

where $\varepsilon \to 0$ as $n \to \infty$. Here we used the fact that density $\phi(\cdot)$ of $N(0,1)$ is bounded, so that the integral $\int_{-\infty}^{\infty} \frac{|y| e^{-\frac{y^2}{2\bar{C}^2(\lfloor \alpha n \rfloor/n)}}}{\sqrt{2\pi \bar{C}^2(\lfloor \alpha n \rfloor/n)}} \phi\left(\frac{\sqrt{n}\cdot cv}{y} + \varepsilon\right)$ is finite.

Therefore, plugging X from Eq. (39) into Eq. (38), we get $P^m = 0.05 + \frac{1}{\sqrt{n}} \times \text{const} + o(n^{-0.5})$. □

Proof of Lemma 7.

Proof. Under H_0, $X_t = X_{t-1} + u_t = \sum_{\tau=0}^{t} u_\tau$, so

$$
\sum_{t=0}^{n} (X_t - X_{t-1})^2 = \sum_{t=0}^{n} u_t^2, \tag{40}
$$

$$
\begin{aligned}
\sum_{t=0}^{n} \left(X_t - \left(1 + \frac{C^*}{n}\right) X_{t-1}\right)^2 &= \sum_{t=0}^{n} \left(u_t - \frac{C^*}{n} \sum_{\tau=0}^{t-1} u_\tau\right)^2 \\
&= \sum_{t=0}^{n} u_t^2 + \sum_{t=0}^{n} \frac{C^*}{n} \sum_{\tau=0}^{t-1} u_\tau \left(\frac{C^*}{n} \sum_{\tau=0}^{t-1} u_\tau - 2u_t\right).
\end{aligned} \tag{41}
$$

Combining equations (40) and (41), we get

$$
S|_{H_0} := \frac{1}{2\sigma^2} \sum_{t=0}^{n} \frac{C^*}{n} \sum_{\tau=0}^{t-1} u_\tau \left(2u_t - \frac{C^*}{n} \sum_{\tau=0}^{t-1} u_\tau\right).
$$

Under H_1,

$$
X_t = \begin{cases}
X_{t-1} + u_t = \sum_{\tau=0}^{t} u_\tau, & t < \lfloor \alpha n \rfloor; \\
\left(1 + \frac{\bar{C}}{n}\right) X_{t-1} + u_t = \sum_{\tau=0}^{t} u_\tau + \frac{\bar{C}}{n} \sum_{\tau=0}^{t-1} u_\tau, & t = \lfloor \alpha n \rfloor; \\
X_{t-1} + u_t = \sum_{\tau=0}^{t} u_\tau + \frac{\bar{C}}{n} \sum_{\tau=0}^{\lfloor \alpha n \rfloor - 1} u_\tau, & t > \lfloor \alpha n \rfloor.
\end{cases}
$$

So we have:

$$\begin{aligned}\sum_{t=0}^{n}(X_t-X_{t-1})^2 &= \sum_{t\neq\lfloor\alpha n\rfloor}^{n} u_t^2+\left(u_{\lfloor\alpha n\rfloor}+\frac{\bar{C}}{n}\sum_{\tau=0}^{\lfloor\alpha n\rfloor-1}u_\tau\right)^2\\ &=\sum_{t=0}^{n}u_t^2+\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\left(2u_{\lfloor\alpha n\rfloor}+\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right);\end{aligned}\tag{42}$$

$$\begin{aligned}&\sum_{t=0}^{n}\left(X_t-\left(1+\frac{C^*}{n}\right)X_{t-1}\right)^2=\sum_{t=0}^{\lfloor\alpha n\rfloor-1}\left(u_t-\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right)^2\\ &+\left(u_{\lfloor\alpha n\rfloor}+\frac{\bar{C}-C^*}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)^2+\sum_{t>\lfloor\alpha n\rfloor}\left(u_t-\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau-\frac{\bar{C}C^*}{n^2}\sum_{\tau=0}^{\lfloor\alpha n\rfloor-1}u_\tau\right)^2\\ &=\sum_{t=0}^{n}u_t^2+\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\left(2u_{\lfloor\alpha n\rfloor}+\frac{\bar{C}}{n}\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)-2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)^2\\ &+\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau-2u_t\right)-2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor\alpha n\rfloor}u_t\\ &+2\frac{\bar{C}C^{*2}}{n^3}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor\alpha n\rfloor}\sum_{\tau=0}^{t-1}u_\tau+\frac{\bar{C}^2C^{*2}}{n^4}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)^2(n-\lfloor\alpha n\rfloor).\end{aligned}\tag{43}$$

