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By

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Limit Theory for Moderate Deviations from a Unit Root under Weak Dependence[1]

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Abstract

An asymptotic theory is given for autoregressive time series with weakly dependent innovations and a root of the form $\rho_n = 1 + c/n^\alpha$, involving moderate deviations from unity when $\alpha \in (0,1)$ and $c \in \mathbb{R}$ are constant parameters. The limit theory combines a functional law to a diffusion on $D[0,\infty)$ and a central limit theorem. For $c > 0$, the limit theory of the first order serial correlation coefficient is Cauchy and is invariant to both the distribution and the dependence structure of the innovations. To our knowledge, this is the first invariance principle of its kind for explosive processes. The rate of convergence is found to be $n^\alpha \rho_n^n$, which bridges asymptotic rate results for conventional local to unity cases (n) and explosive autoregressions ($(1+c)^n$). For $c < 0$, we provide results for $\alpha \in (0,1)$ that give an $n^{(1+\alpha)/2}$ rate of convergence and lead to asymptotic normality for the first order serial correlation, bridging the $\sqrt{n}$ and n convergence rates for the stationary and conventional local to unity cases. Weakly dependent errors are shown to induce a bias in the limit distribution, analogous to that of the local to unity case. Linkages to the limit theory in the stationary and explosive cases are established.

Keywords: Central limit theory; Diffusion; Explosive autoregression, Local to unity; Moderate deviations, Unit root distribution, Weak dependence.

AMS 1991 subject classification: 62M10; JEL classification: C22

Dedication

This paper is dedicated to the loving memory of Michael Magdalinos, whose enthusiasm for econometrics was an inspiration to all and was surpassed only by the devotion he had to his family and friends.

1. Introduction

In time series regression theory, much attention has been given to models with autoregressive roots at unity or in the vicinity of unity. The limit theory has relied on functional laws to Brownian motion and diffusions, and weak convergence to stochastic integrals. The treatment of local to unity roots has relied exclusively on specifications of the form $\rho = 1 + c/n$, where n is the sample size (Phillips, 1987a; Chan and Wei, 1987) or matrix versions of this form (Phillips, 1988). The theory has been particularly useful in defining power functions for unit root tests (Phillips, 1987a) under alternatives that are immediately local to unity.

To characterize greater deviations from unity Phillips and Magdalinos (2004; hereafter simply PM) have recently investigated time series with an autoregressive root of the form $\rho_n = 1 + c/n^{\alpha}$, where the exponent α lies in the interval $(0, 1)$. Such roots represent moderate deviations from unity in the sense that they belong to larger neighborhoods of one than conventional local to unity roots. The parameter α measures the radial width of the neighborhood with smaller values of α being associated with larger neighborhoods. The boundary value as $\alpha \to 1$ includes the conventional local to unity case, whereas the boundary value as $\alpha \to 0$ includes the stationary or explosive AR(1) process, depending on the value of c.

The limit theory developed in PM was derived under the assumption of independent and identically distributed (i.i.d.) innovations. By combining a functional law to a diffusion with a central limit law to a Gaussian random variable, the asymptotic distribution of the normalized and centred serial correlation coefficient $h(n)(\hat{\rho}_n - \rho_n)$ was shown to be Gaussian in the near-stationary ($c < 0$) case and Cauchy in the near-explosive ($c > 0$) case. The normalization $h(n)$ depends on the radial parameter α of the width of the neighborhood of unity and the localizing coefficient c. When $c < 0$, $h(n) = n^{(1+\alpha)/2}$, a rate that bridges the $\sqrt{n}$ and n asymptotics of the stationary ($\alpha = 0$) and conventional local to unity ($\alpha = 1$) cases. When $c > 0$, $h(n) = n^{\alpha}\rho_n^n$, a rate that increases from $O(n)$ when $\alpha \to 1$ to $O((1+c)^n)$ when $\alpha \to 0$, thereby bridging the asymptotics of local to unity and explosive autoregressions.

The present paper extends these results to processes with weakly dependent innovations. We impose a linear process structure on the errors and discuss the effect this type of weak dependence has on the limit theory. The results vary significantly according to the sign of c.

In the near-explosive case, the limit theory can be extended without imposing additional restrictions over those in PM beyond a summability condition on the weak dependence structure. The resulting Cauchy limit law for the normalized serial correlation coefficient shows that the limit theory is invariant to both the distribution and the dependence structure of the innovation errors. To our knowledge, this is the first general invariance principle for explosive processes, all earlier results depending explicitly on distributional assumptions as was emphasized in the original paper by Anderson (1959).

The near-stationary case presents more substantial technical difficulties in making the transition from nonstationarity to stationarity. The results given here have been derived under a stronger summability condition on the weak dependence structure when $\alpha \in \left(0, \frac{1}{3}\right]$. Nonetheless, we provide a full extension of the limit theory to the weakly dependent case, a Gaussian limit law obtained for the serial correlation coefficient with normalisation $n^{(1+\alpha)/2}$ for $\alpha \in (0,1)$. An interesting feature of the near stationary case is that Gaussian asymptotics apply, but with a limiting bias that is analogous to the correction (cf. Phillips, 1987b) that is known to apply in the unit root case. Linkages to the limit theory for the serial correlation coefficient in the stationary case (where $\alpha = 0$) are established.

The paper is organised as follows. Section 2 briefly summarizes the limit theory obtained in PM for autoregressive processes with moderate deviations from unity and i.i.d. errors. This section provides a foundation for the rest of the paper since several asymptotic results for the weakly dependent case are derived as approximations of the relevant results under independence using the Phillips-Solo (1992) device and Theorem 2.1 below. The moderate deviations from unity model under weak dependence is presented in Section 3. This section also describes a blocking method that is central to the derivation of the subsequent limit results, based on a segmentation of the sample size and an embedding of a random walk in a Brownian motion. Sections 4 and 5 provide the limit theory for the near-stationary and the near-explosive case respectively. Section 6 includes some discussion and concluding remarks and Section 7 is a notational glossary. All proofs are collected in Section 8, together with some technical propositions.

2. Moderate deviations with i.i.d. errors

Consider the autoregressive time series

$$x_t = \rho_n x_{t-1} + \varepsilon_t, \quad t = 1, ..., n; \quad \rho_n = 1 + \frac{c}{n^\alpha}, \ \alpha \in (0,1) \tag{1}$$

initialized at some $x_0 = o_p\left(n^{\alpha/2}\right)$ independent of $\sigma\left(\varepsilon_1, ..., \varepsilon_n\right)$, where ε_t is a sequence of i.i.d. $(0, \sigma^2)$ random variables with finite ν'th absolute moment

$$E\left|\varepsilon_1\right|^\nu < \infty \text{ for some } \nu > \frac{2}{\alpha}. \tag{2}$$

PM developed a limit theory for statistics arising from model (1) based on a segmentation of the time series $(x_t)_{t\in\mathbb{N}}$ into blocks[1], the details of which are provided in Section 3. The advantage of this blocking method lies on the fact that it provides

[1]Subsequently, in a revised version of Phillips and Magdalinos (2004) it was shown that the main results could be obtained when the innovations ε_t are iid without using a blocking approach and using only finite second moments. Giraitis and Phillips (2004) derived related limit results for the case of martingale difference errors.

a way to study the asymptotic behavior of x_n via that of the component random elements $x_{\lfloor n^\alpha \cdot \rfloor}$ of the Skorohod space $D[0,\infty)$ (e.g., Pollard, 1984). Denoting by $W_{n^\alpha}(t) := \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor n^\alpha t \rfloor} \varepsilon_i$ the partial sum process on $D[0,\infty)$, it is possible to approximate $x_{\lfloor n^\alpha t \rfloor}$ by the Stieltjes integral

$$U_{n^\alpha}(t) := \int_0^t e^{c(t-r)} dW_{n^\alpha}(r) = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor n^\alpha t \rfloor} e^{\frac{c}{n^\alpha}(n^\alpha t - i)} \varepsilon_i.$$

For each $\alpha \in (0,1)$ and $c < 0$,

$$\sup_{t \in [0, n^{1-\alpha}]} \left| \frac{1}{n^{\alpha/2}} x_{\lfloor n^\alpha t \rfloor} - U_{n^\alpha}(t) \right| = o_p(1) \quad \text{as } n \to \infty. \tag{3}$$

Thus, we are able to operate in the familiar framework of Phillips (1987a) where $U_{n^\alpha}(t)$, and hence the time series x_n with appropriate normalization, converges to the linear diffusion $\int_0^t e^{c(t-s)} dW(s)$, where W is Brownian motion with variance σ^2. However, unlike the local to unity asymptotics of Phillips (1987a), the limiting distribution of the various sample moments of x_n cannot be obtained by the above functional law alone because the series itself is segmented into an asymptotically infinite number of such blocks with this behavior. Accordingly, this approach is combined with an analysis of asymptotic behavior as the number of blocks increases.

We use the fact that, by virtue of the moment condition (2), the Hungarian construction (cf. Csörgõ and Horváth, 1993) ensures the existence of a probability space where $W_{n^\alpha}(t) \longrightarrow_{a.s.} W(t)$ and $U_{n^\alpha}(t) \longrightarrow_{a.s.} \int_0^t e^{c(t-s)} dW(s)$ uniformly on $[0, n^{1-\alpha}]$. For the near-stationary case, this embedding then allows the sample moments of the original time series data to be approximated by normalized sums of functionals of the form $\int_0^t e^{c(t-s)} dW(s)$ which obey a law of large numbers in the case of the sample variance and a central limit theorem in the case of the sample covariance. For the near-explosive case, the limit theory is also derived by using the above embedding in conjunction with the martingale convergence theorem.

The following theorem contains a summary of the main results of PM.

2.1 Theorem. *For model (1) with $\rho_n = 1 + c/n^a$ and $\alpha \in (0,1)$, the following limits apply as $n \to \infty$. When $c < 0$,*

(a) $n^{-\alpha/2} x_{\lfloor n^\alpha t \rfloor} \Longrightarrow \int_0^t e^{c(t-r)} dW(r)$ on $D[0,\infty)$,

(b) $n^{-1-\alpha} \sum_{t=1}^n x_t^2 \longrightarrow_p \frac{\sigma^2}{-2c}$,

(c) $n^{-\frac{1+\alpha}{2}} \sum_{t=1}^n x_{t-1}\varepsilon_t \Longrightarrow N\left(0, \frac{\sigma^4}{-2c}\right)$,

(d) $n^{\frac{1+\alpha}{2}} (\hat{\rho}_n - \rho_n) \Longrightarrow N(0, -2c)$, *where W is Brownian motion with variance σ^2. When $c > 0$*

(e) $\frac{n^{\alpha}\rho_n^n}{2c}\left(\hat{\rho}_n - \rho_n\right) \Longrightarrow C$,

where C is a standard Cauchy variate.

3. Moderate deviations from unity with weakly dependent errors

In this paper we consider the time series

$$y_t = \rho_n y_{t-1} + u_t, \quad t = 1, ..., n; \quad \rho_n = 1 + \frac{c}{n^{\alpha}}, \ \alpha \in (0, 1) \tag{4}$$

initialized at some $y_0 = o_p\left(n^{\alpha/2}\right)$ independent of $\sigma\left(u_1, ..., u_n\right)$, with zero mean, weakly dependent errors u_t that satisfy the following condition.

Assumption LP. *For each $t \in \mathbb{N}$, u_t has Wold representation*

$$u_t = C\left(L\right)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad C(1) \neq 0,$$

where C is the operator $C\left(z\right) = \sum_{j=0}^{\infty} c_j z^j$, $\left(\varepsilon_t\right)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. $(0, \sigma^2)$ random variables satisfying the moment condition (2) and $\left(c_j\right)_{j\in\mathbb{N}}$ is a sequence of constants such that

(i) *when $c > 0$,* $\sum_{j=1}^{\infty} j\left|c_j\right| < \infty$,

(ii) *when $c < 0$,* $\sum_{j=1}^{\infty} j^{1\vee\frac{3-3\alpha+\delta}{2}}\left|c_j\right| < \infty$, *for some $\delta \in (0, 3\alpha)$.*

Note that when $\alpha \in \left(\frac{1}{3}, 1\right)$ in condition (ii) above, we can always choose a small enough δ, $\delta < 3\alpha - 1$, so that $\frac{3-3\alpha+\delta}{2} < 1$ and the usual summability condition $\sum_{j=1}^{\infty} j\left|c_j\right| < \infty$ applies for the near stationary case. The derivation of a limit theory for $\alpha \in \left(0, \frac{1}{3}\right]$ requires the summability assumption

$$\sum_{j=1}^{\infty} j^{\frac{3-3\alpha+\delta}{2}}\left|c_j\right| < \infty, \quad \text{for some } \delta \in (0, 3\alpha)$$

which becomes stronger as we approach the boundary with the stationary region, becoming eventually $\sum_{j=1}^{\infty} j^{3/2}\left|c_j\right| < \infty$ when $\alpha \to 0$.

Under **LP**, u_t has variance $\sigma_u^2 = \sigma^2 \sum_{j=0}^{\infty} c_j^2$, finite ν'th moment $E|u_t|^{\nu} < \infty$ and its partial sums $S_t := \sum_{i=1}^{t} u_i$ satisfy the functional law (cf. Phillips and Solo, 1992)

$$B_{n^\alpha}(\cdot) := \frac{S_{\lfloor n^\alpha \cdot \rfloor}}{n^{\alpha/2}} = \frac{\sum_{i=1}^{\lfloor n^\alpha \cdot \rfloor} u_i}{n^{\alpha/2}} \Longrightarrow B(\cdot),$$

where $B(\cdot)$ is Brownian motion with variance $\omega^2 = \sigma^2 C(1)^2$. Using the Beveridge Nelson (BN) decomposition, we obtain the following representation for u_t

$$u_t = C(1)\varepsilon_t - \Delta\tilde{\varepsilon}_t, \quad \text{for } \tilde{\varepsilon}_t = \sum_{j=0}^{\infty} \tilde{c}_j \varepsilon_{t-j},\ \tilde{c}_j = \sum_{k=j+1}^{\infty} c_k, \tag{5}$$

where $\sum_{j=0}^{\infty} |\tilde{c}_j| < \infty$ is assured by the summability condition $\sum_{j=1}^{\infty} j|c_j| < \infty$. The derivation of (5) as well as the summability of the sequence $(\tilde{c}_j)_{j \geq 0}$ are included in Lemma 2.1 of Phillips and Solo (1992).

A strong approximation over $[0, n^{1-\alpha}]$ for the partial sum process of i.i.d. errors was derived in PM. In the notation of Section 2, we can construct an expanded probability space with a Brownian motion $W(\cdot)$ with variance σ^2 for which

$$\sup_{t \in [0, n^{1-\alpha}]} |W_{n^\alpha}(t) - W(t)| = o_{a.s.}\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{\nu}}}\right) \quad \text{as } n \to \infty. \tag{6}$$

Using the representation (5) and Proposition A3 in the Appendix, it is possible to embed the partial sum process $B_{n^\alpha}(\cdot)$ of the weakly dependent errors in a Brownian motion with variance ω^2, as the following result which is based on Phillips (1999, Lemma D) shows.

3.1 Lemma. *Suppose that the sequence $(u_t)_{t \in \mathbb{N}}$ satisfies Assumption **LP**. Then, the probability space which supports $(u_t)_{t \in \mathbb{N}}$ can be expanded in such a way that there exists a process distributionally equivalent to $B_{n^\alpha}(\cdot)$ and a Brownian motion $B(\cdot)$ with variance ω^2 on the new space for which*

$$\sup_{t \in [0, n^{1-\alpha}]} |B_{n^\alpha}(t) - B(t)| = o_p\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{\nu}}}\right) \quad \textit{as } n \to \infty. \tag{7}$$

In what follows, we will assume that the probability space has been expanded as necessary in order for (7) to apply. Note that the moment condition $\nu > \frac{2}{\alpha}$ in (2) ensures that $o_p\left(1/n^{\frac{\alpha}{2} - \frac{1}{\nu}}\right) = o_p(1)$ in (7). Note also that the argument used in the proof of Lemma 3.1 describes the expanded probability space on which (7) holds explicitly: it is the same as the probability space on which (6) holds with $B(t) = C(1)W(t)$ *a.s.*.

We now employ the same segmentation of the sample size used in PM. The chronological sequence $\{t = 1, ..., n\}$ can be written in blocks of size $\lfloor n^\alpha \rfloor$ as follows. Set $t = \lfloor n^\alpha j \rfloor + k$ for $k = 1, ..., \lfloor n^\alpha \rfloor$ and $j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1$, so that

$$y_{\lfloor n^\alpha j \rfloor + k} = \sum_{i=1}^{\lfloor n^\alpha j \rfloor + k} \rho_n^{\lfloor n^\alpha j \rfloor + k - i} u_i + \rho_n^{\lfloor n^\alpha j \rfloor + k} y_0.$$

This arrangement effectively partitions the sample size into $\lfloor n^{1-\alpha} \rfloor$ blocks each containing $\lfloor n^\alpha \rfloor$ sample points. Since the last element of each block is asymptotically equivalent to the first element of the next block, it is possible to study the asymptotic behavior of the time series $\{y_t : t = 1, ..., n\}$ via the asymptotic properties of the time series $\{y_{\lfloor n^\alpha j \rfloor + k} : j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1,\ k = 1, ..., \lfloor n^\alpha \rfloor\}$.

Letting $k = \lfloor n^\alpha p \rfloor$, for some $p \in [0, 1]$, we obtain

$$\frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor} = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor} \rho_n^{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor - i} u_i + \rho_n^{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor} \frac{y_0}{n^{\alpha/2}}.$$

The random element $y_{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor}$ corresponds to the random element $x_{\lfloor n^\alpha t \rfloor}$ of Section 2 (note that $j + p \in [0, \lfloor n^{1-\alpha} \rfloor])$. As in the case of independent errors, deriving a functional law for $y_{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor}$ provides the first step towards obtaining the limiting distribution of the various statistics arising from (4).

