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THE STRUCTURE OF DECISIVE SETS

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AGGREGATION of PREFERENCES and the STRUCTURE OF DECISIVE SETS

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Dedicated to the Memory of Herbert Scarf : [1930 to 2015]

Abstract: This is the manuscript for the talk that I presented at the Koerner Center's Intellectual Trajectories Seminar in September 2016

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To begin a postdoc with no prior training in a challenging field of research is to embark on many intellectual trajectories. Where to begin?
This is the story of two papers that I wrote, early in my career, to understand Arrow's Impossibility Theorem. I was trained in mathematics and physics where I wrote my Ph.d dissertation in mathematical logic. My academic career in economics begins with my postdoc in mathematical economics at the Cowles Foundation for Research in Economics, during the tenure of Herb Scarf as director.
In the 45 years that we were colleagues on the economics faculty at Yale. Herb was the preeminent intellectual influence on my scholarship. One day, I went to his office to discuss my current research on **Arrow's Impossibility Theorem,** using the ultra-filter construction from mathematical logic to "aggregate" models of syntactic structures. After a few minutes of my incoherent rambling he asked me the question that was to define my scholarship for the rest of my academic life.
" First, what is the question that you are trying to answer ? Let's talk later about what methods you might use to find the answer" . " What problem, relating to Arrow's broad research agenda, are you trying to solve ?"
That was easy, like Arrow, I wanted to know:

If the **Condorcet voting paradox**, where the presence of voting cycles under majority rule, allows the social outcome to be solely determined by the voting agenda, is peculiar to majority rule or is it a fundamental characteristic of every democratic voting rule?

Here is the classic example of the Condorcet's voting paradox, first proposed by Condorcet (***1785)***:

In Condorcet's world there are three voters, A, B and C voting over three social alternatives, X,Y and Z. A ranks the alternatives as X , Y and Z; B ranks the alternatives as Y, Z and X. Finally, C ranks the alternatives as Z, X and Y. If the voting rule is majority rule and the chair of the committee choses the voting agenda, where she first lists the order for voting over pairs of alternatives in each round, where the winning alternative in the current round goes on to the next round to compete against the next alternative on the agenda. If voters are non-strategic and vote their true preferences, then C the chairman of the committee, will chose the agenda [X,Y,Z] , where the social outcome is Z, her preferred outcome. That is, in the first round of voting A and C vote for X over Y, so X is the majority winner. In the second round of voting the alternatives are X and Z. B and C vote for Z over X. Hence for this agenda, under majority rule, Z is the social outcome, the preferred outcome of B, the chair. If the chair was A, what voting agenda would he propose to have his preferred outcome be the social choice?

There is another feature of the Condorcet voting paradox. As is well known, economists define a rational voter as a voter endowed with a strict preference relation P where XPY can be interpreted as the voter votes for X over Y, if given the choice between X and Y. The voter is also endowed with an indifference preference relation I, where XIY can be interpreted as X is not preferred to Y and Y is not preferred to X.The rationality assumptions are that both P and I are assumed to be transitive and that P is irreflexive, i.e. XPX is false for every alternative X. Individual rationality is not preserved under majority voting. In fact, Pm, the social preference relation defined by majority voting, is cyclic.That is, in the Condorcet voting paradox,YPmZ and ZPmX and XPmY. Hence YPmY, contradicting the assumption that Pm is irreflexive. Recall that Pm is transitive, if for every triple of social alternatives X,Y and Z: If XPmY and YPmZ then XPmZ.

My insight was to argue that it is the lack of acyclicity in the social preferences, Pm, not the lack of transitivity, that produces the Condorcet voting paradox in 1785 and subsequently is the root cause of Arrow's Impossibility Theorem in 1951. Arrow like Condorcet had hoped that social preferences defined by voting preserved individual rationality. Is cycling of the social preferences Pm for some profile of individual preferences an important concern for small committees. **YES,** since it allows the chair of the committee to manipulate the agenda and dictate the social outcome.

Hence we should require that the social preferences, Pm, defining the democratic voting rule to be acyclic for all profiles. That is, we need to characterize the democratic voting rules that preserve the acyclicity of rational vote's preferences. In Aggregation of Preferences, I prove an Impossibility Theorem for acyclic voting rules inspired by the voting rule for the Security Council of the United Nations prior to August 31,1965 that provides this characterization

Following the axiomatic methodology of Von Neumann and Morgenstern in the Theory of Games and Economic Behavior (1944), Arrow proposed four axioms that he believed every democratic voting rule should satisfy:

(1) Universal Domain: The democratic voting rule should aggregate every profile of individual preferences into social preferences, Pm and Im, that are transitive and irreflexive.

Hence majority rule fails to satisfy Arrow's first axiom, as demonstrated by the Condorcet voting paradox.

(2) Pareto Optimality: If everyone votes for alternative X over Y in a pairwise choice between X and Y, then the social outcome is X.

(3) Independence of Irrelevant Alternatives: In a pairwise choice between the social alternatives X and Y, the social outcome depends only on X and Y. That is, the social preference of X versus Y depends only on the individual preferences of X versus Y.