Combining equations (42) and (43), we get

$$\begin{aligned}S|_{H_0}:=\frac{1}{2\sigma^2}\Bigg[&\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t-\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right)+2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)^2\\ &+2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor\alpha n\rfloor}u_t-2\frac{\bar{C}C^{*2}}{n^3}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor\alpha n\rfloor}\sum_{\tau=0}^{t-1}u_\tau\\ &-\frac{\bar{C}^2C^{*2}}{n^4}\left(\sum_{t=0}^{\lfloor\alpha n\rfloor-1}u_t\right)^2(n-\lfloor\alpha n\rfloor)\Bigg].\end{aligned}$$

□

Proof of Theorem 5.

Proof. The 5% critical value cv is determined by

$$\mathbb{P}\left(\frac{1}{2\sigma^2}\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\left(2u_t-\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right)>cv\right)=0.05,$$

and cv is bounded in probability as $n \to \infty$ because

$$\frac{1}{\sigma^2}\sum_{t=0}^{n}\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau u_t = C^*\sum_{t=0}^{n}\frac{u_t}{\sigma\sqrt{n}}\frac{u_0+\ldots+u_{t-1}}{\sigma\sqrt{n}} \sim C^*\int_0^1 W_t dW_t = O_p(1);$$

and

$$\frac{C^{*2}}{2\sigma^2}\sum_{t=0}^{n}\left(\frac{1}{n}\sum_{\tau=0}^{t-1}u_\tau\right)^2 = \frac{C^{*2}}{2}\sum_{t=0}^{n}\left(\frac{u_0+\ldots+u_{t-1}}{\sigma\sqrt{n}}\right)^2\frac{1}{n} \sim \frac{C^{*2}}{2}\int_0^1 W_t^2 dt = O_p(1).$$

Note from Lemma 7, that

$$\begin{aligned} S|_{H_1} = S|_{H_0} + \frac{1}{2\sigma^2}\Bigg[&2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)^2 + 2\frac{\bar{C}C^*}{n^2}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor \alpha n\rfloor}u_t \\ &- 2\frac{\bar{C}C^{*2}}{n^3}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor \alpha n\rfloor}\sum_{\tau=0}^{t-1}u_\tau - \frac{\bar{C}^2C^{*2}}{n^4}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)^2(n-\lfloor \alpha n\rfloor)\Bigg]. \end{aligned} \tag{44}$$

We are going to show that $S|_{H_1} = S|_{H_0} + O_p\left(n^{-1}\right)$. To do this we calculate the order of each term in Eq. (44):

$$\frac{1}{\sigma^2 n^2}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)^2 \sim \frac{\alpha}{n}\nu^2, \text{ where } \nu \sim N(0,1);$$

$$\frac{1}{\sigma^2 n^2}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor \alpha n\rfloor}u_t \sim \frac{\sqrt{\alpha(1-\alpha)}}{n}\nu\chi, \text{ where } \chi \sim N(0,1),\ \nu \perp \chi;$$

$$\frac{1}{\sigma^2 n^3}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)\sum_{t>\lfloor \alpha n\rfloor}\sum_{\tau=0}^{t-1}u_\tau \sim \frac{\sqrt{\alpha}}{n}\nu\int_\alpha^1 W_t dt;$$

$$\frac{1}{\sigma^2 n^4}\left(\sum_{t=0}^{\lfloor \alpha n\rfloor-1}u_t\right)^2(n-\lfloor \alpha n\rfloor) \sim \frac{\alpha(1-\alpha)}{n^2}\nu^2.$$

Thus, $S|_{H_1} = S|_{H_0} + \frac{1}{n}O_p(1)$. Therefore, the power of the test is

$$\begin{aligned} P^c &= \mathbb{P}\left(S|_{H_1} > cv\right) = \mathbb{P}\left(S|_{H_0} + \frac{1}{n}O_p(1) > cv\right) \\ &\le \mathbb{P}\left(S|_{H_0} > cv\right) + \text{const}_\varepsilon n^{-1+\varepsilon} = 0.05 + \text{const}_\varepsilon n^{-1+\varepsilon}, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary, and the inequality follows from Lemma 3. □

Analysis and Heuristic Proof of Conjecture 1.