We start with the near stationary case $c < 0$. With a minor abuse of notation, define $x_t := \sum_{i=1}^t \rho_n^{t-i} \varepsilon_i$ and

$$V_{n^\alpha}(t) := \int_0^t e^{c(t-r)} dB_{n^\alpha}(r) = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor n^\alpha t \rfloor} e^{\frac{c}{n^\alpha}(n^\alpha t - i)} u_i.$$

Here, x_t as defined above is simply the time series x_t defined in Section 2 with initialization $x_0 = 0$. Since the limit theory of Section 2 is invariant to the initial condition x_0, the asymptotic behavior of $x_t = \sum_{i=1}^t \rho_n^{t-i} \varepsilon_i$ is given by Theorem 2.1. The random element $V_{n^\alpha}(t)$ is a direct extension of $U_{n^\alpha}(t)$ to the weakly dependent error case. The relationship between the random elements $y_{\lfloor n^\alpha \cdot \rfloor}$, $V_{n^\alpha}(\cdot)$ and their counterparts under independence is given below.

3.2 Lemma. *For each $\alpha \in (0, 1)$ and $c < 0$*

(a) $\sup_{t \in [0, n^{1-\alpha}]} \left| \frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha t \rfloor} - \frac{C(1)}{n^{\alpha/2}} x_{\lfloor n^\alpha t \rfloor} \right| = o_p(1)$

(b) $\sup_{t \in [0, n^{1-\alpha}]} \left| V_{n^\alpha}(t) - C(1) U_{n^\alpha}(t) \right| = o_p(1)$.

Lemma 3.2 together with (3) provide a uniform approximation of $n^{-\alpha/2}y_{\lfloor n^{\alpha}\cdot\rfloor}$ by $V_{n^{\alpha}}(\cdot)$ on $[0,n^{1-\alpha}]$. For each $\alpha\in(0,1)$ and $c<0$

$$\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}t\rfloor}-V_{n^{\alpha}}(t)\right|=o_p(1)\quad\text{as } n\to\infty. \tag{8}$$

The importance of (8) lies in the fact that an embedding of the random element $V_{n^{\alpha}}(t)$ to the linear diffusion $J_c(t):=\int_0^t e^{c(t-r)}dB(r)$ is possible. Using integration by parts as in the proof of Lemma 2.1 of PM, it can be shown that

$$\sup_{t\in[0,n^{1-\alpha}]}|V_{n^{\alpha}}(t)-J_c(t)|\leq 2\sup_{t\in[0,n^{1-\alpha}]}|B_{n^{\alpha}}(t)-B(t)|.$$

Thus, by Lemma 3.1 we obtain

$$\sup_{t\in[0,n^{1-\alpha}]}|V_{n^{\alpha}}(t)-J_c(t)|=o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)\quad\text{as } n\to\infty \tag{9}$$

and, in view of (8),

$$\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}t\rfloor}-J_c(t)\right|=o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)\quad\text{as } n\to\infty, \tag{10}$$

on the same probability space that (7) holds.

It has already been mentioned that the limit theory established in the following sections is derived through a combination of a functional law to a diffusion and a central limit law to a Gaussian random variable. The approximations in (9) and (10) provide the functional law part of the argument. A more immediate consequence of (10) is the limit law of the random element $y_{\lfloor n^{\alpha}\cdot\rfloor}$ on the original probability space (rather than its distributionally equivalent copy on the space where (7) holds). For all $j=0,...,\lfloor n^{1-\alpha}\rfloor-1$ and $p\in[0,1]$ we obtain

$$\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}j\rfloor+\lfloor n^{\alpha}p\rfloor}\Longrightarrow\int_0^{j+p}e^{c(j+p-r)}dB(r)\quad\text{as } n\to\infty.$$

4. Limit theory for the near stationary case

We now develop a limit theory for the centred serial correlation coefficient

$$\hat{\rho}_n-\rho_n=\frac{\sum_{t=1}^n y_{t-1}u_t}{\sum_{t=1}^n y_{t-1}^2}, \tag{11}$$

when $\rho_n=1+\frac{c}{n^{\alpha}}$ and $c<0$. The approach follows PM and uses a segmentation of the y_t series into blocks in which we may utilize the embedding (10) and apply law of large numbers and central limit arguments to the denominator and numerator of (11).

The sample variance of y_t can be rewritten in terms of block components as

$$
\begin{aligned}
\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n} y_t^2 &= \frac{1}{n^{1+\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\sum_{k=1}^{\lfloor n^{\alpha}\rfloor} y_{\lfloor n^{\alpha}j\rfloor+k}^2 + O_p\left(\frac{1}{n^{1-\alpha}}\right) \\
&= \frac{1}{n^{1-\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\frac{1}{n^{2\alpha}}\sum_{k=1}^{\lfloor n^{\alpha}\rfloor} y_{\lfloor n^{\alpha}j\rfloor+k}^2 \\
&= \frac{1}{n^{1-\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\int_0^1\left(\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}j\rfloor+\lfloor n^{\alpha}p\rfloor}\right)^2 dp \\
&= \frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor}\left(\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}\right)^2 dr + o_p(1) \\
&= \frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor} J_c(r)^2\, dr + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right), \qquad (12)
\end{aligned}
$$

by (10) and Proposition A2. By equation (12) of PM, it is possible to replace the Ornstein-Uhlenbeck process $J_c(t)$ in (12) by its stationary version $J_c^*(t)$ with an approximation error of order $O_p\left(n^{-(1-\alpha)}\right)$. If $J_c^*(0)$ is a $N\left(0, \frac{\omega^2}{-2c}\right)$ random variable independent of $B(\cdot)$, $J_c^*(t) := e^{ct}J_c^*(0) + J_c(t)$ is a strictly stationary process with autocovariance function given by

$$
\gamma_{J_c^*}(h) = \frac{\omega^2}{-2c}e^{c|h|} \quad h \in \mathbb{Z}.
$$

The sample variance then becomes

$$
\begin{aligned}
\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n} y_t^2 &= \frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor} J_c^*(r)^2\, dr + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \\
&= \frac{1}{n^{1-\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\int_j^{j+1} J_c^*(r)^2\, dr + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \\
&= \frac{\omega^2}{-2c} + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \qquad (13)
\end{aligned}
$$

by the weak law of large numbers for stationary processes, since $\gamma_{J_c^*}(0) = \omega^2/(-2c)$.

The limit distribution of a suitably standardized version of the sample covariance $\sum_{t=1}^{n} y_{t-1}u_t$ is found by expanding this covariance (see (15) below) in terms of components whose asymptotic behavior can be found directly, such as $\sum_{t=1}^{n} y_{t-1}\varepsilon_t$. The following results help to analyze these components and are proved in the Appendix.

4.1 Lemma. *Define* $\lambda := Eu_t\tilde{\varepsilon}_t = \sigma^2 \sum_{j=0}^{\infty} c_j \tilde{c}_j$. *For each* $\alpha \in (0,1)$ *and* $c < 0$

(a) $n^{-\frac{1+\alpha}{2}} y_n \tilde{\varepsilon}_n = o_p\left(n^{-\frac{1-\alpha}{2}}\right)$

(b) $n^{-\frac{1+\alpha}{2}} \sum_{t=1}^{n} (u_t\tilde{\varepsilon}_t - \lambda) = o_p(1)$

as $n \to \infty$, *where part (b) is valid under the moment condition* $E\varepsilon_0^4 < \infty$.

4.2 Lemma. *For each* $\alpha \in (0,1)$ *and* $c < 0$

$$\frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} y_{t-1}\varepsilon_t \Longrightarrow N\left(0, \frac{C(1)^2 \sigma^4}{-2c}\right) \quad \text{as } n \to \infty.$$

For the next result, it is convenient to introduce some notation used throughout the rest of the paper. Let

$$\begin{aligned} \gamma_m(h) \; &: = E\tilde{\varepsilon}_t u_{t-h} = \sigma^2 \sum_{j=0}^{\infty} c_j \tilde{c}_{j+h}, \quad h \geq 0 \\ m_n \; &: = \sum_{i=1}^{\infty} \rho_n^{i-1} \gamma_m(i). \end{aligned} \tag{14}$$

Proposition A4 in the Appendix shows that $m_n \to \sum_{i=1}^{\infty} \gamma_m(i)$ as $n \to \infty$.

4.3 Lemma. *For each* $c < 0$ *and* $\alpha \in (0,1)$ *we have*

(a) $\sum_{t=1}^{n} y_{t-1}\tilde{\varepsilon}_t = O_p\left(n^{1+\frac{\alpha}{2}}\right)$,

(b) $n^{-\frac{1+3\alpha}{2}} \sum_{t=1}^{n} (y_{t-1}\tilde{\varepsilon}_t - m_n) = o_p(1)$,

as $n \to \infty$, *where part (b) is valid under the moment condition* $E\varepsilon_0^4 < \infty$.

Using the BN decomposition (5) and summation by parts the sample covariance can be decomposed as follows:

$$\begin{aligned} &\frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} y_{t-1}u_t \\ =\; &\frac{C(1)}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} y_{t-1}\varepsilon_t - \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} y_{t-1}\Delta\tilde{\varepsilon}_t \\ =\; &\frac{C(1)}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} y_{t-1}\varepsilon_t - \frac{1}{n^{\frac{1+\alpha}{2}}} y_n\tilde{\varepsilon}_n + \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} \left\{\frac{c}{n^\alpha} y_{t-1} + u_t\right\} \tilde{\varepsilon}_t \\ =\; &\frac{C(1)}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} y_{t-1}\varepsilon_t + \frac{c}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} y_{t-1}\tilde{\varepsilon}_t + \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} u_t\tilde{\varepsilon}_t + o_p(1), \end{aligned} \tag{15}$$

by Lemma 4.1 (a). From Lemmas 4.1, 4.2 and 4.3 (a), it is clear that the leading term in the above expression for the sample covariance will be $n^{-\frac{1+\alpha}{2}}\sum_{t=1}^{n}u_t\tilde{\varepsilon}_t$ with asymptotic order $O_{a.s.}\left(n^{\frac{1-\alpha}{2}}\right)$ given by the ergodic theorem. Thus, if no correction is made to account for weak dependence, the sample covariance will converge to the constant probability limit of the leading term as follows:

$$\frac{1}{n}\sum_{t=1}^{n}y_{t-1}u_t=\frac{1}{n}\sum_{t=1}^{n}u_t\tilde{\varepsilon}_t+O_p\left(n^{-\frac{\alpha\wedge(1-\alpha)}{2}}\right)=\lambda+o_p(1), \tag{16}$$

by ergodicity of $u_t\tilde{\varepsilon}_t$. The above, together with (13), imply that for each $\alpha\in(0,1)$

$$n^{\alpha}(\hat{\rho}_n-\rho_n)=\frac{\frac{1}{n}\sum_{t=1}^{n}y_{t-1}u_t}{\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n}y_{t-1}^2}\longrightarrow_p\frac{\lambda}{\frac{\omega^2}{-2c}}. \tag{17}$$

Note that λ is a one sided long run covariance of u_t (cf. Phillips, 1987b) since, denoting the autocovariance function of u_t by $\gamma_u(h)$, we have

$$\begin{aligned}\sum_{h=1}^{\infty}\gamma_u(h) &= \sigma^2\sum_{j=1}^{\infty}c_j\sum_{h=1}^{\infty}c_{j+h}=\sigma^2\sum_{j=1}^{\infty}c_j\sum_{k=j+1}^{\infty}c_k\\ &= \sigma^2\sum_{j=0}^{\infty}c_j\tilde{c}_j=\lambda.\end{aligned} \tag{18}$$

Obtaining a non degenerate weak limit for the sample covariance requires centering around the asymptotic mean of the terms $\sum_{t=1}^{n}u_t\tilde{\varepsilon}_t$ and $\sum_{t=1}^{n}y_{t-1}\tilde{\varepsilon}_t$. Then, for each $\alpha\in(0,1)$ (15) gives, up to $o_p(1)$,

$$\begin{aligned}\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\left(y_{t-1}u_t-\lambda-\frac{c}{n^{\alpha}}m_n\right) &= \frac{C(1)}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}y_{t-1}\varepsilon_t+\frac{c}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}(y_{t-1}\tilde{\varepsilon}_t-m_n)\\ &\quad+\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}(u_t\tilde{\varepsilon}_t-\lambda)\\ &= \frac{C(1)}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}y_{t-1}\varepsilon_t+o_p(1)\\ &\Longrightarrow N\left(0,\frac{\omega^4}{-2c}\right),\end{aligned} \tag{19}$$

under the moment condition $E\varepsilon_0^4<\infty$, by Lemmas 4.1 (b), 4.2 and 4.3 (b), recalling that $\omega^2=C(1)^2\sigma^2$.

From (13) and (19) it is clear that the weak dependence structure of the innovations induces an asymptotic bias for the least squares estimator $\hat{\rho}_n$, since for each

$\alpha \in (0,1)$,

$$n^{\frac{1+\alpha}{2}}\left[\hat{\rho}_n - \rho_n - \frac{n\left(\lambda + \frac{c}{n^\alpha} m_n\right)}{\sum_{t=1}^n y_{t-1}^2}\right] = \frac{\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^n \left(y_{t-1}u_t - \lambda - \frac{c}{n^\alpha} m_n\right)}{\frac{1}{n^{1+\alpha}}\sum_{t=1}^n y_{t-1}^2} \Longrightarrow N(0, -2c). \tag{20}$$

More explicit calculations of the asymptotic bias of $\hat{\rho}_n$ involve analysis of the limiting distribution of the denominator, $\sum_{t=1}^n y_{t-1}^2$, of $\hat{\rho}_n$ centered around its asymptotic mean. The left side of (20) can be written as

$$n^{\frac{1+\alpha}{2}}\left(\hat{\rho}_n - \rho_n - \frac{\lambda + \frac{c}{n^\alpha} m_n}{\frac{1}{n}\sum_{t=1}^n y_{t-1}^2}\right) = n^{\frac{1+\alpha}{2}}\left[\hat{\rho}_n - \rho_n - \frac{1}{n^\alpha}\frac{\lambda + \frac{c}{n^\alpha} m_n}{\frac{\omega_n^2}{-2c} + \frac{1}{n}\sum_{t=1}^n \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega_n^2}{-2c}\right)}\right],$$

where

$$\omega_n^2 := \frac{\omega^2 + \frac{2c}{n^\alpha}(\lambda + \rho_n m_n)}{1 + \frac{c}{2n^\alpha}}. \tag{21}$$

Part (b) of Theorem 4.4 below gives $n^{-1}\sum_{t=1}^n \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega_n^2}{-2c}\right) = O_p\left(n^{-\frac{1-\alpha}{2}}\right)$, implying that

$$n^{\frac{1+\alpha}{2}}\left[\hat{\rho}_n - \rho_n - \frac{n\left(\lambda + \frac{c}{n^\alpha} m_n\right)}{\sum_{t=1}^n y_{t-1}^2}\right] = n^{\frac{1+\alpha}{2}}\left[\hat{\rho}_n - \rho_n - \frac{1}{n^\alpha}\frac{-2c\left(\lambda + \frac{c}{n^\alpha} m_n\right)}{\omega_n^2}\right] + O_p(1).$$

As we see below, the asymptotic distribution of $\hat{\rho}_n$ depends not only on the probability limit (13) of $n^{-1-\alpha}\sum_{t=1}^n y_t^2$ but also the asymptotic distribution of a centred and standardized version of this sample moment. The latter can be obtained as an approximation of the centered sample covariance, established in the theorem below.

4.4 Theorem. *For model (4) with $\rho_n = 1 + c/n^a$, $c < 0$, $\alpha \in (0,1)$ and weakly dependent errors satisfying Assumption **LP** with $E\varepsilon_0^4 < \infty$, the following limits apply as $n \to \infty$.*

(a) $n^{-\frac{1+\alpha}{2}}\sum_{t=1}^n \left(y_{t-1}u_t - \lambda - \frac{c}{n^\alpha} m_n\right) \Longrightarrow N\left(0, \frac{\omega^4}{-2c}\right)$,

(b) $n^{-\frac{1+\alpha}{2}}\sum_{t=1}^n \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega_n^2}{-2c}\right) = \frac{1}{-c} n^{-\frac{1+\alpha}{2}}\sum_{t=1}^n \left(y_{t-1}u_t - \lambda - \frac{c}{n^\alpha} m_n\right) + o_p(1)$,

where ω_n^2 is given by (21).

