A SHORTCUT TO LAD ESTIMATOR ASYMPTOTICS

by

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0. ABSTRACT

Using generalized functions of random variables and generalized Taylor series expansions, we provide almost trivial demonstrations of the asymptotic theory for the LAD estimator in a regression model setting. The approach is justified by the smoothing that is delivered in the limit by the asymptotics, whereby the generalized functions are forced to appear as linear functionals wherein they become real valued. Models with fixed and random regressors, autoregressions and autoregressions with infinite variance errors are studied. Some new analytic results are obtained including an asymptotic expansion of the distribution of the LAD estimator and the results of some earlier simulation studies are explained.

Keywords. Autoregression, delta sequence, density estimate, domain of attraction, generalized function, generalized Taylor series, LAD estimator, stable process.

Running Head: LAD Asymptotics.
"Shall I refuse my dinner because I do not fully understand the process of digestion." O. Heaviside

1. INTRODUCTION

Classical asymptotic methods such as those given in Cramér (1946, Ch. 33) are usually thought to apply only in cases of regular estimation. Here smoothness conditions facilitate the use of Taylor expansions of the objective function and the associated first order conditions. When supplemented with preliminary consistency arguments, these expansions then yield the required asymptotic distribution theory in a few simple steps. With more tiresome algebra and higher order smoothness conditions, they also supply the formulae for Edgeworth expansions to second and higher orders.

The absence of smoothness in the criterion function that occurs in what are called non—regular cases is generally thought to prevent the use of this classical approach. Early studies that dealt with non—regularity complications by direct methods were Daniels (1961) and Huber (1967). More recently there has been a move towards the use of stochastic equicontinuity arguments and empirical process techniques to address the complications that are presented by the absence of smoothness and continuity in the criterion function. These techniques have been successful in accommodating a wide range of non—regular cases, including simulation based optimization estimators such as those that are employed by McFadden (1989) and Pakes and Pollard (1989). They have also found attractive applications in the development of asymptotics for semiparametric models (Andrews, 1989). The work of Pollard (1984, 1985, 1989a, b) has been especially influential in advancing the use of these techniques in econometric applications. However, as pointed out by Pollard (1989a), stochastic equicontinuity arguments are less accessible to many potential users even though they often capture the key technical difficulty in the asymptotics. Partly in response to this objection, Pollard presents an alternative approach for studying the asymptotic theory of the least absolute deviation (LAD) estimator in a simple regression
context. Pollard's alternative approach builds on the convexity of the LAD criterion function to construct a quadratic approximation whose minimand is close enough to the LAD estimator for the latter to share the same asymptotic normal distribution.

The present note is related closely to Pollard (1989a). However, instead of putting forward an alternative approach, our objective is to show the serviceability of the classical approach in non-regular problems like that of the LAD estimator. The idea we put forward is very simple. If the criterion function has non-regularities like discontinuities in its derivatives, these may be accommodated directly by the use of generalized functions, provided the discontinuities are smoothed out asymptotically. First order conditions and Taylor representations can be written down in the usual way but they take the form of generalized Taylor series. They may be formally interpreted as linear functionals in terms of the empirical distribution function. As the sample size $n \to \infty$, these linear functionals become well behaved provided some basic smoothness conditions are imposed on the underlying probability law of the data. In effect, with this generalization of the classical approach the asymptotics provide the smoothness that is required to justify the Taylor development and thereby the resulting asymptotics.

Our approach is heuristic and we do not claim to deal rigorously with all of the mathematical issues that arise. However, it is hoped that our extended treatment of LAD asymptotics will serve to illustrate the utility of these ideas and to stimulate the interest of others in the use of these methods. To continue the theme put forward by Heaviside in the line that heads this article, we believe there is still good food to enjoy in classical dinners.
2. LAD ASYMPTOTICS: THE HEURISTICS

Suppose $y_t$ is generated by the linear regression

$$(1) \quad y_t = x'_t \beta^0 + u_t \quad (t = 1, \ldots, n),$$

where the parameter vector $\beta^0 \in \mathbb{R}^k$, the errors $u_t$ satisfy $(A_1)$ below and $(x_t)$ constitutes a bounded, deterministic sequence for which

$$(2) \quad Q_n = n^{-1} \Sigma^n_1 x_t x'_t \to Q,$$

a positive definite limit, as $n \to \infty$.