Proof. The statistic for functional point-optimal testing of $H_0 : \theta_{tn} = 1$, for all t, takes the form

$$\begin{aligned} S &= \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - \theta_{tn}X_{t-1})^2 \\ &=_{H_0} \frac{1}{2\sigma^2}\sum_{t=0}^{n}u_t^2 - \left\{\frac{1}{2\sigma^2}\sum_{t=0}^{L}\left(u_t - \frac{\bar{C}}{n}X_{t-1}\right)^2 + \frac{1}{2\sigma^2}\sum_{t=L=1}^{n}u_t^2\right\} \\ &= \frac{1}{\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{L}u_tX_{t-1} - \frac{1}{2\sigma^2}\left(\frac{\bar{C}}{n}\right)^2\sum_{t=0}^{L}X_{t-1}^2 \\ &= \frac{1}{\sigma^2}\frac{\bar{C}L}{n}\frac{1}{L}\sum_{t=0}^{L}u_tX_{t-1} - \frac{1}{2\sigma^2}\left(\frac{\bar{C}L}{n}\right)^2\frac{1}{L^2}\sum_{t=0}^{L}X_{t-1}^2. \end{aligned}$$

$$\sim \frac{1}{\sigma^2}\frac{\bar{C}L}{n}\int_0^1 dB_0B_0 - \frac{1}{2\sigma^2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 B_0^2 = \frac{\bar{C}L}{n}\int_0^1 dW_0W_0 - \frac{1}{2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2,$$

since $\frac{1}{\sqrt{L}}\sum_{t=0}^{\lfloor L\cdot\rfloor}u_t \Rightarrow B_0(\cdot)$ where $B_0 = \sigma W_0$ and W_0 is standard Brownian motion on $[0,1]$. Note that B_0 is independent of the Brownian motion $B = \sigma W$, where $\frac{1}{\sqrt{n}}\sum_{t=0}^{\lfloor n\cdot\rfloor}u_t \Rightarrow B(\cdot)$, since $\frac{L}{n} \to 0$. We may take a probability space in which the weak convergence is replaced by *a.s.* convergence, so that

$$\left(\frac{1}{L}\sum_{t=0}^{L}u_tX_{t-1}, \frac{1}{L^2}\sum_{t=0}^{L}X_{t-1}^2\right) \to_{a.s.} \left(\int_0^1 dB_0B_0, \int_0^1 B_0^2\right). \tag{45}$$

In this space we have the asymptotic representation

$$S =_{H_0} \left\{\frac{\bar{C}L}{n}\int_0^1 dW_0W_0 - \frac{1}{2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2\right\}\{1 + o_{a.s.}(1)\}$$

Critical values cv for the statistic S, which satisfy $\mathbb{P}\{S_n > cv\} = 0.05$, are therefore asymptotically approximately delivered by

$$\mathbb{P}\left\{\frac{\bar{C}L}{n}\int_0^1 dW_0W_0 - \frac{1}{2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2 > acv\right\} = 0.05, \tag{46}$$

which we write as $\mathbb{P}\{\xi_{0n} > acv\} = 0.05$, where $\xi_{0n} := \frac{\bar{C}L}{n}\int_0^1 dW_0W_0 - \frac{1}{2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2$. Since $\int_0^1 dW_0W_0, \int_0^1 W_0^2 = O_p(1)$, it follows that $\xi_{0n} = O_p\left(\frac{L}{n}\right)$ and so, from (46), we have $acv = O\left(\frac{L}{n}\right)$ and then $A_n := \frac{n}{L}acv = O(1)$ and

$$\frac{n}{L}\xi_{0n} = \bar{C}\int_0^1 dW_0W_0 - \frac{\bar{C}^2}{2}\left(\frac{L}{n}\right)\int_0^1 W_0^2 = \bar{C}\int_0^1 dW_0W_0 + O_p\left(\frac{L}{n}\right),$$

as $n \to \infty$.