4.5 Remarks.

(i) Since by Proposition A4

$$\lim_{n\to\infty} m_n = \sum_{i=1}^{\infty} \gamma_m(i) < \infty,$$

we have $n^{-\frac{1+\alpha}{2}} \sum_{t=1}^{n} \frac{c}{n^\alpha} m_n = O\left(n^{-\frac{3\alpha-1}{2}}\right)$. Thus, when $\alpha \in \left(\frac{1}{3}, 1\right)$, part (a) becomes

$$n^{-\frac{1+\alpha}{2}} \sum_{t=1}^{n} (y_{t-1}u_t - \lambda) \Longrightarrow N\left(0, \frac{\omega^4}{-2c}\right).$$

(ii) Convergence of m_n also implies that $\omega_n^2 = \omega^2 + O(n^{-\alpha})$ as $n \to \infty$, giving

$$n^{-\frac{1+\alpha}{2}} \sum_{t=1}^{n} \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega_n^2}{-2c}\right) = n^{-\frac{1+\alpha}{2}} \sum_{t=1}^{n} \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega^2}{-2c}\right) + O\left(n^{-\frac{3\alpha-1}{2}}\right).$$

Thus, when $\alpha \in \left(\frac{1}{3}, 1\right)$, part (b) becomes

$$n^{-\frac{1+\alpha}{2}} \sum_{t=1}^{n} \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega^2}{-2c}\right) = \frac{1}{-c} n^{-\frac{1+\alpha}{2}} \sum_{t=1}^{n} (y_{t-1}u_t - \lambda) + o_p(1).$$

We are now in a position to provide a nonrandom expression for the asymptotic bias term in (20) and hence derive the limit distribution of the normalized and centered serial correlation coefficient. Letting

$$\bar{\rho}_n := \rho_n + \frac{1}{n^\alpha}\left(\lambda + \frac{c}{n^\alpha} m_n\right) \frac{-2c}{\omega_n^2}$$

we obtain

$$\begin{aligned}
\hat{\rho}_n - \bar{\rho}_n &= \frac{\sum_{t=1}^{n} y_t y_{t-1}}{\sum_{t=1}^{n} y_{t-1}^2} - \bar{\rho}_n = \frac{\sum_{t=1}^{n} y_t y_{t-1} - \bar{\rho}_n \sum_{t=1}^{n} y_{t-1}^2}{\sum_{t=1}^{n} y_{t-1}^2} \\
&= \frac{\sum_{t=1}^{n} y_{t-1}u_t - \left(\lambda + \frac{c}{n^\alpha} m_n\right) \frac{-2c}{\omega_n^2} \sum_{t=1}^{n} \frac{y_{t-1}^2}{n^\alpha}}{\sum_{t=1}^{n} y_{t-1}^2} \\
&= \frac{\sum_{t=1}^{n} y_{t-1}u_t - \left(\lambda + \frac{c}{n^\alpha} m_n\right) \frac{-2c}{\omega_n^2} \sum_{t=1}^{n} \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega_n^2}{-2c}\right) - n\left(\lambda + \frac{c}{n^\alpha} m_n\right)}{\sum_{t=1}^{n} y_{t-1}^2} \\
&= \frac{\sum_{t=1}^{n} \left(y_{t-1}u_t - \lambda - \frac{c}{n^\alpha} m_n\right) - \left(\lambda + \frac{c}{n^\alpha} m_n\right) \frac{-2c}{\omega_n^2} \sum_{t=1}^{n} \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega^2}{-2c}\right)}{\sum_{t=1}^{n} y_{t-1}^2}. \quad (22)
\end{aligned}$$

Normalizing and using Theorem 4.4, yields as $n \to \infty$

$$\begin{aligned} n^{\frac{1+\alpha}{2}} \left(\hat{\rho}_n - \bar{\rho}_n\right) &= \frac{\left[1 - \frac{2\left(\lambda + \frac{c}{n^\alpha} m_n\right)}{\omega_n^2}\right] \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^n \left(y_{t-1}u_t - \lambda - \frac{c}{n^\alpha} m_n\right)}{\frac{1}{n^{1+\alpha}} \sum_{t=1}^n y_{t-1}^2} + o_p(1) \\ &\Longrightarrow \left[1 - \frac{2\lambda}{\omega^2}\right] N(0, -2c) \equiv \frac{\sigma_u^2}{\omega^2} N(0, -2c) \equiv N\left(0, -2c \frac{\sigma_u^4}{\omega^4}\right), \end{aligned}$$

since $\omega^2 = \sigma_u^2 + 2\lambda$. We have thus obtained the asymptotic distribution of the normalized and centered serial correlation coefficient, presented in the following theorem.

4.6 Theorem. *For model (4) with $\rho_n = 1 + c/n^a$, $c < 0$, $\alpha \in (0,1)$ and weakly dependent errors satisfying Assumption **LP** with $E\varepsilon_0^4 < \infty$*

$$n^{\frac{1+\alpha}{2}} \left[\hat{\rho}_n - \rho_n - \frac{1}{n^\alpha} \frac{-2c}{\omega_n^2} \left(\lambda + \frac{c}{n^\alpha} m_n\right)\right] \Longrightarrow N\left(0, -2c \frac{\sigma_u^4}{\omega^4}\right) \quad \text{as } n \to \infty. \tag{23}$$

4.7 Remarks.

(i) Since m_n is a convergent sequence and $\omega_n^2 = \omega^2 + O(n^{-\alpha})$ as $n \to \infty$ we have

$$n^{\frac{1+\alpha}{2}} \left[\frac{1}{n^\alpha} \frac{-2c}{\omega_n^2} \left(\lambda + \frac{c}{n^\alpha} m_n\right)\right] = n^{\frac{1+\alpha}{2}} \left(\frac{1}{n^\alpha} \frac{-2c\lambda}{\omega^2}\right) + O\left(\frac{1}{n^{\frac{3\alpha - 1}{2}}}\right),$$

which implies that, for $\alpha \in \left(\frac{1}{3}, 1\right)$, (23) becomes

$$n^{\frac{1+\alpha}{2}} \left(\hat{\rho}_n - \rho_n - \frac{1}{n^\alpha} \frac{-2c\lambda}{\omega^2}\right) \Longrightarrow N\left(0, -2c \frac{\sigma_u^4}{\omega^4}\right) \quad \text{as } n \to \infty. \tag{24}$$

(ii) By a simple rearrangement, the bias term in (23) can be written as

$$\frac{1}{n^\alpha} \frac{-2c}{\omega_n^2} \left(\lambda + \frac{c}{n^\alpha} m_n\right) = \frac{\left(1 - \rho_n^2\right) \left[\lambda + (\rho_n - 1) m_n\right]}{\sigma_u^2 + 2\rho_n \left[\lambda + (\rho_n - 1) m_n\right]} = \frac{\left(1 - \rho_n^2\right) \sum_{i=1}^\infty \rho_n^{i-1} \gamma_u(i)}{\sigma_u^2 + 2 \sum_{i=1}^\infty \rho_n^i \gamma_u(i)},$$

using the identity $\sum_{i=1}^\infty \rho_n^{i-1} \gamma_u(i) = \lambda + (\rho_n - 1) m_n$. This corresponds to the asymptotic bias term of the serial correlation coefficient of a stationary first order autoregression with linear process errors. To see this, fix $\rho \in (-1, 1)$ and consider the process

$$y_t = \rho y_{t-1} + u_t, \quad u_t = \sum_{j=0}^\infty c_j \varepsilon_{t-j}, \ \sum_{j=1}^\infty j \left|c_j\right| < \infty$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. $(0,\sigma^2)$ random variables. Then y_t is itself a linear process,

$$y_t = \sum_{j=0}^{\infty} \overline{c}_j \varepsilon_{t-j}, \quad \overline{c}_j = \sum_{k=0}^{j} \rho^{j-k} c_k,$$

satisfying $\sum_{j=1}^{\infty} j\left|\overline{c}_j\right| < \infty$. Thus, denoting by $\rho_y(j)$ the autocorrelation function of y_t, equation (29) of Phillips and Solo (1992) implies that $\sqrt{n}\left[\hat{\rho}_n - \rho_y(1)\right]$ has a $N(0, w(1))$ limiting distribution, where

$$w(1) = \sum_{r=1}^{\infty} \left\{\rho_y(r+h) + \rho_y(h-r) - 2\rho_y(h)\rho_y(r)\right\}^2. \tag{25}$$

It is then an easy matter to obtain

$$\rho_y(1) = \rho + \frac{Eu_t y_{t-1}}{Ey_t^2} = \rho + \frac{(1-\rho^2)\sum_{i=1}^{\infty}\rho^{i-1}\gamma_u(i)}{\sigma_u^2 + 2\sum_{i=1}^{\infty}\rho^i\gamma_u(i)},$$

showing that the asymptotic bias term in Theorem 4.6 coincides with the asymptotic bias under stationarity.

(iii) The bias/inconsistency arising from weak dependence, as calculated in (23), has the same order $O(n^{-\alpha})$ as the moderate deviation departure from unity itself. When $\alpha \in \left(\frac{1}{3}, 1\right)$ (24) shows that the parameter determining the bias is the one sided long run covariance λ of the errors u_t, precisely the same parameter that appears in the limiting bias of the least squares estimator in the unit root case (cf. Phillips, 1987b). Although the term $\frac{c}{n^\alpha}m_n$ in (23) is of a smaller order than that involving λ, the effect of $m_n = \sum_{i=1}^{\infty}\rho_n^{i-1}E\tilde{\varepsilon}_t u_{t-i}$ on the asymptotic bias increases as ρ_n approaches the stationary region (i.e., as $\alpha \to 0$).

(iv) When the innovation errors u_t are i.i.d., λ and m_n are identically equal to 0, $\sigma_u^2 = \omega^2$, and (23) reduces to

$$n^{\frac{1+\alpha}{2}}\left(\hat{\rho}_n - \rho_n\right) \Longrightarrow N(0, -2c) \quad \textit{as } n \to \infty, \tag{26}$$

which is part (d) of Theorem 2.1 from PM. Thus, Theorem 4.6 generalizes that moderate deviation limit theory to the case of weak dependence. Comparing the asymptotic variances between (23) and (26), we conclude that, while weak dependence introduces a limiting bias, it also changes the asymptotic variance of the centered least squares estimator. Indeed, when $\omega^2 > \sigma_u^2$ (or when $\lambda > 0$) the limiting variance is reduced. Thus, stronger long run dependence in the series reduces the variance in the limit distribution serial correlation coefficient, as might be anticipated by heuristic arguments.

4.8 The Stationary Case

When $\alpha = 0$, $\rho_n = \rho = 1 + c$ and the model (4) is stationary for $c \in (-2, 0)$. As we have seen in Remark 4.7 (ii), centering in (23) corresponds to the usual centering for the serial correlation coefficient in the stationary case and we have $\hat{\rho}_n \to_p \rho_y(1)$.

For the limit distribution theory we may set, without loss of generality, $c = -1$ and $\rho = 0$, so that $y_t = u_t$ in (4) and then y_t is a weakly dependent time series. We note that equation (22) reduces as follows

$$\begin{aligned} \hat{\rho}_n - \rho_u(1) &= \frac{\sum_{t=1}^n \left(y_{t-1}u_t - \lambda - \frac{c}{n^\alpha} m_n\right) - \left(\lambda + \frac{c}{n^\alpha} m_n\right) \frac{-2c}{\omega_n^2} \sum_{t=1}^n \left(\frac{y_{t-1}^2}{n^\alpha} - \frac{\omega^2}{-2c}\right)}{\sum_{t=1}^n y_{t-1}^2} \\ &= \frac{\sum_{t=1}^n \left(u_t u_{t-1} - \gamma_u(1)\right) - \frac{\gamma_u(1)}{\gamma_u(0)} \sum_{t=1}^n \left(u_{t-1}^2 - \gamma_u(0)\right)}{\sum_{t=1}^n u_{t-1}^2}, \end{aligned} \tag{27}$$

so that by standard limit results for serial correlations (e.g. Phillips and Solo, 1992) we have

$$\begin{aligned} & \sqrt{n}\left(\hat{\rho}_n - \rho_u(1)\right) \\ = \; & \frac{n^{-1/2} \sum_{t=1}^n \left(u_t u_{t-1} - \gamma_u(1)\right) - \rho_u(1)\, n^{-1/2} \sum_{t=1}^n \left(u_{t-1}^2 - \gamma_u(0)\right)}{n^{-1} \sum_{t=1}^n u_{t-1}^2} && (28) \\ \Longrightarrow \; & N(0, w(1)), && (29) \end{aligned}$$

where $w(1)$ is as in (25) with ρ_y replaced by ρ_u. Thus, in contrast to the case $\alpha > 0$ where the terms in the numerator of (22) are asymptotically collinear after standardization (as implied by Theorem 4.4 (b) and as used in the limit distribution (23) for this case), the terms in the numerator of (28) are no longer asymptotically collinear. Instead, the terms in the numerator of (28) have a common component involving the term $\gamma_u(1)\, n^{-1/2} \sum_{t=1}^n \left(\varepsilon_t^2 - \sigma^2\right)/\sigma^2$ which cancels out, ensuring that the limiting variance (25) depends only on second order moments.

Thus, the limit distribution theory in Theorem 4.6 for the moderate deviations case does not specialize directly to the stationary case. Instead, when $\alpha = 0$ some additional terms enter the calculations that are $o_p(1)$ when $\alpha > 0$. For instance, when y_t is a moderate deviations from unity process, the sample covariance $\sum_{t=1}^n y_{t-1}u_t$ can be approximated, after appropriate centering, by the martingale $\sum_{t=1}^n y_{t-1}\varepsilon_t$:

$$\begin{aligned} \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^n \left(y_{t-1}u_t - \lambda - \frac{c}{n^\alpha} m_n\right) &= \frac{C(1)}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^n y_{t-1}\varepsilon_t + \frac{c}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^n \left(y_{t-1}\tilde{\varepsilon}_t - m_n\right) \\ &\quad + \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^n \left(u_t \tilde{\varepsilon}_t - \lambda\right), \end{aligned}$$

the last two terms on the right side being asymptotically negligible for each $\alpha > 0$. When $\alpha = 0$, however, both $n^{-1/2} \sum_{t=1}^n \left(y_{t-1}\tilde{\varepsilon}_t - m_n\right)$ and $n^{-1/2} \sum_{t=1}^n \left(u_t \tilde{\varepsilon}_t - \lambda\right)$ contribute to the Gaussian limit distribution of the centered sample covariance.

Also, in the proof of Theorem 4.4 we now have in place of (56)

$$\begin{aligned}
\sum_{t=1}^{n} y_{t-1}^{2} &= \frac{1}{-2c\left(1+\frac{c}{2}\right)}\left\{2\left(1+c\right)\sum_{t=1}^{n} y_{t-1}u_t+\sum_{t=1}^{n} u_t^2\right\}+O_p\left(1\right)\\
&= \frac{1}{-2c\left(1+\frac{c}{2}\right)}\left\{2\left(1+c\right)\sum_{t=1}^{n}\left(y_{t-1}u_t-\lambda-cm_n\right)+\sum_{t=1}^{n}\left(u_t^2-\sigma_u^2\right)\right\}\\
&\quad+\frac{n}{-2c}\frac{2\left(1+c\right)\left(\lambda+cm_n\right)+\sigma_u^2}{1+\frac{c}{2}}+O_p\left(1\right)\\
&= \frac{1}{-2c\left(1+\frac{c}{2}\right)}\left\{2\left(1+c\right)\sum_{t=1}^{n}\left(y_{t-1}u_t-\lambda-cm\right)+\sum_{t=1}^{n}\left(u_t^2-\sigma_u^2\right)\right\}\\
&\quad+\frac{n}{-2c}\frac{2\left(1+c\right)\left(\lambda+cm\right)+\sigma_u^2}{1+\frac{c}{2}}+O_p\left(1\right), \qquad (30)
\end{aligned}$$

where

$$m=\sum_{i=1}^{\infty}\left(1+c\right)^{i-1}\gamma_m\left(i\right).$$

When $c=-1$, (30) simplifies to

$$\sum_{t=1}^{n} y_{t-1}^{2}=\sum_{t=1}^{n}\left(u_t^2-\sigma_u^2\right)+n\sigma_u^2+O_p\left(1\right).$$

Theorem 4.4 (b) fails in both cases because of the presence of the term $\sum_{t=1}^{n}\left(u_t^2-\sigma_u^2\right)$ which remains important asymptotically, unlike the $\alpha>0$ case. Nonetheless, the correct limit distribution theory still follows from (22) as shown above in (27) - (29).

5. Limit theory for the near explosive case

We now turn to the limit behavior of $\hat{\rho}_n-\rho_n$ when $\rho_n=1+c/n^a$ and $c>0$. The approach follows PM closely and adjustments in the arguments of that paper are needed only to allow for weakly dependent u_t in the derivations. First, the weak convergence of $V_{n^\alpha}\left(t\right)$ to $J_c\left(t\right)$ still holds on $D\left[0,\infty\right)$. $J_c\left(t\right)\equiv N\left(0,\frac{\omega^2}{2c}\left(e^{2ct}-1\right)\right)$ is not bounded in probability as $t\to\infty$, so for $t\in\left[0,n^{1-\alpha}\right]$ a normalization of $O\left(\exp\left\{-cn^{1-\alpha}\right\}\right)$ is used to achieve a weak limit for $V_{n^\alpha}\left(t\right)$. A similar normalization is needed for $n^{-\alpha/2}y_{\lfloor n^\alpha t\rfloor}$, namely ρ_n^{-n}. The notational conventions introduced in PM, $\kappa_n:=n^\alpha\lfloor n^{1-\alpha}\rfloor$ and $q:=n^{1-\alpha}-\lfloor n^{1-\alpha}\rfloor$, are used throughout the paper.

The following lemma shows the continued validity of two functional approximations for the near explosive case that were used in PM.

5.1 Lemma. *For each* $\alpha \in (0,1)$ *and* $c > 0$

(a) $\sup_{t\in[0,n^{1-\alpha}]} \left| \int_0^t \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB(s) \right| = o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)$

(b) $\sup_{t\in[0,n^{1-\alpha}]} \left| \int_0^t \rho_n^{-(\lfloor n^\alpha t\rfloor - \lfloor n^\alpha s\rfloor)} dB_{n^\alpha}(s) - J_{-c}(t) \right| = o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)$

as $n \to \infty$, *on the same probability space that (7) holds.*

For the sample variance, note first that, unlike the near-stationary case, the limit theory is not determined exclusively from the blocks $\{y^2_{\lfloor n^\alpha j\rfloor + k} : j = 0, ..., \lfloor n^{1-\alpha}\rfloor - 1, k = 1, ..., \lfloor n^\alpha \rfloor\}$. From Proposition A3 of PM, we can write the sample variance as

$$\frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{t=1}^n y_t^2 = \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor - 1} \sum_{k=1}^{\lfloor n^\alpha \rfloor} y^2_{\lfloor n^\alpha j\rfloor + k} + \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{t=\lfloor \kappa_n\rfloor}^n y_t^2 + O_p\left(\frac{1}{n^\alpha}\right). \tag{31}$$

We denote by U_{1n} and U_{2n} the first and second term on the right side of (31) respectively. Since U_{2n} is almost surely positive with limiting expectation $\frac{\sigma^2}{4c^2}\left(e^{2cq} - 1\right) > 0$ when $q > 0$, we conclude that it contributes to the limit theory whenever $n^{1-\alpha}$ is not an integer.