(4) Non-Dictatorship: If only one voter prefers X to Y and everyone else prefers Y to X, then the social outcome is Y.

To his surprise and I expect his dismay, Arrow proved the following remarkable result.

Arrow's Impossibility Theorem (1951) If there are at least three social alternatives, then there are no voting rules that satisfy all four of Arrow's conditions.

If only the first three axioms are required to hold, then we have an important corollary of the Impossibility Theorem, Arrow's Possibility Theorem

Arrow's Possibility Theorem: The only voting rule that satisfies Arrows first three axioms is dictatorial.

As might be expected, Arrow's theorems inspired new fields of research in the social sciences spanning economics, political science, and game theory.

I shared a parking lot with Abraham Robinson,[1925-1965]one of the great mathematicians of the twentieth century, who formalized Leibinitz's calculus of infinitesimals, (1675) as an alternative to Newton's calculus (1665-1666). Robinson's model of infinitesimals is exposited in his magnus opus, Nonstandard Analysis (1966). Robinson knew my background in mathematical logic and asked if I wanted to collaborate on using nonstandard analysis as a model for Edgeworth's conjecture.

Edgeworth's conjecture is that economic equilibrium resulting from bargaining in markets with large numbers of economic agents is equivalent to economic equilibria realized by the invisible hand of competition, introduced by Adam Smith in Wealth of Nations (1776). It was Herb, who urged me to drop my ongoing research on Arrow's theorem and accept Robinson's generous offer.

His reason was simple. Robinson was a great man. He was and we wrote several published papers, that subsequently produced my first cohort of Ph.D. students at Yale and jump-started my career as an academic economist.

Later, another director of Cowles, Bill Brainard, took me aside and told me that my tenure review would happen within the year, but my chances for tenure at Yale were not good. The issue was that despite my introduction, with Robinson, of Nonstandard Analysis as a new and exciting tool in mathematical economics. All my publications were methodological, there was no substantive contribution to economics.

Bill's advice was to write a paper on some important issues in economics that the senior faculty could read and understand, without a Phd ina abstract mathematics.

That is, no equations.

I used to play tennis with Bob Dahl. I confessed to Bob my ignorance of the institutional aspects of political power using voting and told him tht I wanted to define an abstract index of the distribution of the political power between winning coalitions of voters that might suggest a new and more intuitive proof of Arrow's Impossibility Theorem.

Dahl offered to meet with me over lunch to tutor me on aspects of democracy and voting. I learned quite a bit.

What did I learn? In particular, after reading some of the history of the Roman Senate, suggested by Bob. I discovered that the Romans used the veto as their primary system of checks and balances on the political power of the Roman Senate

Maybe, I could use veto players to prevent voting cycles. Bob and I did meet and I published without equations: **Aggregation of Preferences** in the Quarterly Journal of Economics JE,1975, where veto players played an essential role in preventing voting cycles.Yes, I did get tenure.

Certainly, my best known publications in economics are in general equilibrium theory. See the selected articles listed in chronological order under references at the end of the paper

In the next section, I give a brief summary of the ideas and results in my paper: Aggregation of Preferences. In his classic essay, Social Choice and Individual Values (1951), Arrow uses a notion that is central to his proof. That is, decisive sets of voters (or winning coalitions) defined by the voting rule:. Society prefers X over Y, iff there is a decisive set that prefers X to Y. In the Condorcet voting paradox the decisive sets are each two voter majority and the coalition of the whole. Here is the formal definition of decisive sets.

DECISIVE SET: A subset, W, or coalition of voters is decisive for a voting rule if everyone in W prefers the social alternative X to Y then the social outcome is X. That is, each decisive set is an oligarchy. Any voting rule that satisfies the Pareto optimality axiom has at least one decisive set: The coalition of the whole. Moreover, every superset of a decisive set is a decisive set. My approach was inspired by the preservation theorems in model theory, a branch of mathematical logic and one of Robinson's principal research interests.

HERE IS MY RESEARCH QUESTION. What is the structure of the family of decisive sets for democratic voting rules that preserve acyclicity of individual voter's preferences?

A filter, F, is an abstraction of a family of large subsets of some universe U. That is,

(1) U is in F

(2) If W and V are in F then W intersect V is in F

(3) If W in F and V is a superset of W then V is in F.

Finally, the empty set is not in F.

Every family of decisive sets defines a social welfare preference relation over the family of social outcomes, where X is socially preferred to Y iff a decisive set V prefers X to Y

This construction need not generate rational individualistic preferences but if the decisive sets form a filter then they do form an oligarchy. That is, the oligarchy is the smallest decisive set. Of course, an oligarchy of one is a dictator. An ultra filter F is a filter where for every subset of voters W, either W or its complement is in F. The only ultra filters on a finite universe of voters are dictatorial, i.e there exists a decisive set with a single voter. Hence Arrow's Impossibility Theorem on a finite universe proves that the family of decisive sets for a democratic voting rule is a dictatorial ultra-filter.