Under the specific functional alternative hypothesis $H_1 : \theta_{tn} = \left(1 + \frac{\bar{C}}{n}\mathbf{1}\{t \leq L\}\right)$ we have the test statistic

$$\begin{aligned}
S &= \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - \theta_{tn}X_{t-1})^2 \\
&=_{H_1} \frac{1}{2\sigma^2}\sum_{t=0}^{n}(u_t + \frac{\bar{C}}{n}\mathbf{1}\{t \leq L\}X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}u_t^2 \\
&= \frac{1}{2\sigma^2}\sum_{t=0}^{L}(u_t + \frac{\bar{C}}{n}X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{L}u_t^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sigma^2}\frac{\bar{C}}{n}\sum_{t=0}^{L}u_tX_{t-1} + \frac{1}{2\sigma^2}\left(\frac{\bar{C}}{n}\right)^2\sum_{t=0}^{L}X_{t-1}^2 \\
&= \frac{1}{\sigma^2}\frac{\bar{C}L}{n}\frac{1}{L}\sum_{t=0}^{L}u_tX_{t-1} + \frac{1}{2\sigma^2}\left(\frac{\bar{C}L}{n}\right)^2\frac{1}{L^2}\sum_{t=0}^{L}X_{t-1}^2 \\
&\sim \frac{1}{\sigma^2}\frac{\bar{C}L}{n}\int_0^1 B_0(t)dB_0(t) + \frac{1}{\sigma^2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1\int_0^t(1-s)dB_0(s)dB_0(t) \\
&+ \frac{1}{2\sigma^2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 B_0^2(t)dt \\
&= \frac{\bar{C}L}{n}\int_0^1 W_0(t)dW_0(t) + \frac{1}{2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2(t)dt \\
&+ \left(\frac{\bar{C}L}{n}\right)^2\int_0^1\int_0^t(1-s)dW_0(s)dW_0(t)
\end{aligned} \tag{47}$$

The approximation in the penultimate line holds because under H_1 we have $\theta_{tn}\mathbf{1}\{t \leq L\} = \left(1 + \frac{\bar{C}}{n}\right)\mathbf{1}\{t \leq L\}$ and then

$$\begin{aligned}
L^{-1/2}X_{\lfloor L\cdot\rfloor} &= \frac{1}{\sqrt{L}}\sum_{j=0}^{\lfloor L\cdot\rfloor}\left(1 + \frac{\bar{C}}{n}\right)^j u_{\lfloor L\cdot\rfloor - j} \\
&= \frac{1}{\sqrt{L}}\sum_{j=0}^{\lfloor L\cdot\rfloor}\left(u_{\lfloor L\cdot\rfloor - j} + \frac{j\bar{C}}{n}u_{\lfloor L\cdot\rfloor - j}\right) + o\left(\frac{L}{n}\right) \\
&= B_0(\cdot) + \frac{L}{n}\bar{C}\int_0^{\cdot}(1-s)dB_0 + o\left(\frac{L}{n}\right).
\end{aligned} \tag{48}$$

Using $\xi_{0n} = \frac{\bar{C}L}{n}\int_0^1 dW_0W_0 - \frac{1}{2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2$, it follows from (46) and (47) that asymptotic power is given by

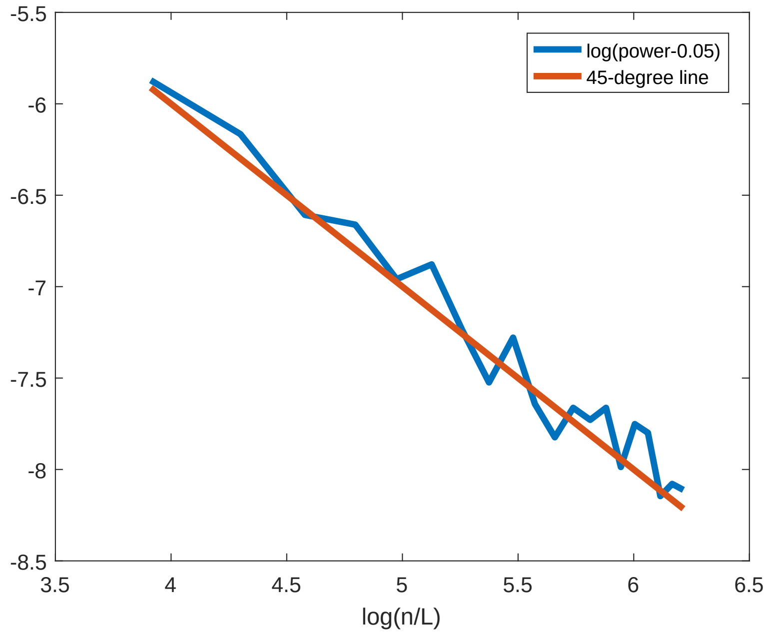

FIGURE 12. $\log\left(P^c - 0.05\right)$ as a function of $\log\left(\frac{n}{L}\right)$.