We will analyze each of the two terms on the right of (31) separately. The term containing the block components can be written as

$$\begin{aligned} U_{1n} &= \rho_n^{-2\kappa_n} \sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor - 1} \frac{1}{n^{2\alpha}} \sum_{k=1}^{\lfloor n^\alpha\rfloor} y^2_{\lfloor n^\alpha j\rfloor + k} \\ &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha}\rfloor} \left(\int_0^r \rho_n^{\lfloor n^\alpha r\rfloor - n^\alpha s} dB_{n^\alpha}(s) \right)^2 dr + o_p(1). \end{aligned}$$

Taking the inner integral along $[0, r] = [0, \lfloor n^{1-\alpha}\rfloor] \setminus (r, \lfloor n^{1-\alpha}\rfloor]$ we have, up to $o_p(1)$,

$$U_{1n} = \left(\int_0^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) \right)^2 \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{2\lfloor n^\alpha r\rfloor} dr + R_n, \tag{32}$$

where the remainder term R_n is shown in the Appendix to be $o_p(1)$. The second integral on the right side of (32) can be evaluated directly to obtain

$$\int_0^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{2\lfloor n^\alpha r\rfloor} dr = \frac{\rho_n^{2\kappa_n}}{2c} [1 + o(1)] \quad \text{as } n \to \infty. \tag{33}$$

Using (33) and part (a) of Lemma 5.1, (32) yields

$$\begin{aligned} U_{1n} &= \frac{1}{2c} \left(\int_0^{\lfloor n^{1-\alpha}\rfloor} e^{-cs} dB(s) \right)^2 + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \\ &= \frac{1}{2c} \left(\int_0^{\infty} e^{-cs} dB(s) \right)^2 + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \end{aligned} \tag{34}$$

on the same probability space that (7) holds.

For the second term on the right of (31), noting that $\lfloor n-\kappa_n\rfloor = \lfloor n^{\alpha}q\rfloor$, $q\in[0,1)$, we obtain

$$
\begin{aligned}
U_{2n} &= \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}}\sum_{i=0}^{n-\lfloor\kappa_n\rfloor} y_{i+\lfloor\kappa_n\rfloor}^2 \\
&= \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}}\sum_{i=1}^{\lfloor n^{\alpha}q\rfloor} y_{i+\lfloor\kappa_n\rfloor-1}^2 + O_p\left(\frac{1}{n^{\alpha}}\right) \\
&= \frac{\rho_n^{-2\kappa_n}}{n^{\alpha}}\int_0^q y_{\lfloor\kappa_n\rfloor+\lfloor n^{\alpha}p\rfloor}^2 dp - \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}}\left(q-\frac{\lfloor n^{\alpha}q\rfloor}{n^{\alpha}}\right) y_{\lfloor\kappa_n\rfloor+\lfloor n^{\alpha}q\rfloor}^2 \\
&= \int_0^q\left(\frac{\rho_n^{-\kappa_n}}{n^{\alpha/2}}y_{\lfloor\kappa_n\rfloor+\lfloor n^{\alpha}p\rfloor}\right)^2 dp + O_p\left(\frac{1}{n^{2\alpha}}\right). \qquad (35)
\end{aligned}
$$

Now for each $p\in[0,q]$, $q\in[0,1)$, the following functional approximation is established in the Appendix:

$$
\frac{\rho_n^{-\kappa_n}}{n^{\alpha/2}}y_{\lfloor\kappa_n\rfloor+\lfloor n^{\alpha}p\rfloor} = e^{cp}\int_0^{\infty} e^{-cs}dW(s) + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \quad \text{as } n\to\infty \qquad (36)
$$

on the same probability space that (7) holds. Thus, applying the dominated convergence theorem to (35) yields

$$
\begin{aligned}
U_{2n} &= \left(\int_0^{\infty} e^{-cs}dW(s)\right)^2\int_0^q e^{2cp}dp + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \\
&= \frac{1}{2c}\left(\int_0^{\infty} e^{-cs}dW(s)\right)^2\left(e^{2cq}-1\right) + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right). \qquad (37)
\end{aligned}
$$

The asymptotic distribution of the sample variance in the near explosive case can be derived directly from the limit results (34) and (37) for the two terms of (31). Letting $X := \int_0^{\infty} e^{-cs}dB(s) \equiv N\left(0,\frac{\omega^2}{2c}\right)$, and using the asymptotic equivalence $\rho_n^{-2\kappa_n}e^{-2cq} = \rho_n^{-2n}[1+o(1)]$, we conclude that

$$
\frac{\rho_n^{-2n}}{n^{2\alpha}}\sum_{t=1}^{n} y_t^2 = \frac{1}{2c}X^2 + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right),
$$

on the same probability space that (7) holds, and hence

$$
\frac{\rho_n^{-2n}}{n^{2\alpha}}\sum_{t=1}^{n} y_t^2 \Longrightarrow \frac{1}{2c}X^2 \quad \text{as } n\to\infty \qquad (38)
$$

on the original space.

As in the case of the sample variance, the asymptotic behavior of the sample covariance is partly determined by elements of the time series $y_{t-1}u_t$ that do not belong to the block components $\left\{y_{\lfloor n^\alpha j\rfloor+k-1}u_{\lfloor n^\alpha j\rfloor+k} : j=0,...,\lfloor n^{1-\alpha}\rfloor-1, k=1,...,\lfloor n^\alpha\rfloor\right\}$. Obtaining limits for the block components and the remaining time series separately in a method similar to that used for the sample variance will work. It is, however, more efficient to derive the limiting distribution of the sample covariance by using a direct argument on $\rho_n^{-n}n^{-\alpha}\sum_{t=1}^n y_{t-1}u_t$.

Using the initial condition $y_0 = o_p\left(n^{\alpha/2}\right)$ and equation (46) in the Appendix, the sample variance can be written as

$$\begin{aligned}
\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^n y_{t-1}u_t &= \frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{n-1} y_t u_{t+1} + o_p\left(\frac{\rho_n^{-n}}{n^{\alpha/2}}\right) \\
&= \frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{\left\lfloor n^\alpha\left(n^{1-\alpha}-\frac{1}{n^\alpha}\right)\right\rfloor} y_t u_{t+1} \\
&= \rho_n^{-n}\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}} \frac{1}{n^{\alpha/2}} y_{\left\lfloor n^\alpha\left(r-\frac{1}{n^\alpha}\right)\right\rfloor} dB_{n^\alpha}(r) \\
&= \rho_n^{-n}\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}}\int_0^{r-\frac{1}{n^a}} \rho_n^{\lfloor n^\alpha r\rfloor - n^\alpha s - 1} dB_{n^a}(s)\, dB_{n^\alpha}(r) + o_p(1).
\end{aligned}$$

Taking the inner integral along $\left[0, r-\frac{1}{n^a}\right] = \left[0, n^{1-\alpha}\right] \setminus \left(r-\frac{1}{n^a}, n^{1-\alpha}\right]$ we have, up to $o_p(1)$,

$$\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^n y_{t-1}u_t = \rho_n^{-1}\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}} \rho_n^{-n^\alpha s} dB_{n^a}(s)\int_0^{n^{1-\alpha}} \rho_n^{-(\lfloor\kappa_n\rfloor - \lfloor n^\alpha r\rfloor)} dB_{n^a}(r) - I_n, \quad (39)$$

where the remainder term I_n is shown in the Appendix to be $o_p(1)$. Part (b) of Lemma 5.1 implies

$$\int_0^{n^{1-\alpha}} \rho_n^{-(\lfloor\kappa_n\rfloor - \lfloor n^\alpha r\rfloor)} dB_{n^a}(r) = J_{-c}\left(n^{1-\alpha}\right) + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right).$$

The rest of the argument is identical to that in PM for the i.i.d. error case. In particular, $J_{-c}(t)$ is a L_2-bounded martingale on $[0,\infty)$, and the martingale convergence theorem implies that there exists an almost surely finite random variable Y such that

$$J_{-c}\left(n^{1-\alpha}\right) \longrightarrow_{a.s.} Y \quad \text{as } n\to\infty.$$

Since $J_{-c}\left(n^{1-\alpha}\right) \equiv N\left(0, \frac{\omega^2}{2c}\left(1-e^{-2cn^{1-\alpha}}\right)\right)$, we deduce that $Y \equiv N\left(0, \frac{\omega^2}{2c}\right)$. Thus, if $X = \int_0^\infty e^{-cs}dB(s)$ as in (38), (39) yields

$$\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^n y_{t-1}u_t = XY + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \quad \text{as } n\to\infty$$

on the same probability space that (7) holds. The latter strong approximation implies that the asymptotic distribution of the sample covariance is given in the original space by

$$\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{n} y_{t-1}u_t \Longrightarrow XY \quad X,Y \equiv N\left(0,\frac{\omega^2}{2c}\right). \tag{40}$$

As in PM, the asymptotic behavior of the serial correlation coefficient now follows from the strong approximations leading to (38) and (40) and the fact that the limiting random variables X and Y are independent.

5.2 Theorem. *For model (4) with $\rho_n = 1 + c/n^a$, $c > 0$, $\alpha \in (0,1)$ and weakly dependent errors satisfying Assumption **LP**,*

$$\frac{n^\alpha \rho_n^n}{2c}\left(\hat{\rho}_n - \rho_n\right) \Longrightarrow C \quad \text{as } n \to \infty \tag{41}$$

where C is a standard Cauchy variate.

5.3 Remarks.

(i) Other than the allowance for weakly dependent errors, the statement of theorem 5.2 is identical to that of Theorem 4.3 of PM. As discussed in PM, the Cauchy limit theory relates to much earlier work (White, 1958; Anderson, 1959; Basawa and Brockwell, 1984) on the explosive Gaussian AR(1) process. In particular, for the first order autoregressive process with fixed $|\rho| > 1$, i.i.d. Gaussian innovation errors and initialization $y_0 = 0$, White showed that

$$\frac{\rho^n}{\rho^2-1}\left(\hat{\rho}_n - \rho\right) \Longrightarrow C \quad \text{as } n \to \infty. \tag{42}$$

Replacing ρ by $\rho_n = 1 + c/n^a$, we obtain $\rho^2 - 1 = \frac{2c}{n^\alpha}[1 + o(1)]$. Hence, the normalizations in Theorem 5.2 and (42) are asymptotically equivalent as $n \to \infty$. Anderson (1959) showed that $\frac{\rho^n}{\rho^2-1}(\hat{\rho}_n - \rho)$ has a limit distribution that depends on the distribution of the errors u_t when $\rho > 1$ and that no central limit theory or invariance principle is applicable.

(ii) By contrast, an invariance principle does apply in Theorem 5.2 and the limit theory is not restricted to Gaussian processes. In particular, the Cauchy limit result (41) holds for $\rho_n = 1+c/n^a$, $\alpha \in (0,1)$, and weakly dependent innovations u_t satisfying Assumption **LP**, thereby including a much wider class of processes. At the boundary where $\alpha \to 0$, Theorem 5.2 reduces to (42) with $\rho = 1 + c$ and primitive errors ε_t with infinitely many moments, as under Gaussianity. In summary, the limit theory in the moderate deviation explosive autoregression is invariant to both the distribution and the dependence structure of the innovation errors.

(iii) The limit theory of Theorem 5.2 is also invariant to the initial condition y_0 being any fixed constant value or random variable of smaller asymptotic order than $n^{\alpha/2}$. This property is also not shared by explosive autoregressions where y_0 does influence the limit theory even in the case of i.i.d. Gaussian errors, as shown by Anderson (1959).

5.4. The explosive case

When $\alpha = 0$, the process (4) has an explosive root $\rho = 1 + c$, $c > 0$. As in the case of explosive autoregressions with independent innovations (cf. Anderson, 1959), the asymptotic behavior of the serial correlation coefficient can be derived by investigating the limiting properties of the stochastic sequences

$$Z_n := \sum_{j=1}^{n} \rho^{-j} u_j \text{ and } \Psi_n := \sum_{j=1}^{n} \rho^{-(n-j)-1} u_j. \tag{43}$$

The results of this subsection are valid for $y_0 = 0$ and weakly dependent innovations u_t satisfying assumption **LP** with the moment condition (2) relaxed to $E\varepsilon_1^2 = \sigma^2 < \infty$.

From the monotone convergence theorem

$$E \sum_{j=1}^{\infty} \left|\rho^{-j} u_j\right| = \sum_{j=1}^{\infty} |\rho|^{-j} E\left|u_j\right| = \frac{E\left|u_0\right|}{|\rho| - 1} < \infty,$$

which implies that $\sum_{j=1}^{\infty} \left|\rho^{-j} u_j\right| < \infty$ $a.s$ so that $Z_n \to_{a.s.} Z = \sum_{j=1}^{\infty} \rho^{-j} u_j$. Next, since $\{u_t\}$ is strictly stationary we may construct another strictly stationary time series $\{u_t'\}$ with identical marginal distributions to those of $\{u_t\}$ and a corresponding sequence $\Psi_n' = \sum_{j=1}^{n} \rho^{-(n-j)-1} u_{n-j+1}' = \sum_{j=1}^{n} \rho^{-j} u_j'$ for which $\Psi_n' =_d \Psi_n$ for all n. Then, $\Psi_n' \to_{a.s.} \Psi = \sum_{j=1}^{\infty} \rho^{-j} u_j'$, and it follows by the Skorohod representation theorem that $\Psi_n \to_d \Psi$. Joint weak convergence of Ψ_n and Z_n then follows and we have $(Z_n, \Psi_n) \Longrightarrow (Z, \Psi)$, as $n \to \infty$, with $Z =_d \Psi$.

The limiting random variables Ψ and Z can be shown to be independent by modifying Anderson's (1959, Theorem 2.3) argument adjusted for weakly dependent errors. The idea is that, as $n \to \infty$, Z_n can be approximated by the first $\lfloor L_n \rfloor$ elements of the sum $\sum_{j=1}^{n} \rho^{-j} u_j$ whereas Ψ_n can be approximated by the last $\lfloor L_n \rfloor$ elements of the sum $\sum_{j=1}^{n} \rho^{-(n-j)-1} u_j$ in (43), where $(L_n)_{n\in\mathbb{N}}$ is a sequence increasing to ∞ with $L_n \leq n/3$ for each n. Accordingly, define

$$Z_n^* := \sum_{j=1}^{\lfloor L_n \rfloor} \rho^{-j} u_j \text{ and } \Psi_n^* := \sum_{j=n-\lfloor L_n \rfloor+1}^{n} \rho^{-(n-j)-1} u_j = \sum_{k=1}^{\lfloor L_n \rfloor - 1} \rho^{-k} u_{n-k+1}.$$

We may further approximate Ψ_n^* as follows

$$\begin{aligned}
\Psi_n^* &= \sum_{k=1}^{\lfloor L_n \rfloor -1} \rho^{-k} \sum_{s=0}^{\infty} c_s \varepsilon_{n-k+1-s} \\
&= \sum_{k=1}^{\lfloor L_n \rfloor -1} \rho^{-k} \sum_{s=0}^{\lfloor L_n \rfloor} c_s \varepsilon_{n-k+1-s} + \sum_{k=1}^{\lfloor L_n \rfloor -1} \rho^{-k} \sum_{s=\lfloor L_n \rfloor+1}^{\infty} c_s \varepsilon_{n-k+1-s} \\
&= \Psi_n^{**} + \sum_{k=1}^{\lfloor L_n \rfloor -1} \rho^{-k} \sum_{s=\lfloor L_n \rfloor+1}^{\infty} c_s \varepsilon_{n-k+1-s},
\end{aligned}$$

where $\Psi_n^{**} = \sum_{k=1}^{\lfloor L_n \rfloor -1} \rho^{-k} \sum_{s=0}^{\lfloor L_n \rfloor} c_s \varepsilon_{n-k+1-s}$. Now for each $s \leq \lfloor L_n \rfloor$ and $k \leq \lfloor L_n \rfloor - 1$,

$$n - k + 1 - s > n + 1 - 2 \lfloor L_n \rfloor \geq \lfloor L_n \rfloor + 1,$$

since $L_n \leq n/3$, showing that Ψ_n^{**} is independent of $\sigma\left(\varepsilon_{\lfloor L_n \rfloor}, \varepsilon_{\lfloor L_n \rfloor -1}, ...\right)$ and hence of Z_n^*. Moreover, $\Psi_n^* - \Psi_n^{**} = o_p(1)$ since

$$\begin{aligned}
E\left|\sum_{k=1}^{\lfloor L_n \rfloor -1} \rho^{-k} \sum_{s=\lfloor L_n \rfloor+1}^{\infty} c_s \varepsilon_{n-k+1-s}\right| &\leq E\left|\varepsilon_1\right| \sum_{k=1}^{\lfloor L_n \rfloor -1} |\rho|^{-k} \sum_{s=\lfloor L_n \rfloor+1}^{\infty} |c_s| \\
&\leq \frac{E\left|\varepsilon_1\right|}{|\rho| - 1} \sum_{s=\lfloor L_n \rfloor+1}^{\infty} |c_s| \to 0,
\end{aligned}$$

as $n \to \infty$ in view of **LP**. So, Z_n^* is asymptotically independent of Ψ_n^*. Next, $\Psi_n - \Psi_n^* = \sum_{k=\lfloor L_n \rfloor}^{n} \rho^{-k} u_{n-k+1}$, and so

$$E\left|\Psi_n - \Psi_n^*\right| \leq E\left|u_1\right| \sum_{k=\lfloor L_n \rfloor+1}^{n} |\rho|^{-k} = O\left(|\rho|^{-L_n}\right),$$

so that $\Psi_n - \Psi_n^* = o_p(1)$. In a similar fashion, $Z_n - Z_n^* = o_p(1)$. It follows that Z_n and Ψ_n are asymptotically independent since they differ from the independent variates Z_n^* and Ψ_n^{**} by terms that converge in probability to zero.