Fishburn (1970) observed that on an infinite universe the existence of free ultra-filters, i.e., an ultra-filter, where the intersection of all the decisive sets is the empty set, that there exists democratic voting rules that satisfy all four of Arrow's axioms. Hence Arrow's possibility theorem on an infinite universe proves the family of decisive sets for a democratic social welfare function on an infinite universe is a free ultra-filter.

Now for something new.. That is, my version of Arrow's Possibility Theorem using veto voters where I replace Arrow's axioms with the following axioms:

(1) Universal Domain: The voting rule maps profiles of voters with acyclic preferences into an acyclic preference relation over the social alternatives

(2) Pareto Optimality

(3) Independence of Irrelevant Alternatives

(4) Non-Dictatorship

COLLEGIAL POLITIES

The decisive sets for voting rules satisfying this family of axioms are defined as follows: If there is a finite number of voters then choose some proper subset of voters, the colloquium, C, pick an integer K less than the number of voters outside the collegium. The social alternative X is socially preferred to Y if the collegium prefers X to Y, and at least K voters outside the collegium prefers X to Y. Every member of the collegium has a veto, but no member of the collegium is a dictator.

This voting rule was inspired by the original voting rule in the U.N. Security Council and is acyclic, but the subsequent voting rule is not. Both voting rules and the principal contribution of my paper, Aggregation of Preferences**,** that collegial polities preserve acyclicity of voter's individual preferences, are presented and discussed by Feldman and Serrano in their lucid monograph Welfare Economics and Social Choice (second edition).

In Blau and Brown (1989), we extend the analysis of preservation theorems in social choice theory, using families of decisive sets, that I first introduced in my Aggregation of Preferences paper. In our paper, the primitive notion is the family of decisive sets and the voting rule is explicitly derived from the decisive sets. That is, if the prescribed family of decisive sets is Q, then in pairwise voting between social alternatives X and Y, X is socially preferred to Y, if and only if the set of voters who prefer X to Y is a decisive set in Q. This is not the case in Arrow and more generally in the social choice literature, where the the primitive notion is the voting rule and decisive sets are derived from the voting rule.

With the kind permission of the Koerner Center at Yale and the journal of Social Choice and Welfare, I close with, the abstract and preface to my paper with Julian in fond memory of the end of this intellectual trajectory.

The Structure of Neutral Monotonic Social Functions

by Julian Blau and Donald J. Brown (1989)

Abstract: " In this paper, we show that neutral monotonic social functions and their specializations to social decision functions, quasi-transitive social decisions, and social welfare functions can be uniquely represented as a collection of overlapping simple games, each of which is defined on a finite set of concerned individuals. Moreover, each simple game satisfies certain intersection conditions depending on the number of social alternatives; the number individuals belonging to

the concerned set under consideration; and the collective rationality assumption."

Preface: " I first met Julian Blau at the ***1977*** Public Choice Meeting in New Orleans. As I recall he chaired the session where I presented earlier version of what was to become the joint paper presented here.

At those same meetings, John Ferejohn and Peter Fishburn presented their joint paper on the representation of social decision functions, possibly in the same session of my paper .I remember several long walks with Julian where we discussed extensions of my paper in the direction of the Ferejohn and -Fishburn paper, but emphasizing the role of neutrality, but the importance of neutrality in social choice theory had been a dominant theme in Julians earlier researches. [Neutrality means equal treatment of alternatives.]

It was during these conversations that our collaboration began Over the next year, we corresponded and talked over the phone. I am sorry now that I didn't save those letters. Julian was a perfectionist and we argued long and hard over definitions — he didn't like the term direct sum of games or the proofs. When the paper was essentially done, it was decided that I would send it off for publication. Julian by that time was quite ill.

That was my last conversation with Julian. I submitted the paper to the Review of Economic Studies and a year later received two excellent referees' reports. By then I was actively at work on increasing returns to scale in general equilibrium theory and never got around to making the suggested revisions and sending it back to the Review of Economic Studies. I am, therefore, quite pleased to have this opportunity to share with Julian's friends and colleagues one of his last contributions to his chosen field of scholarship.

EPILOGUE

The evening before my talk at the Koerner Center, I decided to review the discussion of my paper Aggregation of Preferences in the monograph on Welfare Economics and Social Choice by Feldman and Serrano, second edition. On the adjoining shelf in the library was a text entitled:

LIVES OF THE LAUREATES (5^{th} edition), consisting of autobiographical accounts of the careers of 23 economists who had received the Nobel prize in Economics. Despite having the office next to Arrow's during my 7 ear tenure at Stanford, we never discussed social choice or his Impossibility Theorem. So the evening before my talk, I read the chapter on Arrow. His memory of the discovery of his Impossibility Theorem differs significantly from the intellectual history of Arrow's Impossibility Theorem that I present here. In particular, he discovered Condorcet's account of the voting paradox well after he proved his Impossibility Theorem. Moreover, he learned that his Impossibility Theorem had been proven independently by the English economist Ducan Black more or less at the same time.You should read his essay.

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