$(A_1)$ The sequence $(u_t)$ is iid with zero median and probability density $f(\cdot)$ that is positive and analytic at zero.

The LAD estimator $\hat{\beta}_n$ is chosen as a solution of the extremum problem

$$(3) \quad \hat{\beta}_n = \text{argmin} \{n^{-1} \Sigma^n_1 |y_t - x'_t \beta|\}.$$ 

This is the standard framework for deriving the consistency and asymptotic normality of the LAD estimator. The original work is due to Bassett and Koenker (1978). An extensive study with an analysis of autoregressions as well as the regression model (1) is provided by the monograph by Bloomfield and Steiger (1983). A recent and novel treatment of the subject that includes an historical overview of research and some additional references is Pollard (1989a). Our condition $(A_1)$ is stronger than Pollard's "error assumption" on $u_t$ in that we require the density $f(\cdot)$ to be analytic rather than simply continuous at the origin. But there will be gains to making this stronger assumption. Not only does it help in developing generalized Taylor series but in so doing it facilitates the subsequent development of higher order asymptotics.

In most cases of interest and certainly under the standard assumptions of the regression model given above, the consistency of $\hat{\beta}_n$ is easily established by conventional arguments that involve the limit of the objective function in (3). Amemiya (1985, pp.
152–153) is a convenient source for the details of this approach. In what follows, we shall assume this argument has already been made and that \( \hat{\beta}_n \to_P \beta^0 \) as \( n \to \infty \).

Our concern is with the asymptotic distribution of \( \hat{\beta}_n \). Our approach is to proceed as if the problem were regular and the objective function were differentiable in \( \beta \). Although the derivatives do not exist in the usual sense, they do have a meaning as generalized functions. Moreover, they are real valued and unique when they appear in an appropriate linear functional form. Since this is precisely how they do arise in the limit as \( n \to \infty \), it turns out that we may proceed with the usual Taylor series expansion of the first order conditions to extract the asymptotic theory.

We start with the first order conditions for \( \hat{\beta}_n \) from (3). These are

\[
(4) \quad n^{-1} \Sigma_1^n \text{sgn}(y_t - x_t^T \hat{\beta}_n) x_t = 0 ,
\]

where \( \text{sgn}(X) = -1 \) for \( X < 0 \) and \( = 1 \) for \( X \geq 0 \). We now expand (4) in a Taylor series about its value at \( \beta^0 \). Note that \( d/dX(\text{sgn}(X)) = 2\delta(X) \) where \( \delta(X) \) is the delta (generalized) function (Gelfand and Shilov (1964), hereafter simply GS, p. 4). We denote successive derivatives of the delta function by \( \delta^{(k)}(X) \) for \( k = 1, 2, \ldots \). Proceeding in a purely formal way by treating \( \text{sgn}(-) \) as analytic (it is already piecewise analytic) and its derivatives as ordinary functions, we would have the expansion

\[
(5) \quad 0 = n^{-1} \Sigma_1^n \text{sgn}(u_t) x_t - 2n^{-1} \Sigma_1^n \delta(u_t) x_t x_t^T (\hat{\beta}_n - \beta^0) \\
+ 2 \Sigma_{k=2}^n (-1)^{k-1} (1/k!) n^{-1} \Sigma_1^n \delta^{(k-1)}(u_t) x_t x_t^T (\hat{\beta}_n - \beta^0)^k .
\]

Let us now suppose that we can ignore all but the first two terms of (5) as \( n \to \infty \). Scaling by \( n^{1/2} \) and taking the error to be \( o_p(1) \), we have

\[
(6) \quad n^{-1/2} \Sigma_1^n \text{sgn}(u_t) x_t = 2 \left[ -n^{-1} \Sigma_1^n \delta(u_t) x_t x_t^T \right] n^{1/2}(\hat{\beta}_n - \beta^0) + o_p(1) .
\]

Since \( (\text{sgn}(u_t)) \) is iid\((0,1)\), the left side of (6) satisfies a multivariate extension of the Lindeberg–Lévy theorem, leading to
\[ n^{-1/2} \sum_1^n \text{sgn}(u_t)x_t \sim_d N(0, Q). \]

On the right hand side of (6) the matrix factor in square brackets satisfies a weak law of large numbers, viz.