The variance of Z (and Ψ) can be calculated directly as

$$\begin{aligned}
E\left(\sum_{j=1}^{\infty} \rho^{-j} u_j\right)^2 &= \sigma_u^2 \sum_{j=1}^{\infty} \rho^{-2j} + 2 \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} \rho^{-j-k} \gamma_u(k-j) \\
&= \sum_{j=1}^{\infty} \rho^{-2j} \left\{\sigma_u^2 + 2 \sum_{i=1}^{\infty} \rho^{-i} \gamma_u(i)\right\} = \frac{\widetilde{\omega}^2}{\rho^2 - 1},
\end{aligned}$$

where $\widetilde{\omega}^2 = \sigma_u^2 + 2\sum_{i=1}^{\infty} \rho^{-i}\gamma_u(i)$. Since $E\left|\rho^{-n}\sum_{t=1}^{n}\sum_{j=t}^{n}\rho^{t-j-1}u_j u_t\right| = O(\rho^{-n}n)$ as $n \to \infty$, we can write the sample covariance as

$$\rho^{-n}\sum_{t=1}^{n} y_{t-1}u_t = \rho^{-n}\sum_{t=1}^{n}\sum_{j=1}^{t-1}\rho^{t-j-1}u_j u_t = Z_n\Psi_n + o_p(1).$$

By a standard argument (e.g. Anderson, 1959), $\rho^{-2n}\sum_{t=1}^{n} y_{t-1}^2 = Z_n^2/(\rho^2-1) + O_p(\rho^{-n})$. Thus, joint convergence of Ψ_n and Z_n implies that

$$\left(\rho^{-n}\sum_{t=1}^{n} y_{t-1}u_t, \rho^{-2n}\sum_{t=1}^{n} y_{t-1}^2\right) \Longrightarrow \left(Z\Psi, Z^2\right) \quad \text{as } n \to \infty. \tag{44}$$

When $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a Gaussian sequence, Z and Ψ are independent Gaussian random variables and (44) yields the standard Cauchy limit

$$\frac{\rho^n}{\rho^2-1}(\hat{\rho}_n - \rho) \Longrightarrow C. \tag{45}$$

Note that, when $\rho_n = 1 + c/n^{\alpha}$,

$$\sum_{i=1}^{\infty} \rho_n^{-i}\gamma_u(i) \to \sum_{i=1}^{\infty}\gamma_u(i) = \lambda \quad \text{as } n \to \infty$$

for each $c, \alpha > 0$ by an identical argument to that used in the proof of Proposition A4 (b). Thus, when y_t is a near explosive moderate deviations from unity process, (45) agrees with Theorem 5.2 and $\widetilde{\omega}^2 = \sigma_u^2 + 2\sum_{i=1}^{\infty}\rho_n^{-i}\gamma_u(i) \to \omega^2$, the long run variance of u_t.

6. Discussion

When there are moderate deviations from unity, the derivations of Sections 4 and 5 reveal that both functional approximations to a diffusion and standard laws of large numbers and central limit theorems contribute to the limit theory. The functional law provides in each case a limiting subsidiary process whose elements form the components that upon further summation satisfy a law of large numbers and a central limit law. While there is only one limiting process involved as $n \to \infty$, it is convenient to think of the functional law operating within blocks of length $\lfloor n^{\alpha} \rfloor$ and the law of large numbers and central limit laws operating across the $\lfloor n^{1-\alpha} \rfloor$ blocks. The moment condition in (2) ensures the validity of the embedding argument that makes this segmentation rigorous as $n \to \infty$.

Theorem 4.6 provides a bridge between stationary and local to unity autoregressions with weakly dependent innovation errors. When the innovation error sequence is a linear process, the least squares estimator has been found to satisfy a Gaussian limit theory with an asymptotic bias. A convergence rate of $n^{\frac{1}{2}+\frac{\alpha}{2}}$ has been obtained, which for $\alpha \in (0,1)$ covers the interval $\left(n^{1/2}, n\right)$, providing a link between $\sqrt{n}$ and n asymptotics. As shown in Section 4, there is also a close connection between the asymptotic bias in the serial correlation coefficient and the second order bias that arises in local to unity and unit root asymptotics.

Theorem 5.2 provides a bridge between local to unity and explosive autoregressions with weakly dependent innovation errors. In particular, when $\alpha = 1$,

$$\rho_n^n = \left(1 + \frac{c}{n}\right)^n = e^c \left[1 + o\left(1\right)\right] \text{ and } \frac{n^\alpha \rho_n^n}{2c} = O\left(n\right).$$

Thus, ignoring multiplicative constants, the convergence rate of the serial correlation coefficient takes values on (n, ρ^n) as α ranges from 1 to 0, where $\rho := 1 + c$ is an explosive autoregressive root when $\alpha = 0$. Thus, the convergence rate of the serial correlation coefficient covers the interval (n, ρ^n), establishing a link between the asymptotic behavior of local to unity and explosive autoregressions.

As discussed in PM, the bridging asymptotics are not continuous at the stationary boundary of α, at least without some modification. In the stationary case where $c < 0$ and $\alpha = 0$, the probability limit of the serial correlation coefficient is correctly captured in the limit of the moderate deviation theory as is the $\sqrt{n}$ rate of convergence, but the moderate deviation limit distribution does not continuously merge into the limit theory for the stationary case although the limit distributions are both normal with compatible centering. In the explosive case when $\alpha \to 0$, the bridging asymptotics are continuous at the boundary in the case of weak dependence, yielding the standard Cauchy limit (which applies in the boundary case under Gaussian errors).

For the limit as $\alpha \to 1$, we have $n^{1-\alpha} \to 1$, and so $\lfloor n^{1-\alpha} \rfloor = 1$ for $\alpha = 1$, in which case $j = 0$ necessarily in the blocking scheme of Section 3. The invariance principle of Phillips (1987a) $n^{-1/2} y_{\lfloor np \rfloor} \Longrightarrow J_c(p)$ on $D\left[0,1\right]$ together with the argument preceding (32) and (39) with $\alpha = 1$ and $j = 0$ yield the usual local to unity limit result (cf. Phillips, 1987a)

$$n\left(\hat{\rho}_n - \rho_n\right) \Longrightarrow \frac{\int_0^1 J_c\left(r\right) dB\left(r\right)}{\int_0^1 J_c\left(r\right)^2 dr}.$$

Thus, as in PM, continuity in the limit theory cannot be achieved at the (inside) boundary with the conventional local to unity asymptotics, at least without using the blocking construction.

7. Notation

$\lfloor \cdot \rfloor$	integer part of	$a \wedge b$	$\min(a,b)$
$:=$	definitional equality	$a \vee b$	$\max(a,b)$
$C(z)$	$:= \sum_{j=0}^{\infty} c_j z^j$	$\mathbf{1}\{\cdot\}$	indicator function
ω^2	$:= \sigma^2 C(1)^2$	$\longrightarrow_{a.s.}$	almost sure convergence
λ	$:= E u_t \tilde{\varepsilon}_t = \sigma^2 \sum_{j=0}^{\infty} c_j \tilde{c}_j$	$\longrightarrow_p$	convergence in probability
$\gamma_m(h)$	$:= E\tilde{\varepsilon}_t u_{t-h} = \sigma^2 \sum_{j=0}^{\infty} c_j \tilde{c}_{j+h},\ h \geq 0$	$\longrightarrow_{L_p}$	convergence in L_p norm
y_{nt}^*	$:= \sum_{i=0}^{n} \rho_n^i u_{t-i}$	$\Longrightarrow$	weak convergence
m_n	$:= \sum_{i=1}^{\infty} \rho_n^{i-1} \gamma_m(i)$	$\equiv$	distributional equivalence
ω_n^2	$:= \left(1 + \frac{c}{2n^\alpha}\right)^{-1} \left[\omega^2 + \frac{2c}{n^\alpha}(\lambda + \rho_n m_n)\right]$	$o_p(1)$	tends to zero in probability
$\bar{\rho}_n$	$:= \rho_n + \frac{1}{n^\alpha}\left(\lambda + \frac{c}{n^\alpha} m_n\right) \frac{-2c}{\omega_n^2}$	$o_{a.s.}(1)$	tends to zero almost surely
$\rho_z(\cdot)$	correlation of the process z_t	$a.s.$	almost surely
$W(\cdot)$	Brownian motion with variance σ^2	κ_n	$:= n^\alpha \lfloor n^{1-\alpha} \rfloor$
$B(\cdot)$	Brownian motion with variance ω^2	q	$:= n^{1-\alpha} - \lfloor n^{1-\alpha} \rfloor$
		$J_c(\cdot)$	Ornstein-Uhlenbeck process

8. Technical appendix and proofs

Propositions A1 and A2 below are proved in PM. The remainder of this section contains Propositions A3 and A4 as well as the proofs of the various statements made in the paper.

For the sake of brevity we introduce the following notation for the rest of the section.

$$C_1 := \sum_{j=0}^{\infty} |c_j| \qquad C_2 := \sum_{j=0}^{\infty} |\tilde{c}_j|$$
$$C_3 := \sum_{j=0}^{\infty} c_j^2 \qquad C_4 := \sum_{j=0}^{\infty} \tilde{c}_j^2$$
$$C_5 := \sum_{j=1}^{\infty} j^{1/2} |c_j| \qquad C_6 := \sum_{j=1}^{\infty} j |c_j|$$

and

$$C_{\alpha\delta} := \sum_{j=1}^{\infty} j^{\frac{3-3\alpha+\delta}{2}} |c_j|,$$

where δ is the positive constant of Assumption **LP**. Under **LP**, C_i, $i = 1, ..., 6, \alpha\delta$ are all finite constants.

Proposition A1. *For each $x \in [0, M]$, $M > 0$, possibly depending on n, and real valued, measurable function f* on $[0, \infty)$

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor x n^\alpha \rfloor} f\left(\frac{i}{n^\alpha}\right) u_i = \int_0^x f(r)\, dB_{n^\alpha}(r).$$

An immediate consequence of Proposition A1 is the following useful identity. For each $x \in [0, n^{1-\alpha}]$ and $m \in \mathbb{N}$

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor xn^{\alpha} \rfloor} f\left(\frac{i}{n^{\alpha}}\right) u_{i+m} = \int_0^x f(r)\, dB_{n^{\alpha}}\left(r + \frac{m}{n^{\alpha}}\right). \tag{46}$$

Proposition A2. *For $c < 0$, $\sup_{t>0} |J_c(t)| < \infty$ a.s.*

Proposition A3. *For each $\alpha \in (0,1)$*

$$\max_{0 \le t \le n} \left| \frac{\tilde{\varepsilon}_t}{n^{\alpha/2}} \right| = o_p(1) \quad \text{as } n \to \infty.$$

Proof. The argument follows Phillips (1999). Summability of $\sum_{j=1}^{\infty} j |c_j|$ ensures that $\tilde{\varepsilon}_t = \sum_{j=0}^{\infty} \tilde{c}_j \varepsilon_{t-j}$ converges absolutely almost surely. Thus, Fatou's lemma and the Minkowski inequality give

$$\begin{aligned} E|\tilde{\varepsilon}_t|^{\nu} &\le \operatorname{liminf}_{N\to\infty} E\left|\sum_{j=0}^{N} \tilde{c}_j \varepsilon_{t-j}\right|^{\nu} \le \operatorname{liminf}_{N\to\infty} \left[\sum_{j=0}^{N} \left(E|\tilde{c}_j \varepsilon_{t-j}|^{\nu}\right)^{\frac{1}{\nu}}\right]^{\nu} \\ &= E|\varepsilon_0|^{\nu} \operatorname{liminf}_{N\to\infty} \left(\sum_{j=0}^{N} |\tilde{c}_j|\right)^{\nu} = E|\varepsilon_0|^{\nu} C_2^{\nu}, \end{aligned}$$

where $C_2 = \sum_{j=0}^{\infty} |\tilde{c}_j| < \infty$ and $E|\varepsilon_0|^{\nu} < \infty$ by (2). Thus, for any $\delta > 0$ the Markov inequality gives

$$\begin{aligned} P\left(\max_{0 \le t \le n} |\tilde{\varepsilon}_t| > \delta n^{\alpha/2}\right) &\le \sum_{t=0}^{n} P\left(|\tilde{\varepsilon}_t| > \delta n^{\alpha/2}\right) \le \sum_{t=0}^{n} \frac{E|\tilde{\varepsilon}_t|^{\nu}}{\delta^{\nu} n^{\nu\alpha/2}} \\ &\le \frac{E|\varepsilon_0|^{\nu} C_2^{\nu}}{\delta^{\nu}} \frac{n+1}{n^{\nu\alpha/2}} = o(1) \end{aligned}$$

if and only if $\frac{\nu\alpha}{2} > 1$, which holds by (2). ■

Proposition A4.

(a) *Let $y_{nt}^* := \sum_{i=0}^{n} \rho_n^i u_{t-i}$. Then for each $\alpha \in \left(0, \frac{1}{2}\right]$*

$$\frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} y_{t-1} \tilde{\varepsilon}_t = \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} y_{nt-1}^* \tilde{\varepsilon}_t + o_p(1) \quad \text{as } n \to \infty.$$

(b) *Let $\gamma_m(h) = E\tilde{\varepsilon}_t u_{t-h} = \sigma^2 \sum_{j=0}^{\infty} c_j \tilde{c}_{j+h}$ for $h \ge 0$ and $m_n = \sum_{i=1}^{\infty} \rho_n^{i-1} \gamma_m(i)$. Then*

$$\lim_{n\to\infty} m_n = \sum_{i=1}^{\infty} \gamma_m(i).$$

Proof. For part (a), we can write

$$\begin{aligned}\frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\left(y_{nt-1}^{*}-y_{t-1}\right)\tilde{\varepsilon}_t &= \frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\left[\left(\sum_{i=t}^{n}\rho_n^i u_{t-i-1}-y_0\rho_n^t\right)\right]\tilde{\varepsilon}_t \\ &= \frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\sum_{i=t}^{n}\rho_n^i u_{t-i-1}\tilde{\varepsilon}_t+o_p\left(\frac{1}{n^{\frac{1-\alpha}{2}}}\right),\end{aligned}$$

since, by Proposition A3 and the fact that $\sum_{t=1}^{n}|\rho_n|^t=O(n^{\alpha})$,

$$\left|\frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}y_0\rho_n^t\tilde{\varepsilon}_t\right|\leq\left|\frac{y_0}{n^{\alpha/2}}\right|\max_{1\leq t\leq n}\left|\frac{\tilde{\varepsilon}_t}{n^{\alpha/2}}\right|\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}|\rho_n|^t=o_p\left(\frac{1}{n^{\frac{1-\alpha}{2}}}\right).$$

Also, since

$$\left|\frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\sum_{i=t}^{n}\rho_n^i u_{t-i-1}\tilde{\varepsilon}_t\right|\leq\max_{1\leq t\leq n}\left|\frac{\tilde{\varepsilon}_t}{n^{\alpha/2}}\right|\frac{1}{n^{\frac{1}{2}+\alpha}}\sum_{t=1}^{n}\sum_{i=t}^{n}|\rho_n|^i\,|u_{t-i-1}|,$$

part (a) will follow from $n^{-\left(\frac{1}{2}+\alpha\right)}\sum_{t=1}^{n}\sum_{i=t}^{n}|\rho_n|^i\,|u_{t-i-1}|<\infty$ *a.s.*. The latter holds since

$$\begin{aligned}E\left(\frac{1}{n^{\frac{1}{2}+\alpha}}\sum_{t=1}^{n}\sum_{i=t}^{n}|\rho_n|^i\,|u_{t-i-1}|\right) &\leq E|\varepsilon_0|\sum_{j=0}^{\infty}|c_j|\frac{1}{n^{\frac{1}{2}+\alpha}}\sum_{t=1}^{n}\sum_{i=t}^{n}|\rho_n|^i \\ &= O\left(\frac{1}{n^{\frac{1}{2}-\alpha}}\right)=O(1)\end{aligned}$$

when $\alpha\in\left(0,\frac{1}{2}\right]$. This completes the proof of part (a).