$$\mathbb{P}\left\{\frac{\bar{C}L}{n}\int_0^1 dW_0W_0 + \frac{1}{2}\left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2 + \left(\frac{\bar{C}L}{n}\right)^2\int_0^1\int_0^t (1-s)dW_0dW_0 > acv\right\}$$
$$= \mathbb{P}\left\{\xi_{0n} + \left(\frac{\bar{C}L}{n}\right)^2\int_0^1 W_0^2 + \left(\frac{\bar{C}L}{n}\right)^2\int_0^1\int_0^t (1-s)dW_0dW_0 > acv\right\}$$
$$\geq \mathbb{P}\left\{\xi_{0n} > acv\right\} + \text{const}_\varepsilon\left(\frac{L}{n}\right)^{1+\varepsilon} = 0.05 + \text{const}_\varepsilon\left(\frac{L}{n}\right)^{1+\varepsilon}.$$

The inequality in the penultimate line above is unproved, therefore leading to an argument that is 'heuristic'.

Although we do not provide an explicit proof of the last inequality, we provide simulations, which confirm it. Figure 12 corresponds to $\bar{C} = 1$. The blue line represents $\log\left(P^c - 0.05\right)$ as a function of $\log\left(\frac{n}{L}\right)$. If the argument is correct, we expect $\log\left(P^c - 0.05\right)$ to be a linear function of $\log\left(\frac{n}{L}\right)$ with the slope$= -1$. The red line is a 45−degree line. Evidently from the figure the blue line closely follows the red one, corroborating our hypothesis that the implied relationship is correct.

Thus, we get the same result as when L was fixed and equal to one, i.e. the maximal power is bounded from below by $0.05 + \text{const}_\varepsilon(\frac{L}{n})^{1+\varepsilon}$.

Next consider the pseudo-point optimal test test based on the use of a constant LUR specification $\theta'_{tn} = \left(1 + \frac{C^*}{n}\right)$ for all $t = 0, ..., n$. The true underlying data generating process has $\theta_{tn} = \left(1 + \frac{\bar{c}(t/n)}{n}\right)$, so whenever $\bar{c}(\cdot) \neq C^*$ the specification is incorrect.

The test statistic for testing $H_0 : \theta_{tn} = 1$, for all t, takes the form

$$S' = \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t - \theta'_{tn}X_{t-1}\right)^2$$

and under the null hypothesis H_0 we have

$$\begin{aligned} S' &=_{H_0} \frac{1}{2\sigma^2}\sum_{t=0}^{n}u_t^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(u_t - \frac{C^*}{n}X_{t-1}\right)^2 \\ &= \frac{1}{\sigma^2}\frac{C^*}{n}\sum_{t=0}^{n}u_tX_{t-1} - \frac{1}{2\sigma^2}\left(\frac{C^*}{n}\right)^2\sum_{t=0}^{n}X_{t-1}^2 \\ &= \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(\frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right)\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1}u_\tau\right) \qquad (49) \\ &\sim \frac{1}{\sigma^2}C^*\int_0^1 dBB - \frac{1}{2\sigma^2}C^{*2}\int_0^1 B^2 = C^*\int_0^1 dWW - \frac{1}{2}C^{*2}\int_0^1 W^2. \qquad (50) \end{aligned}$$

As before, we may assume that the probability space is suitably expanded so that (50) holds $a.s.$, and the null H_0 is again rejected when S' is large. Critical values $\overline{cv}$ for the statistic S', which satisfy $\mathbb{P}\left\{S'_n > \overline{cv}\right\} = 0.05$, are therefore asymptotically approximately delivered by

$$\mathbb{P}\left\{C^*\int_0^1 dWW - \frac{1}{2}C^{*2}\int_0^1 W^2 > \overline{acv}\right\} = 0.05, \qquad (51)$$

which we write as $\mathbb{P}\left\{\xi_0 > \overline{acv}\right\} = 0.05$, where $\xi_0 := C^*\int_0^1 dWW - \frac{1}{2}C^{*2}\int_0^1 W^2 = O_p(1)$ so that $\overline{acv} = O(1)$.