\[ n^{-1} \sum_1^n \delta(u_t)x_t x'_t \rightarrow_p \lim_{n \to \infty} n^{-1} \sum_1^n E(\delta(u_t))x_t x'_t = f(0)Q. \]

Putting (6), (7) and (8) together we deduce directly the limit theory for the LAD estimator, i.e.

\[ n^{1/2}(\hat{\beta}_n - \beta^0) \sim_d N \left[ 0, (1/2 f(0))^2 Q^{-1} \right]. \]

3. AN ATTEMPT AT RIGOR

To attach some rigor to this heuristic skeleton, we need to justify three of the steps just taken. Specifically, (i) the Taylor expansion (5); (ii) the \( o_p(1) \) error in (6); and (iii) the weak law of large numbers (8).

Let us start with (i). Obviously, (5) has no meaning as an ordinary equation or as an ordinary Taylor series expansion. But it can be interpreted in terms of generalized functions and as a generalized Taylor series. Thus, suppose \( \varphi \) is a suitable test function for linear functionals of a generalized function \( g \). Suppose, for instance, that \( \varphi \) belongs to the space \( S \) of entire functions which, together with their derivatives, approach zero more rapidly than any power of \( 1/|u| \) as \( |u| \to \infty \) (e.g. \( e^{-u^2} \)) and that the linear functional

\[ (g(u+h), \varphi) = \int_R g(u+h)\varphi(u)du = \int_R g(s)\varphi(s-h)ds \]

is an ordinary analytic function of \( h \) in some neighborhood of \( h = 0 \) for all \( \varphi \). Then, \( g_h(u) = g(u+h) \) is a generalized analytic function of \( h \) (GS, pp. 149–150). Indeed, by expanding \( \varphi(s-h) \) about its value at \( h = 0 \) in (10) and noting that \( (g^{(j)}, \varphi) = (g, (-1)^j \varphi^{(j)}) \), we obtain

\[ (g(u+h), \varphi) = \sum_{j=0}^{\infty} (1/j!)(-h)^j (g, \varphi^{(j)}) = \sum_{j=0}^{\infty} (1/j!)(g^{(j)}, \varphi) \]
which we can write in formal terms as
\[ g_h = \sum_{j=0}^{\infty} (1/j!) g_{h}^{(j)} j^j. \]
Setting \( g_h(u) = \text{sgn}(u+h), \ h = x_t^t (\hat{\beta}_n - \beta^0) \) and noting that \( \int_{-\infty}^{\infty} \text{sgn}(u+h) \varphi(u) du \) is analytic in \( h \) for all \( \varphi \in S \), we deduce the expansion given in (5) above.

Next consider (iii). The limit given on the right of (8) is well defined because the generalized function \( \delta(\cdot) \) arises only through the linear functional \( E(\delta(u)) = \int \delta(u)f(u) du = f(0) \). Note that this last expression remains true even though the density \( f(\cdot) \) may exist only in a neighborhood of zero. This is because generalized functions like \( \delta(\cdot) \) may be defined locally in terms of their operation on test functions with support in arbitrarily small given neighborhoods of every point (see GS, p. 140).

To be more complete in deducing (8) we may replace \( \delta(\cdot) \) with its inverse Fourier transform representation, which we signify by "\( F^{-1}(\cdot) \)", i.e.
\[ \delta(u_t) = F^{-1}(1) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iu_t \sigma} d\sigma. \]
Again, the integral is formal and the correspondence produces the Fourier transform pair of generalized functions \( (\delta(\cdot), 1) \) (GS, p. 168). In place of (8), we may now show that
\[ n^{-1} \sum_{t=1}^{n} x_t x_t^t - \lim_{n \to \infty} n^{-1} \sum_{t=1}^{n} E(e^{-iu_t \sigma}) x_t x_t^t = c_{f_{u}}(-\sigma)Q. \]
But this follows immediately because \( (e^{-iu_t \sigma} - E(e^{-iu_t \sigma})) \) is iid with zero mean and finite variance. Upon inversion of (11) we get (8), since \( f(0) = (1/2\pi) \int_{-\infty}^{\infty} c_{f_{u}}(\sigma) d\sigma \). In fact, (11) may be regarded as the appropriate way to interpret (8) as a weak law for generalized random variables.