For part (b), first note that $\gamma_m(\cdot)$ is summable, since

$$\sum_{h=0}^{\infty}|\gamma_m(h)|\leq\sigma^2\sum_{j=0}^{\infty}|c_j|\sum_{h=0}^{\infty}|\tilde{c}_h|=\sigma^2C_1C_2<\infty.$$

The limit of m_n is obtained by an application of the Toeplitz lemma (see e.g. Hall and Heyde, 1980), as we now show. Letting for each $i\in\mathbb{N}$, $S_m(i)=\sum_{k=1}^{i}\gamma_m(k)$, $S_m(0)=0$ and using summation by parts we obtain

$$\begin{aligned}\lim_{n\to\infty}m_n &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho_n^{i-1}\Delta S_m(i)=\lim_{n\to\infty}\rho_n^nS_m(n)-\lim_{n\to\infty}\sum_{i=1}^{n}\left(\rho_n^i-\rho_n^{i-1}\right)S_m(i) \\ &= \lim_{n\to\infty}\frac{-2c}{n^{\alpha}}\sum_{i=1}^{n}\rho_n^{i-1}S_m(i)=\lim_{n\to\infty}\sum_{i=1}^{n}z_{ni}S_m(i),\end{aligned}$$

where $z_{ni} := \frac{-2c}{n^{\alpha}}\rho_n^{i-1}$, since $\rho_n^n = o(1)$ and $S_m(n) \to \sum_{k=1}^{\infty}\gamma_m(k) < \infty$. Since $z_{ni} \to 0$ for each fixed i, $\sum_{i=1}^{n}|z_{ni}|$ is bounded by a finite constant, and

$$\sum_{i=1}^{n} z_{ni} = \frac{-2c}{n^{\alpha}}\sum_{i=1}^{n}\rho_n^{i-1} = \frac{-2c}{n^{\alpha}}\frac{1-\rho_n^n}{1-\rho_n} = 1-\rho_n^n = 1+o(1) \quad \text{as } n\to\infty,$$

the Toeplitz lemma implies that

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty}\sum_{i=1}^{n} z_{ni}S_m(i) = \lim_{n\to\infty} S_m(n) = \sum_{i=1}^{\infty}\gamma_m(i).$$

This completes the proof of the proposition. ■

Proof of Lemma 3.1. Using the BN decomposition (5) we can write

$$\begin{aligned} B_{n^{\alpha}}(t) &= \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor} u_i = \frac{C(1)}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\varepsilon_i - \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\Delta\tilde{\varepsilon}_i \\ &= C(1)W_{n^{\alpha}}(t) - \frac{1}{n^{\alpha/2}}\left(\tilde{\varepsilon}_{\lfloor tn^{\alpha}\rfloor} - \tilde{\varepsilon}_1\right). \end{aligned}$$

Letting $B(t) = C(1)W(t)$ on the probability space where (6) holds, we obtain

$$\begin{aligned} \sup_{t\in[0,n^{1-\alpha}]}|B_{n^{\alpha}}(t) - B(t)| &\leq C(1)\sup_{t\in[0,n^{1-\alpha}]}|W_{n^{\alpha}}(t) - W(t)| + \frac{1}{n^{\alpha/2}}\sup_{t\in[0,n^{1-\alpha}]}\left|\tilde{\varepsilon}_{\lfloor tn^{\alpha}\rfloor} - \tilde{\varepsilon}_1\right| \\ &\leq C(1)\sup_{t\in[0,n^{1-\alpha}]}|W_{n^{\alpha}}(t) - W(t)| + 2\max_{0\leq k\leq n}\left|\frac{\tilde{\varepsilon}_k}{n^{\alpha/2}}\right| \\ &= o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right), \end{aligned}$$

by (6) and Proposition A3. ■

Proof of Lemma 3.2. For part (a), we can use the BN decomposition (5) to write

$$\begin{aligned} \frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}t\rfloor} &= \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^{\alpha}t\rfloor}\rho_n^{\lfloor n^{\alpha}t\rfloor - i}u_i + \frac{y_0}{n^{\alpha/2}}\rho_n^{\lfloor n^{\alpha}t\rfloor} \\ &= \frac{C(1)}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^{\alpha}t\rfloor}\rho_n^{\lfloor n^{\alpha}t\rfloor - i}\varepsilon_i - \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^{\alpha}t\rfloor}\rho_n^{\lfloor n^{\alpha}t\rfloor - i}\Delta\tilde{\varepsilon}_i + \frac{y_0}{n^{\alpha/2}}\rho_n^{\lfloor n^{\alpha}t\rfloor} \\ &= \frac{C(1)}{n^{\alpha/2}}x_{\lfloor n^{\alpha}t\rfloor} - \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^{\alpha}t\rfloor}\rho_n^{\lfloor n^{\alpha}t\rfloor - i}\Delta\tilde{\varepsilon}_i + \frac{y_0}{n^{\alpha/2}}\rho_n^{\lfloor n^{\alpha}t\rfloor}. \end{aligned}$$

Since $\frac{y_0}{n^{\alpha/2}}\rho_n^{\lfloor n^\alpha t\rfloor}=o_p(1)$ uniformly in $t\geq 0$, it is enough to show that

$$\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^\alpha t\rfloor}\rho_n^{\lfloor n^\alpha t\rfloor-i}\Delta\tilde{\varepsilon}_i\right|=o_p(1). \tag{47}$$

Summation by parts gives

$$\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^\alpha t\rfloor}\rho_n^{\lfloor n^\alpha t\rfloor-i}\Delta\tilde{\varepsilon}_i=\frac{\tilde{\varepsilon}_{\lfloor n^\alpha t\rfloor}}{n^{\alpha/2}}-\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^\alpha t\rfloor}\left(\Delta\rho_n^{\lfloor n^\alpha t\rfloor-i}\right)\tilde{\varepsilon}_i,$$

so that

$$\begin{aligned}\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^\alpha t\rfloor}\rho_n^{\lfloor n^\alpha t\rfloor-i}\Delta\tilde{\varepsilon}_i\right| &\leq \max_{0\leq k\leq n}\left|\frac{\tilde{\varepsilon}_k}{n^{\alpha/2}}\right|+\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^\alpha t\rfloor}\left(\Delta\rho_n^{\lfloor n^\alpha t\rfloor-i}\right)\tilde{\varepsilon}_i\right|\\ &\leq \max_{0\leq k\leq n}\left|\frac{\tilde{\varepsilon}_k}{n^{\alpha/2}}\right|+\max_{0\leq k\leq n}\left|\frac{\tilde{\varepsilon}_k}{n^{\alpha/2}}\right|\sup_{t\geq 0}\frac{|c|}{n^\alpha}\sum_{i=1}^{\lfloor n^\alpha t\rfloor}|\rho_n|^{\lfloor n^\alpha t\rfloor-i}\\ &\leq \max_{0\leq k\leq n}\left|\frac{\tilde{\varepsilon}_k}{n^{\alpha/2}}\right|\left\{1+\frac{|c|}{n^\alpha}\sup_{t\geq 0}\frac{1-|\rho_n|^{\lfloor n^\alpha t\rfloor}}{1-|\rho_n|}\right\}\\ &\leq \max_{0\leq k\leq n}\left|\frac{\tilde{\varepsilon}_k}{n^{\alpha/2}}\right|\left\{1+\frac{|c|}{(1-|\rho_n|)\,n^\alpha}\right\}=o_p(1).\end{aligned}$$

For part (b), the BN decomposition implies that

$$\sup_{t\in[0,n^{1-\alpha}]}|V_{n^\alpha}(t)-C(1)\,U_{n^\alpha}(t)|=\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^\alpha t\rfloor}e^{\frac{c}{n^\alpha}(n^\alpha t-i)}\Delta\tilde{\varepsilon}_i\right|=o_p(1)$$

by an identical argument to the proof of (47). ■

Proof of Lemma 4.1 (a). In view of Proposition A3, it is enough to show that $\frac{y_n}{\sqrt{n}}=O_p\left(n^{-\frac{1-\alpha}{2}}\right)$. Using (5) and the fact that $y_0=o_p\left(n^{\alpha/2}\right)$ we can write

$$\begin{aligned}\frac{y_n}{\sqrt{n}} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\rho_n^{n-i}u_i+o_p\left(n^{-\frac{1-\alpha}{2}}\right)\\ &= \frac{C(1)}{\sqrt{n}}\sum_{i=1}^{n}\rho_n^{n-i}\varepsilon_i-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\rho_n^{n-i}\Delta\tilde{\varepsilon}_i=O_p\left(n^{-\frac{1-\alpha}{2}}\right),\end{aligned}$$

since $\frac{1}{n^{\alpha/2}}\sum_{i=1}^{n}\rho_n^{n-i}\varepsilon_i=O_p(1)$ as an L_2-bounded martingale and $\frac{1}{n^{\alpha/2}}\sum_{i=1}^{n}\rho_n^{n-i}\Delta\tilde{\varepsilon}_i=o_p(1)$ by (47). ■

Proof of Lemma 4.1 (b). Since $\lambda = \sigma^2 \sum_{j=0}^{\infty} c_j \tilde{c}_j$, we use the definition of $\tilde{\varepsilon}_t$ in (5) to write

$$\frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} (u_t \tilde{\varepsilon}_t - \lambda) = \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} \sum_{j=0}^{\infty} c_j \tilde{c}_j \left(\varepsilon_{t-j}^2 - \sigma^2\right) + \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} c_j \tilde{c}_i \varepsilon_{t-j} \varepsilon_{t-i} + \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} c_j \tilde{c}_i \varepsilon_{t-j} \varepsilon_{t-i}. \tag{48}$$

We consider each of the terms of (48) in turn. First, define the linear process $\zeta_t = \sum_{j=0}^{\infty} c_j \tilde{c}_j \left(\varepsilon_{t-j}^2 - \sigma^2\right)$. The quantity

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sum_{j=0}^{\infty} c_j \tilde{c}_j \left(\varepsilon_{t-j}^2 - \sigma^2\right) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \zeta_t$$

satisfies a central limit theorem for sample means of linear processes (Phillips and Solo, 1992, Theorem 3.4) since $E\left(\varepsilon_0^4\right) < \infty$ and

$$\begin{aligned} \sum_{j=0}^{\infty} j^2 c_j^2 \tilde{c}_j^2 &= \sum_{j=0}^{\infty} j^2 c_j^2 \left(\sum_{k=j+1}^{\infty} c_k\right)^2 \leq \sum_{j=0}^{\infty} j^2 c_j^2 \left(\sum_{k=j+1}^{\infty} |c_k|\right)^2 \\ &= \sum_{j=0}^{\infty} c_j^2 \left(\sum_{k=j+1}^{\infty} j\,|c_k|\right)^2 < \sum_{j=0}^{\infty} c_j^2 \left(\sum_{k=j+1}^{\infty} k\,|c_k|\right)^2 \\ &\leq C_3 C_6^2 < \infty. \end{aligned}$$

Thus, the first term of (48) has order $O_p\left(n^{-\alpha/2}\right)$.

For the second term of (48), we have

$$\frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} c_j \tilde{c}_i \varepsilon_{t-j} \varepsilon_{t-i} = \frac{1}{n^{\frac{1+\alpha}{2}}} \sum_{t=1}^{n} \xi_t,$$

where $\xi_t := \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} c_j \tilde{c}_i \varepsilon_{t-j} \varepsilon_{t-i}$ is a stationary process with autocovariance function

$$\gamma_\xi(h) = \sigma^4 \sum_{j=h}^{\infty} \sum_{i=j+1}^{\infty} c_{j-h} \tilde{c}_{i-h} c_j \tilde{c}_i \quad h \in \mathbb{Z}.$$

Thus, by Theorem 18.2.1 of Ibragimov and Linnik (1971)

$$E\left[\left(\sum_{t=1}^{n} \xi_t\right)^2\right] \leq n \sum_{h=-\infty}^{\infty} \left|\gamma_\xi(h)\right|,$$

so

$$E\left[\left(\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\xi_t\right)^2\right] \leq \frac{1}{n^\alpha}\sum_{h=-\infty}^{\infty}\left|\gamma_\xi(h)\right| < \frac{2}{n^\alpha}\sum_{h=0}^{\infty}\left|\gamma_\xi(h)\right| = O\left(\frac{1}{n^\alpha}\right) \tag{49}$$

provided that $\sum_{h=0}^{\infty}\left|\gamma_\xi(h)\right| < \infty$. To show summability of γ_ξ, write

$$\begin{aligned}\sum_{h=0}^{\infty}\left|\gamma_\xi(h)\right| &\leq \sigma^4\sum_{h=0}^{\infty}\sum_{j=h}^{\infty}\left|c_{j-h}c_j\right|\left|\sum_{i=j+1}^{\infty}\tilde{c}_{i-h}\tilde{c}_i\right| \\ &\leq \sigma^4\sum_{h=0}^{\infty}\sum_{j=h}^{\infty}\left|c_{j-h}c_j\right|\left(\sum_{i=j+1}^{\infty}\tilde{c}_{i-h}^2\right)^{1/2}\left(\sum_{i=j+1}^{\infty}\tilde{c}_i^2\right)^{1/2} \\ &\leq \sigma^4\left(\sum_{i=0}^{\infty}\tilde{c}_i^2\right)\sum_{h=0}^{\infty}\sum_{j=h}^{\infty}\left|c_{j-h}\right|\left|c_j\right| \\ &= \sigma^4 C_4\sum_{j=0}^{\infty}\left|c_j\right|\sum_{h=j}^{\infty}\left|c_h\right| \leq \sigma^4 C_4 C_1^2 < \infty.\end{aligned}$$

Hence, (49) holds and the second term of (48) converges to 0 in L_2.

Finally, the third term of (48) can be written as

$$\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty}c_j\tilde{c}_i\varepsilon_{t-j}\varepsilon_{t-i} = \frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\eta_t,$$

where $\eta_t := \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty}c_j\tilde{c}_i\varepsilon_{t-j}\varepsilon_{t-i}$ is a stationary process with autocovariance function

$$\gamma_\eta(h) = \sigma^4\sum_{i=h}^{\infty}\sum_{j=i+1}^{\infty}c_{j-h}\tilde{c}_{i-h}c_j\tilde{c}_i \quad h \in \mathbb{Z}.$$

A similar calculation to that used to establish the summability of γ_ξ yields

$$\begin{aligned}\sum_{h=0}^{\infty}\left|\gamma_\eta(h)\right| &\leq \sigma^4\sum_{h=0}^{\infty}\sum_{i=h}^{\infty}\left|\tilde{c}_{i-h}\tilde{c}_i\right|\sum_{j=i+1}^{\infty}\left|c_{j-h}\right|\left|c_j\right| \\ &\leq \sigma^4\sum_{h=0}^{\infty}\sum_{i=h}^{\infty}\left|\tilde{c}_{i-h}\tilde{c}_i\right|\left(\sum_{j=i+1}^{\infty}c_{j-h}^2\right)^{1/2}\left(\sum_{j=i+1}^{\infty}c_j^2\right)^{1/2} \\ &\leq \sigma^4 C_3\sum_{h=0}^{\infty}\sum_{i=h}^{\infty}\left|\tilde{c}_{i-h}\right|\left|\tilde{c}_i\right| \leq \sigma^4 C_3 C_2^2 < \infty,\end{aligned}$$

which implies that $E\left(n^{-\frac{1+\alpha}{2}}\sum_{t=1}^{n}\eta_t\right)^2 = O\left(n^{-\alpha}\right)$ by (49). Thus,

$$n^{-\frac{1+\alpha}{2}}\sum_{t=1}^{n}\left(u_t\tilde{\varepsilon}_t-\lambda\right)=o_p\left(1\right)$$

and Lemma 4.1 (b) holds. ■

Proof of Lemma 4.2. In view of Theorem 2.1 (c), it is enough to show that

$$\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}y_{t-1}\varepsilon_t = C\left(1\right)\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}x_{t-1}\varepsilon_t+o_p\left(1\right) \tag{50}$$

where $x_t=\sum_{i=1}^{t}\rho_n^{t-i}\varepsilon_i$. Using the BN decomposition (5) we can write

$$y_t=\sum_{i=1}^{t}\rho_n^{t-i}u_i+\rho_n^t y_0 = C\left(1\right)x_t-\sum_{i=1}^{t}\rho_n^{t-i}\Delta\tilde{\varepsilon}_i+\rho_n^t y_0.$$

Summation by parts gives

$$\begin{aligned}\sum_{i=1}^{t}\rho_n^{t-i}\Delta\tilde{\varepsilon}_i &= \tilde{\varepsilon}_t-\sum_{i=1}^{t}\left(\Delta\rho_n^{t-i}\right)\tilde{\varepsilon}_{i-1}=\tilde{\varepsilon}_t-\left(1-\rho_n\right)\sum_{i=1}^{t}\rho_n^{t-i}\tilde{\varepsilon}_{i-1}\\ &= \tilde{\varepsilon}_t+\frac{c}{n^{\alpha}}\sum_{i=1}^{t}\rho_n^{t-i}\tilde{\varepsilon}_{i-1}.\end{aligned}$$

Since

$$E\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\rho_n^{t-1}\varepsilon_t\right)^2=\frac{\sigma^2}{n}\sum_{t=1}^{n}\rho_n^{2t-2}=\frac{\sigma^2}{n}\frac{\rho_n^{2n}-1}{\rho_n^2-1}=O\left(\frac{1}{n^{1-\alpha}}\right)$$

we can write

$$\begin{aligned}&\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\left[y_{t-1}-C\left(1\right)x_{t-1}\right]\varepsilon_t\\ =\ & -\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\sum_{i=1}^{t-1}\rho_n^{t-i-1}\Delta\tilde{\varepsilon}_i\varepsilon_t+\frac{y_0}{n^{\alpha/2}}\frac{1}{n^{1/2}}\sum_{t=1}^{n}\rho_n^{t-1}\varepsilon_t\\ =\ & -\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\left(\tilde{\varepsilon}_{t-1}+\frac{c}{n^{\alpha}}\sum_{i=1}^{t-1}\rho_n^{t-i-1}\tilde{\varepsilon}_{i-1}\right)\varepsilon_t+o_p\left(\frac{1}{n^{\frac{1-\alpha}{2}}}\right)\\ =\ & -\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\tilde{\varepsilon}_{t-1}\varepsilon_t-\frac{c}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\left(\sum_{i=1}^{t-1}\rho_n^{t-i-1}\tilde{\varepsilon}_i\right)\varepsilon_t+o_p\left(\frac{1}{n^{\frac{1-\alpha}{2}}}\right).\end{aligned}$$

Both terms in the above expression are martingales. For the first term, we have

$$E\left(\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\tilde{\varepsilon}_{t-1}\varepsilon_t\right)^2=\frac{\sigma^2}{n^{1+\alpha}}\sum_{t=1}^{n}E\tilde{\varepsilon}_{t-1}^2=\frac{\sigma^2E\tilde{\varepsilon}_0^2}{n^{\alpha}}=O\left(n^{-\alpha}\right).$$

The second term also converges to 0 in L_2 since, by Minkowski's inequality, we have

$$\begin{aligned}E\left[\frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\left(\sum_{i=1}^{t-1}\rho_n^{t-i-1}\tilde{\varepsilon}_i\right)\varepsilon_t\right]^2 &= \frac{\sigma^2}{n^{1+3\alpha}}\sum_{t=1}^{n}E\left(\sum_{i=1}^{t-1}\rho_n^{t-i-1}\tilde{\varepsilon}_i\right)^2\\ &\leq \frac{\sigma^2}{n^{1+3\alpha}}\sum_{t=1}^{n}\left[\sum_{i=1}^{t-1}\left\{E\left(\rho_n^{2(t-i-1)}\tilde{\varepsilon}_i^2\right)\right\}^{1/2}\right]^2\\ &= \frac{\sigma^2E\tilde{\varepsilon}_0^2}{n^{1+3\alpha}}\sum_{t=1}^{n}\left(\sum_{i=1}^{t-1}\left|\rho_n^{t-i-1}\right|\right)^2\\ &= \frac{\sigma^2E\tilde{\varepsilon}_0^2}{n^{1+3\alpha}\left(1-\rho_n\right)^2}\left[n+O\left(n^{\alpha}\right)\right]=O\left(n^{-\alpha}\right).\end{aligned}$$

This shows (50) and the lemma follows. ■

Proof of Lemma 4.3 (a). Using the Cauchy-Schwarz inequality we can write

$$\begin{aligned}\left|\frac{1}{n^{1+\frac{\alpha}{2}}}\sum_{t=1}^{n}y_{t-1}\tilde{\varepsilon}_t\right| &\leq \frac{1}{n^{1+\frac{\alpha}{2}}}\sum_{t=1}^{n}\left|y_{t-1}\tilde{\varepsilon}_t\right|\leq\left(\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n}y_{t-1}^2\right)^{1/2}\left(\frac{1}{n}\sum_{t=1}^{n}\tilde{\varepsilon}_t^2\right)^{1/2}\\ &= \left(\frac{\omega^2}{-2c}E\tilde{\varepsilon}_0^2\right)^{1/2}+o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)\end{aligned}$$

by the ergodic theorem applied to $\tilde{\varepsilon}_t^2$ and by (13). ■

Proof of Lemma 4.3 (b). For $\alpha\in\left(\frac{1}{2},1\right)$ the result follows immediately from part (a). It is therefore enough to show the result for $\alpha\in\left(0,\frac{1}{2}\right]$.