Next, under the functional alternative hypothesis $H_1 : \theta_{tn} = \left(1 + \frac{\bar{C}}{n}\mathbf{1}\{t \le L\}\right)$, we have

$$\begin{aligned} S &= \frac{1}{2\sigma^2}\sum_{t=0}^{n}(X_t - X_{t-1})^2 - \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(X_t - \theta'_{tn}X_{t-1}\right)^2 \\ &\underset{H_1}{=} \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(u_t + \frac{\bar{C}}{n}\mathbf{1}\{t \le L\}X_{t-1}\right)^2 \\ &- \frac{1}{2\sigma^2}\sum_{t=0}^{n}\left(\left[u_t + \frac{\bar{C}}{n}\mathbf{1}\{t \le L\}X_{t-1}\right] - \frac{C^*}{n}X_{t-1}\right)^2 \\ &= \frac{1}{2\sigma^2}\sum_{t=0}^{L}\frac{C^*}{n}X_{t-1}\left(2u_t + \frac{2\bar{C} - C^*}{n}X_{t-1}\right) + \frac{1}{2\sigma^2}\sum_{t=L+1}^{n}\frac{C^*}{n}X_{t-1}\left(2u_t - \frac{C^*}{n}X_{t-1}\right) \\ &= S^1 + S^2. \end{aligned}$$

After simplifications, we can rewrite S^1 as

(52)

$$2\sigma^2 S^1 = \sum_{t=0}^{L} \left(2u_t + \frac{2\bar{C} - C^*}{n}\left(u_{t-1} + \left(1+\frac{\bar{C}}{n}\right)u_{t-2} + \cdots + \left(1+\frac{\bar{C}}{n}\right)^{t-1} u_0\right)\right)$$
$$\cdot \frac{C^*}{n}\left(u_{t-1} + \left(1+\frac{\bar{C}}{n}\right)u_{t-2} + \cdots + \left(1+\frac{\bar{C}}{n}\right)^{t-1} u_0\right)$$
$$= \sum_{t=0}^{L}\left(\frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right)\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right) + O\left(\left(\frac{L}{n}\right)^2\right).$$

Similarly, we simplify and rewrite S^2:

(53)

$$2\sigma^2 S^2 = \sum_{t=L+1}^{n} \frac{C^*}{n}\left(\left(1+\frac{\bar{C}}{n}\right)^{L} u_0 + \cdots + \left(1+\frac{\bar{C}}{n}\right)u_{L-1} + u_L + \cdots + u_{t-1}\right)$$
$$\cdot\left(2u_t - \frac{C^*}{n}\left(\left(1+\frac{\bar{C}}{n}\right)^{L} u_0 + \cdots + \left(1+\frac{\bar{C}}{n}\right)u_{L-1} + u_L + \cdots + u_{t-1}\right)\right)$$
$$= \sum_{t=L+1}^{n}\left(\frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right)\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right) + O\left(\left(\frac{L}{n}\right)^{\frac{3}{2}}\right).$$

Finally, combining equations (52) and (53), we get

$$S = \sum_{t=0}^{n}\left(\frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right)\left(2u_t - \frac{C^*}{n}\sum_{\tau=0}^{t-1} u_\tau\right) + O\left(\left(\frac{L}{n}\right)^{\frac{3}{2}}\right) = S' + O\left(\left(\frac{L}{n}\right)^{\frac{3}{2}}\right).$$

Thus, the power for this test is

$$\mathbb{P}\left\{S > \overline{cv}\right\} = \mathbb{P}\left\{S' + O\left(\left(\frac{L}{n}\right)^{\frac{3}{2}}\right) > \overline{cv}\right\} \le 0.05 + \mathrm{const}_\varepsilon \left(\frac{L}{n}\right)^{3/2-\varepsilon}, \tag{54}$$

where the inequality follows from Lemma 3, where instead of n we work with n/L. □

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