First note that, if δ is the positive constant in **LP**,

$$\begin{aligned}\sum_{j=1}^{\infty}j^{2-3\alpha+\frac{\delta}{2}}\tilde{c}_j^2 &\leq \sum_{j=1}^{\infty}j^{-1-\frac{\delta}{2}}\left(\sum_{k=j+1}^{\infty}j^{\frac{3-3\alpha+\delta}{2}}\left|c_k\right|\right)^2\\ &\leq \sum_{j=1}^{\infty}j^{-1-\frac{\delta}{2}}\left(\sum_{k=j+1}^{\infty}k^{\frac{3-3\alpha+\delta}{2}}\left|c_k\right|\right)^2\\ &\leq C_{\alpha\delta}^2\sum_{j=1}^{\infty}j^{-1-\frac{\delta}{2}}<\infty. \qquad (51)\end{aligned}$$

We can use Proposition 3.7.5 (a) to write for each $\alpha \in \left(0, \frac{1}{2}\right]$

$$\frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \left(y_{t-1}\tilde{\varepsilon}_t - m_n\right) = \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \left(y^*_{nt-1}\tilde{\varepsilon}_t - m_n\right) + o_p\left(1\right).$$

From the definitions of y^*_{nt} and m_n we obtain

$$\begin{aligned} \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \left(y^*_{nt-1}\tilde{\varepsilon}_t - m_n\right) &= \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \left\{ \sum_{i=0}^{n} \rho_n^i u_{t-i-1}\tilde{\varepsilon}_t - \sum_{i=0}^{\infty} \rho_n^i \gamma_m\left(i+1\right) \right\} \\ &= \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=0}^{n} \rho_n^i \left\{u_{t-i-1}\tilde{\varepsilon}_t - \gamma_m\left(i+1\right)\right\} + o\left(1\right) \end{aligned}$$

as $n \to \infty$, because

$$\left| \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=n+1}^{\infty} \rho_n^i \gamma_m\left(i+1\right) \right| \leq n^{\frac{1-3\alpha}{2}} \sum_{i=n+1}^{\infty} \left|\rho_n^i \gamma_m\left(i+1\right)\right| = o\left(1\right)$$

by summability of $\gamma_m\left(\cdot\right)$ when $\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right]$ and Assumption **LP** when $\alpha \in \left(0, \frac{1}{3}\right)$ since

$$\begin{aligned} n^{\frac{1-3\alpha}{2}} \sum_{i=n+1}^{\infty} \left|\rho_n^i \gamma_m\left(i+1\right)\right| &\leq \sum_{i=n+1}^{\infty} i^{\frac{1-3\alpha}{2}} \left|\gamma_m\left(i+1\right)\right| \leq \sigma^2 \sum_{i=n+1}^{\infty} i^{\frac{1-3\alpha}{2}} \sum_{j=0}^{\infty} \left|c_j\right| \left|\tilde{c}_{j+i+1}\right| \\ &\leq \sigma^2 \sum_{i=n+1}^{\infty} i^{\frac{1-3\alpha}{2}} \sum_{j=0}^{\infty} \left|c_j\right| \sum_{k=j+i+2}^{\infty} \left|c_k\right| \\ &\leq \sigma^2 C_1 \sum_{i=n+1}^{\infty} i^{\frac{1-3\alpha}{2}} \sum_{k=i+2}^{\infty} \left|c_k\right| \\ &\leq \sigma^2 C_1 \sum_{i=n+1}^{\infty} i^{-1-\frac{\delta}{2}} \sum_{k=n+3}^{\infty} k^{\frac{3-3\alpha+\delta}{2}} \left|c_k\right| = o\left(1\right). \end{aligned}$$

Thus, by using the definition of $\tilde{\varepsilon}_t$ we obtain, up to $o_p\left(1\right)$,

$$\begin{aligned} \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \left(y_{t-1}\tilde{\varepsilon}_t - m_n\right) &= \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=0}^{n} \rho_n^i \sum_{j=0}^{\infty} c_j \tilde{c}_{j+i+1} \left(\varepsilon^2_{t-1-i-j} - \sigma^2\right) \\ &\quad + \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=0}^{n} \rho_n^i \sum_{j=0}^{\infty} c_j \sum_{k=j+i+2}^{\infty} \tilde{c}_k \varepsilon_{t-1-i-j} \varepsilon_{t-k} \\ &\quad + \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=0}^{n} \rho_n^i \sum_{j=0}^{\infty} c_j \sum_{k=0}^{j+i} \tilde{c}_k \varepsilon_{t-1-i-j} \varepsilon_{t-k}. \qquad (52) \end{aligned}$$

Denote the three terms on the right side of (52) by S_{n1}, S_{n2}, S_{n3} according to the order of appearance. The following result is useful in the discussion of S_{n1}.

$$\frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\sum_{i=0}^{n}\rho_n^i\sum_{k=n+1}^{\infty}c_{k-i-1}\tilde{c}_k\left(\varepsilon_{t-k}^2-\sigma^2\right)\longrightarrow_{L_1}0\quad\text{as } n\to\infty. \tag{53}$$

To establish (53), note that

$$E\left|\frac{1}{n^{\frac{1+3\alpha}{2}}}\sum_{t=1}^{n}\sum_{i=0}^{n}\rho_n^i\sum_{k=n+1}^{\infty}c_{k-i-1}\tilde{c}_k\left(\varepsilon_{t-k}^2-\sigma^2\right)\right|\leq 2\sigma^2 n^{\frac{1-3\alpha}{2}}\sum_{i=0}^{n}|\rho_n|^i\sum_{k=n+1}^{\infty}|c_{k-i-1}|\,|\tilde{c}_k|$$

and also

$$\begin{aligned}
n^{\frac{1-3\alpha}{2}}\sum_{i=0}^{n}|\rho_n|^i\sum_{k=n+1}^{\infty}|c_{k-i-1}|\,|\tilde{c}_k| &= \frac{1}{n^{\alpha}}\sum_{i=0}^{n}|\rho_n|^i\sum_{k=n+1}^{\infty}n^{\frac{1-\alpha}{2}}|c_{k-i-1}|\,|\tilde{c}_k| \\
&\leq \frac{1}{n^{\alpha}}\sum_{i=0}^{n}|\rho_n|^i\sum_{k=n+1}^{\infty}|c_{k-i-1}|\,k^{\frac{1-\alpha}{2}}|\tilde{c}_k| \\
&\leq \frac{1}{n^{\alpha}}\sum_{i=0}^{n}|\rho_n|^i\left(\sum_{k=n+1}^{\infty}c_{k-i-1}^2\right)^{1/2}\left(\sum_{k=n+1}^{\infty}k^{1-\alpha}\tilde{c}_k^2\right)^{1/2} \\
&\leq KC_3^{1/2}\left(\sum_{k=n+1}^{\infty}k^{1-\alpha}\tilde{c}_k^2\right)^{1/2} \\
&\leq KC_3^{1/2}\left\{\sum_{k=n+1}^{\infty}k^{1-\alpha}\left(\sum_{j=k+1}^{\infty}|c_j|\right)^2\right\}^{1/2} \\
&= KC_3^{1/2}\left\{\sum_{k=n+1}^{\infty}k^{-1-\alpha}\left(\sum_{j=k+1}^{\infty}k\,|c_j|\right)^2\right\}^{1/2} \\
&\leq KC_3^{1/2}\left(\sum_{j=n+2}^{\infty}j\,|c_j|\right)\left(\sum_{k=n+1}^{\infty}k^{-1-\alpha}\right)^{1/2}=o(1),
\end{aligned}$$

since $n^{-\alpha}\sum_{i=0}^{n}|\rho_n|^i$ is bounded by a finite constant K and $\sum_{j=1}^{\infty}j\,|c_j|$ and $\sum_{k=1}^{\infty}k^{-1-\alpha}$ are convergent series.

Using (53), the first term on the right side of (52) can be written as

$$\begin{aligned} S_{n1} &= \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=0}^{n} \rho_n^i \sum_{k=i+1}^{\infty} c_{k-i-1} \tilde{c}_k \left(\varepsilon_{t-k}^2 - \sigma^2 \right) \\ &= \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{i=0}^{n} \rho_n^i \sum_{k=i+1}^{n} c_{k-i-1} \tilde{c}_k \left(\varepsilon_{t-k}^2 - \sigma^2 \right) + o_p(1) \\ &= \frac{1}{n^{\frac{1+3\alpha}{2}}} \sum_{t=1}^{n} \sum_{j=1}^{n} \sum_{i=0}^{j-1} \rho_n^{j-1-i} c_i \tilde{c}_j \left(\varepsilon_{t-j}^2 - \sigma^2 \right) \\ &= \frac{1}{n^{\frac{1}{2}+\frac{\delta}{4}}} \sum_{t=1}^{n} \sum_{j=1}^{n} \widehat{c}_j \left(\varepsilon_{t-j}^2 - \sigma^2 \right), \end{aligned}$$

where $\widehat{c}_j := n^{-\frac{3\alpha}{2}+\frac{\delta}{4}} \tilde{c}_j \sum_{i=0}^{j-1} \rho_n^{j-1-i} c_i$ and δ is the positive constant of **LP**. Now $\zeta_{nt} := \sum_{j=1}^{n} \widehat{c}_j \left(\varepsilon_{t-j}^2 - \sigma^2 \right)$ is a linear process satisfying a central limit theorem for sample means (Phillips and Solo, 1992, Theorem 3.4) provided that $E\varepsilon_0^4 < \infty$ and $\sum_{j=1}^{n} j^2 \widehat{c}_j^2 < \infty$. The latter holds by (51) since, for each $\delta \in (0, 3\alpha)$, we obtain

$$\begin{aligned} \sum_{j=1}^{n} j^2 \widehat{c}_j^2 &\leq n^{-3\alpha+\frac{\delta}{2}} \sum_{j=1}^{n} j^2 \tilde{c}_j^2 \left(\sum_{i=0}^{j-1} |\rho_n|^{j-1-i} |c_i| \right)^2 \\ &\leq \sum_{j=1}^{n} j^2 j^{-3\alpha+\frac{\delta}{2}} \tilde{c}_j^2 \left(\sum_{i=0}^{j-1} |c_i| \right)^2 \\ &\leq C_1^2 \sum_{j=1}^{\infty} j^{2-3\alpha+\frac{\delta}{2}} \tilde{c}_j^2 \leq C_1^2 C_{\alpha\delta}^2 \sum_{j=1}^{\infty} j^{-1-\frac{\delta}{2}} < \infty. \end{aligned}$$

This shows that the first term in (52) has order $O_p\left(n^{-\delta/4}\right)$, $\delta \in (0, 3\alpha)$.

The second term in (52) can be written as $S_{n2} = n^{-\frac{1+3\alpha}{2}} \sum_{t=1}^{n} \xi_{nt}$, where

$$\xi_{nt} := \sum_{i=0}^{n} \rho_n^i \sum_{j=0}^{\infty} c_j \sum_{k=j+i+2}^{\infty} \tilde{c}_k \varepsilon_{t-1-i-j} \varepsilon_{t-k},$$

and, for each $h \geq 0$,

$$\gamma_\xi(h, n) := E\xi_{nt}\xi_{nt-h} = \sigma^4 \sum_{i,l=0}^{n} \rho_n^{i+l} \sum_{j=(l+h-i)\vee 0}^{\infty} c_j c_{i+j-l-h} \sum_{k=i+j+2}^{\infty} \tilde{c}_k \tilde{c}_{k-h}.$$

Since $\gamma_\xi(h, n)$ does not depend on t, we have that

$$E\left(\sum_{t=1}^{n} \xi_{nt} \right)^2 \leq 2n \sum_{h=0}^{\infty} \left| \gamma_\xi(h, n) \right| \quad \text{or} \quad E\left(S_{n2}^2 \right) \leq \frac{2}{n^{3\alpha}} \sum_{h=0}^{\infty} \left| \gamma_\xi(h, n) \right|. \tag{54}$$

By using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\sum_{h=0}^{\infty}\left|\gamma_{\xi}(h,n)\right| &\leq \sigma^4 \sum_{i,l=0}^{n} |\rho_n|^{i+l} \sum_{h=0}^{\infty} \sum_{j=(l+h-i)\vee 0}^{\infty} |c_j| \, |c_{i+j-l-h}| \left| \sum_{k=i+j+2}^{\infty} \tilde{c}_k \tilde{c}_{k-h} \right| \\
&\leq \sigma^4 \sum_{i,l=0}^{n} |\rho_n|^{i+l} \sum_{h=0}^{\infty} \sum_{j=(l+h-i)\vee 0}^{\infty} |c_j| \, |c_{i+j-l-h}| \times \\
&\quad \left(\sum_{k=i+j+2}^{\infty} \tilde{c}_k^2 \right)^{1/2} \left(\sum_{k=i+j+2}^{\infty} \tilde{c}_{k-h}^2 \right)^{1/2} \\
&\leq \sigma^4 C_4 \sum_{i,l=0}^{n} |\rho_n|^{i+l} \sum_{h=0}^{\infty} \sum_{j=(l+h-i)\vee 0}^{\infty} |c_j| \, |c_{i+j-l-h}| .
\end{aligned}$$

We will show that

$$\varphi_n := \sum_{i,l=0}^{n} |\rho_n|^{i+l} \sum_{h=0}^{\infty} \sum_{j=(l+h-i)\vee 0}^{\infty} |c_j| \, |c_{i+j-l-h}| = O\left(n^{2\alpha}\right) \tag{55}$$

separately for $l+h \geq i$ and $l+h < i$.

When $l+h \geq i$,

$$\begin{aligned}
\varphi_n &= \sum_{l=0}^{n} \sum_{h=0}^{\infty} \sum_{i=0}^{l+h} |\rho_n|^{i+l} \sum_{j=l+h-i}^{\infty} |c_j| \, |c_{i+j-l-h}| \\
&\leq \sum_{i=0}^{\infty} \sum_{l=0}^{n} |\rho_n|^{i+l} \sum_{h=0}^{\infty} \mathbf{1}\{i \leq l+h\} \sum_{j=l+h-i}^{\infty} |c_j| \, |c_{i+j-l-h}| \\
&\leq \sum_{i=0}^{\infty} \sum_{l=0}^{n} |\rho_n|^{i+l} \sum_{h=i-l}^{\infty} \sum_{j=l+h-i}^{\infty} |c_j| \, |c_{i+j-l-h}| \\
&= \sum_{i=0}^{\infty} \sum_{l=0}^{n} |\rho_n|^{i+l} \sum_{m=0}^{\infty} |c_m| \sum_{k=m}^{\infty} |c_k| \\
&\leq C_1^2 \sum_{i=0}^{\infty} \sum_{l=0}^{n} |\rho_n|^{i+l} = O\left(n^{2\alpha}\right).
\end{aligned}$$

When $l+h<i$, we have

$$\begin{aligned}
\varphi_n &= \sum_{l=0}^{n}\sum_{h=0}^{\infty}\sum_{i=l+h+1}^{\infty}\left|\rho_n\right|^{i+l}\sum_{j=0}^{\infty}\left|c_j\right|\left|c_{i+j-l-h}\right| \\
&\leq \sum_{l=0}^{n}\sum_{i=0}^{\infty}\left|\rho_n\right|^{i+l}\sum_{h=0}^{\infty}\mathbf{1}\left\{i>l+h\right\}\sum_{j=0}^{\infty}\left|c_j\right|\left|c_{i+j-l-h}\right| \\
&\leq \sum_{l=0}^{n}\sum_{i=0}^{\infty}\left|\rho_n\right|^{i+l}\sum_{j=0}^{\infty}\left|c_j\right|\sum_{h=0}^{i-l-1}\left|c_{i+j-l-h}\right| \\
&= \sum_{l=0}^{n}\sum_{i=0}^{\infty}\left|\rho_n\right|^{i+l}\sum_{j=0}^{\infty}\left|c_j\right|\sum_{h=j+1}^{j+i-l}\left|c_h\right| \\
&\leq C_1^2\sum_{l=0}^{n}\sum_{i=0}^{\infty}\left|\rho_n\right|^{i+l}=O\left(n^{2\alpha}\right).
\end{aligned}$$

Thus, by (54), S_{n2} converges to 0 in L_2, since

$$E\left(S_{n2}^2\right)\leq\frac{2}{n^{3\alpha}}\sum_{h=0}^{\infty}\left|\gamma_\xi\left(h,n\right)\right|=\frac{2}{n^{3\alpha}}O\left(n^{2\alpha}\right)=O\left(\frac{1}{n^\alpha}\right).$$

Finally, the third term in (52) can be written as $S_{n3}=n^{-\frac{1+3\alpha}{2}}\sum_{t=1}^{n}\eta_{nt}$, where

$$\eta_{nt}:=\sum_{i=0}^{n}\rho_n^i\sum_{j=0}^{\infty}c_j\sum_{k=0}^{j+i}\tilde{c}_k\varepsilon_{t-1-i-j}\varepsilon_{t-k},$$

and for each $h\geq 0$,

$$\gamma_\eta\left(h,n\right):=E\eta_{nt}\eta_{nt-h}=\sigma^4\sum_{i,l=0}^{n}\rho_n^{i+l}\sum_{j=(l+h-i)\vee 0}^{\infty}c_jc_{i+j-l-h}\sum_{k=h}^{j+i}\tilde{c}_k\tilde{c}_{k-h}.$$

Thus, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\sum_{h=0}^{\infty}\left|\gamma_\eta\left(h,n\right)\right| &\leq \sigma^4\sum_{i,l=0}^{n}\left|\rho_n\right|^{i+l}\sum_{h=0}^{\infty}\sum_{j=(l+h-i)\vee 0}^{\infty}\left|c_j\right|\left|c_{i+j-l-h}\right|\left|\sum_{k=h}^{j+i}\tilde{c}_k\tilde{c}_{k-h}\right| \\
&\leq \sigma^4C_1\sum_{i,l=0}^{n}\left|\rho_n\right|^{i+l}\sum_{h=0}^{\infty}\sum_{j=(l+h-i)\vee 0}^{\infty}\left|c_j\right|\left|c_{i+j-l-h}\right|=\sigma^4C_1\varphi_n=O\left(n^{2\alpha}\right),
\end{aligned}$$

from (55). This shows that $ES_{n3}^2=O\left(n^{-\alpha}\right)$ and completes the proof. ∎

Proof of Theorem 4.4. Part (a) is given by (19). The moment condition $E\varepsilon_0^4 < \infty$ is essential for all $\alpha \in (0,1)$ as a consequence of using Lemma 4.1 (b).

For part (b), by squaring (4) we obtain

$$\begin{aligned}\left(1-\rho_n^2\right)\sum_{t=1}^{n} y_{t-1}^2 &= y_0^2 - y_n^2 + 2\rho_n \sum_{t=1}^{n} y_{t-1}u_t + \sum_{t=1}^{n} u_t^2 \\ &= 2\rho_n \sum_{t=1}^{n} y_{t-1}u_t + \sum_{t=1}^{n} u_t^2 + O_p\left(n^{\alpha}\right),\end{aligned}$$

since $y_n = O_p\left(n^{\alpha/2}\right)$ by (10). Writing $1-\rho_n^2 = \frac{-2c}{n^{\alpha}}\left(1+\frac{c}{2n^{\alpha}}\right)$ we obtain

$$\begin{aligned}\sum_{t=1}^{n} \frac{y_{t-1}^2}{n^{\alpha}} &= \frac{1}{-2c\left(1+\frac{c}{2n^{\alpha}}\right)}\left\{2\rho_n \sum_{t=1}^{n} y_{t-1}u_t + \sum_{t=1}^{n} u_t^2\right\} + O_p\left(n^{\alpha}\right) \\ &= \frac{1}{-2c\left(1+\frac{c}{2n^{\alpha}}\right)}\left\{2\rho_n \sum_{t=1}^{n}\left(y_{t-1}u_t - \lambda - \frac{c}{n^{\alpha}}m_n\right) + \sum_{t=1}^{n}\left(u_t^2 - \sigma_u^2\right)\right\} \\ &\quad + \frac{n}{-2c}\frac{2\rho_n\left(\lambda + \frac{c}{n^{\alpha}}m_n\right) + \sigma_u^2}{1+\frac{c}{2n^{\alpha}}} + O_p\left(n^{\alpha}\right) \\ &= \frac{1+o(1)}{-2c}\left\{2\rho_n \sum_{t=1}^{n}\left(y_{t-1}u_t - \lambda - \frac{c}{n^{\alpha}}m_n\right) + O_p\left(n^{1/2}\right)\right\} \\ &\quad + \frac{n\omega_n^2}{-2c} + O_p\left(n^{\alpha}\right),\end{aligned} \tag{56}$$

since, under the assumption $E\varepsilon_0^4 < \infty$, $n^{-1/2}\sum_{t=1}^{n}\left(u_t^2 - \sigma_u^2\right)$ satisfies a CLT for sample variances (Theorem 3.8 of Phillips and Solo, 1992). Hence, as $n \to \infty$

$$\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n}\left(\frac{y_{t-1}^2}{n^{\alpha}} - \frac{\omega_n^2}{-2c}\right) = \frac{1+o(1)}{-c}\sum_{t=1}^{n}\left(y_{t-1}u_t - \lambda - \frac{c}{n^{\alpha}}m_n\right) + O_p\left(n^{-\frac{\alpha\wedge(1-\alpha)}{2}}\right). \quad \blacksquare$$

Proof of Lemma 5.1. For part (a) we can write, using Proposition A1 and the BN decomposition (5),

$$\begin{aligned}\int_0^t \rho_n^{-n^{\alpha}s} dB_{n^{\alpha}}(s) &= \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\rho_n^{-i}u_i = \frac{C(1)}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\rho_n^{-i}\varepsilon_i - \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\rho_n^{-i}\Delta\tilde{\varepsilon}_i \\ &= C(1)\int_0^t \rho_n^{-n^{\alpha}s} dW_{n^{\alpha}}(s) - \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\rho_n^{-i}\Delta\tilde{\varepsilon}_i.\end{aligned}$$

By Lemma 4.1 of PM

$$\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-n^{\alpha}s} dW_{n^{\alpha}}(s) - \int_0^t e^{-cs} dW(s)\right| = o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right),$$

on the probability space that (6) holds, which is the same space that (7) holds with $B(t) = C(1) W(t)$ (see the proof of Lemma 3.1). Therefore, it is enough to show that

$$\frac{1}{n^{\alpha/2}} \max_{0\le k\le n} \left| \sum_{i=1}^{k} \rho_n^{-i} \Delta\tilde{\varepsilon}_i \right| = o_p(1). \tag{57}$$

Summation by parts gives

$$\sum_{i=1}^{k} \rho_n^{-i} \Delta\tilde{\varepsilon}_i = \rho_n^{-k}\tilde{\varepsilon}_k - \sum_{i=1}^{k} \left(\Delta\rho_n^{-i}\right) \tilde{\varepsilon}_{i-1} = \rho_n^{-k}\tilde{\varepsilon}_k + \frac{c}{n^{\alpha}} \sum_{i=1}^{k} \rho_n^{-i}\tilde{\varepsilon}_{i-1}$$

so that

$$\begin{aligned} \frac{1}{n^{\alpha/2}} \max_{0\le k\le n} \left| \sum_{i=1}^{k} \rho_n^{-i} \Delta\tilde{\varepsilon}_i \right| &\le \max_{0\le k\le n} \left| \frac{\tilde{\varepsilon}_k}{n^{\alpha/2}} \right| \left\{ \max_{0\le k\le n} \left|\rho_n^{-k}\right| + \frac{|c|}{n^{\alpha}} \sum_{i=1}^{n} \left|\rho_n^{-i}\right| \right\} \\ &\le 2 \max_{0\le k\le n} \left| \frac{\tilde{\varepsilon}_k}{n^{\alpha/2}} \right| = o_p(1), \end{aligned}$$

by Proposition A3. For part (b) a similar argument gives

$$\int_0^t \rho_n^{-(\lfloor n^{\alpha}t\rfloor - \lfloor n^{\alpha}s\rfloor)} dB_{n^{\alpha}}(s) = C(1) \int_0^t \rho_n^{-(\lfloor n^{\alpha}t\rfloor - \lfloor n^{\alpha}s\rfloor)} dW_{n^{\alpha}}(s) - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor tn^{\alpha}\rfloor} \rho_n^{-(\lfloor n^{\alpha}t\rfloor - i)} \Delta\tilde{\varepsilon}_i.$$

By Lemma 4.2 of PM

$$\sup_{t\in[0,n^{1-\alpha}]} \left| \int_0^t \rho_n^{-(\lfloor n^{\alpha}t\rfloor - \lfloor n^{\alpha}s\rfloor)} dW_{n^{\alpha}}(s) - \int_0^t e^{-c(t-s)} dW(s) \right| = o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}} \right),$$

again on the probability space that (6) and (7) with $B(t) = C(1) W(t)$ hold. Summation by parts again shows that

$$\frac{1}{n^{\alpha/2}} \max_{0\le k\le n} \left| \sum_{i=1}^{k} \rho_n^{-(k-i)} \Delta\tilde{\varepsilon}_i \right| \le 2 \max_{0\le k\le n} \left| \frac{\tilde{\varepsilon}_k}{n^{\alpha/2}} \right| = o_p(1),$$

and part (b) follows. ■

Proof of (36). Proposition A1 and Lemma 5.1 (a) give for each $p \in [0, q]$

$$\begin{aligned}
\frac{\rho_n^{-\kappa_n}}{n^{\alpha/2}} y_{\lfloor \kappa_n \rfloor + \lfloor n^\alpha p \rfloor} &= \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor \kappa_n + n^\alpha p \rfloor} \rho_n^{\lfloor n^\alpha p \rfloor - i} u_i + o_p(1) \\
&= \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor n^\alpha \left(\lfloor n^{1-\alpha} \rfloor + p \right) \rfloor} \rho_n^{\lfloor n^\alpha p \rfloor - i} u_i \\
&= \rho_n^{\lfloor n^\alpha p \rfloor} \int_0^{\lfloor n^{1-\alpha} \rfloor + p} \rho_n^{-n^\alpha s} dB_{n^a}(s) \\
&= e^{cp} \int_0^\infty e^{-cs} dB(s) + o_p\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{\nu}}} \right)
\end{aligned}$$

on the probability space that (7) holds. ■

Proof of asymptotic negligibility of R_n. Write $R_n = R_{1n} - 2R_{2n}$, where

$$\begin{aligned}
R_{1n} &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \left(\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right)^2 dr \\
R_{2n} &= \rho_n^{-2\kappa_n} \left(\int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right) \\
&\quad \times \int_0^{\lfloor n^{1-\alpha} \rfloor} \left(\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right) dr \\
&= \left(\int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right) \overline{R}_{2n},
\end{aligned}$$

where

$$\overline{R}_{2n} := \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \, dr.$$

In PM, it is shown that

$$\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dW_{n^\alpha}(s) = O_p(1), \quad \text{uniformly on } r \in \left[0, \lfloor n^{1-\alpha} \rfloor \right].$$

Using Proposition A1 and the BN decomposition we obtain

$$\begin{aligned}
&\int_r^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{-n^\alpha(s-r)} dB_{n^\alpha}(s) \\
&= \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor\kappa_n\rfloor-\lfloor n^\alpha r\rfloor} \rho_n^{-i} u_{\lfloor n^\alpha r\rfloor+i} \\
&= \frac{C(1)}{n^{\alpha/2}} \sum_{i=1}^{\lfloor\kappa_n\rfloor-\lfloor n^\alpha r\rfloor} \rho_n^{-i} \varepsilon_{\lfloor n^\alpha r\rfloor+i} - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor\kappa_n\rfloor-\lfloor n^\alpha r\rfloor} \rho_n^{-i} \Delta\tilde{\varepsilon}_{\lfloor n^\alpha r\rfloor+i} \\
&= C(1) \int_r^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{-n^\alpha(s-r)} dW_{n^\alpha}(s) - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor\kappa_n\rfloor-\lfloor n^\alpha r\rfloor} \rho_n^{-i} \Delta\tilde{\varepsilon}_{\lfloor n^\alpha r\rfloor+i},
\end{aligned}$$

where, from (57),

$$\frac{1}{n^{\alpha/2}} \sup_{r\in[0,\lfloor n^{1-\alpha}\rfloor]} \left| \sum_{i=1}^{\lfloor\kappa_n\rfloor-\lfloor n^\alpha r\rfloor} \rho_n^{-i} \Delta\tilde{\varepsilon}_{\lfloor n^\alpha r\rfloor+i} \right| \leq \frac{1}{n^{\alpha/2}} \max_{0\leq k\leq n} \left| \sum_{i=1}^{k} \rho_n^{-i} \Delta\tilde{\varepsilon}_i \right| = o_p(1).$$

Thus,

$$\int_r^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{-n^\alpha(s-r)} dB_{n^\alpha}(s) = O_p(1), \quad \text{uniformly on } r \in \left[0, \lfloor n^{1-\alpha}\rfloor\right]. \tag{58}$$

The uniform boundedness in (58) together with the fact that $\rho_n^{-\kappa_n} = o(n^{-1})$ give

$$\begin{aligned}
R_{1n} &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha}\rfloor} \left(\int_r^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{-n^\alpha(s-r)} dB_{n^\alpha}(s) \right)^2 dr \\
&= O_p(1) \times O\left(\rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha}\rfloor} dr \right) \\
&= O_p\left(\rho_n^{-2\kappa_n} \lfloor n^{1-\alpha}\rfloor \right) = o_p\left(\frac{1}{n^\alpha}\right).
\end{aligned}$$

For the second remainder term, we obtain from (58)

$$\begin{aligned}
\overline{R}_{2n} &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha}\rfloor} \int_r^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{-n^\alpha(s-r)} dB_{n^\alpha}(s)\, dr \\
&= O_p\left(\rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha}\rfloor} dr \right) = o_p\left(\frac{1}{n^\alpha}\right),
\end{aligned}$$

so that

$$R_{2n} = \left(\int_0^{\lfloor n^{1-\alpha}\rfloor} \rho_n^{-n^\alpha(s-r)} dB_{n^\alpha}(s) \right) \overline{R}_{2n} = o_p\left(\frac{1}{n^\alpha}\right),$$

by part (b) of Lemma 5.1. Thus, $R_n = R_{1n} - 2R_{2n} = o_p(1)$ follows. ■

Proof of asymptotic negligibility of I_n. Using Proposition A1, I_n can be written as

$$\begin{aligned} I_n &= \rho_n^{-n-1} \int_{\frac{1}{n^a}}^{n^{1-\alpha}} \int_{r-\frac{1}{n^a}}^{n^{1-\alpha}} \rho_n^{\lfloor n^a r \rfloor - n^a s} dB_{n^a}(s)\, dB_{n^a}(r) \\ &= \rho_n^{-n-1} \int_{\frac{1}{n^a}}^{n^{1-\alpha}} \frac{1}{n^{\alpha/2}} \sum_{i=\lfloor n^\alpha r \rfloor}^{n} \rho_n^{\lfloor n^a r \rfloor - i} u_i dB_{n^a}(r) \\ &= \rho_n^{-n-1} \frac{1}{n^\alpha} \sum_{j=2}^{n} \sum_{i=j}^{n} \rho_n^{j-i} u_i u_j. \end{aligned}$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned} E\left|I_n\right| &\leq \rho_n^{-n-1} \frac{1}{n^\alpha} \sum_{j=2}^{n} \sum_{i=j}^{n} \rho_n^{j-i} E\left|u_i u_j\right| \\ &\leq \rho_n^{-n-1} \frac{\sigma_u^2}{n^\alpha} \sum_{j=2}^{n} \sum_{i=j}^{n} \rho_n^{j-i} \\ &= O\left(\rho_n^{-n} n\right) = o(1), \end{aligned}$$

since $\rho_n^{-n} = o\left(n^{-1}\right)$ and $\sigma_u^2 < \infty$. Thus, $I_n \to 0$ in L_1. ■

Proof of Theorem 5.2. This follows precisely as in Theorem 4.3 of PM. In particular, since (38) and (40) have been established, it simply remains to show that the Gaussian random variables X and Y are independent, or equivalently, that $E(XY) = 0$. Since $X = \lim_{n\to\infty} \int_0^{n^{1-\alpha}} e^{-cs} dB(s)$ *a.s.*, $Y = \lim_{n\to\infty} J_{-c}\left(n^{1-\alpha}\right)$ *a.s.* the dominated convergence theorem gives

$$\begin{aligned} E(XY) &= \lim_{n\to\infty} E\left(\int_0^{n^{1-\alpha}} e^{-cs} dB(s)\, J_{-c}\left(n^{1-\alpha}\right)\right) \\ &= \lim_{n\to\infty} e^{-cn^{1-\alpha}} E\left(\int_0^{n^{1-\alpha}} e^{-cs} dB(s) \int_0^{n^{1-\alpha}} e^{cr} dB(r)\right) \\ &= \omega^2 \lim_{n\to\infty} e^{-cn^{1-\alpha}} \int_0^{n^{1-\alpha}} dr = \omega^2 \lim_{n\to\infty} e^{-cn^{1-\alpha}} n^{1-\alpha} = 0, \end{aligned}$$

so X and Y are independent. ■

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