This leaves us with (ii). Working from (5) we have
\[ n^{-1/2} \sum_{t=1}^{n} \text{sgn}(u_t) x_t = 2A_n n^{1/2} (\hat{\beta}_n - \beta^0), \]
where
\[ A_n = n^{-1} \sum_{i=1}^{n} \delta(u_i)x_i x'_i + \sum_{j=1}^{\infty} (-1)^{j+1}((j+1)!)^{-1} n^{-1} \sum_{i=1}^{n} \delta^{(j)}(u_i)x_i x'_i \left[ x'_i (\hat{\beta}_n - \beta^0) \right]^{j} \]
and we have to show that the second term of \( A_n \) is \( o_p(1) \) as \( n \to \infty \).

As with (8) we have
\[ n^{-1} \sum_{i=1}^{n} \delta^{(k)}(u_i)x_i x'_i \xrightarrow{p} \lim_{n \to \infty} \left[ n^{-1} \sum_{i=1}^{n} \mathbb{E}(\delta^{(k)}(u_i))x_i x'_i \right] = (-1)^k f^{(k)}(0)Q, \]
where we use the fact that
\[ (\delta^{(k)}, f) = \int_{-\infty}^{\infty} \delta^k(u)f(u)du = (\delta, (-1)^k f^{(k)}) = (-1)^k f^{(k)}(0), \]
(GS, p. 26). Moreover, since \( f(\cdot) \) is analytic at zero, power series such as
\[ \sum_{k=2}^{\infty} (1/k!) f^{(k)}(0) \frac{\epsilon^{k-1}}{k-1} = (f(\epsilon) - f(0) - f'(0)\epsilon)/\epsilon \]
are convergent and of \( O(\epsilon) \) for all \( \epsilon \) in the vicinity of zero. However, \( \hat{\beta}_n \xrightarrow{p} \beta^0 \) and \( x'_i (\hat{\beta}_n - \beta^0) \xrightarrow{p} 0 \) uniformly in \( t \). It follows that the second term of \( A_n \) converges in probability to zero as required.

4. AN ASYMPTOTIC EXPANSION

One advantage of the above approach is that it lends itself to the development of higher order asymptotic expansions. To see how to proceed we set \( q_n = n^{1/2} (\hat{\beta}_n - \beta^0), \)
\[ \ell_1 = n^{-1/2} \sum_{i=1}^{n} \text{sgn}(u_i)x_i, \]
scale the expansion (5) by \( n^{1/2} \) and write it in the form
\[ 0 = \ell_{i1} + \ell_{2ij} q_{nj} + n^{-1/2} \ell_{3ijk} q_{nj} q_{nk} + n^{-1} \ell_{4ijkl} q_{nj} q_{nk} q_{nm} + o_p(n^{-1}), \]
where we use the summation convention. Inverting (13), we have up to \( O_p(n^{-1/2}) \)
\[ q_{ni} = \ell_{2ij} \ell_{i1j} + n^{-1/2} \ell_{2ij} \ell_{3ijk} (\ell_{2km} \ell_{1m})(\ell_{2ln} \ell_{1n}) + o_p(n^{-1}). \]
Next observe that the distribution of \( \text{sgn}(u_i) \) is symmetric, that the distribution of \( \ell_1 \) will admit a valid Edgeworth expansion and that, because of the symmetry of the distribution of \( \text{sgn}(u_i) \), there will be no skewness term in this expansion. Thus, we may write
symbolically
\[ \ell_1 = d N(0,Q) + O_p(n^{-1}) = \ell_1 + O_p(n^{-1}), \text{ say.} \]

The second term of (14) is \( O_p(n^{-1/2}) \) and
\[ \ell_{3jk\ell} = n^{-1} \sum_1^n \delta^{(i)}(u_{t_i}) x_{t_i}^j x_{t_i}^k x_{t_i}^\ell \to_p (\ell')^{(0)} r_{jk\ell} = \ell_{3jk\ell}, \text{ say,} \]
where
\[ r_{jk\ell} = \lim_{n \to \infty} \frac{1}{n^2} \sum_1^n x_{t_i}^j x_{t_i}^k x_{t_i}^\ell. \]
Thus (14) would appear to yield a conventional Edgeworth expansion for the distribution of \( q_n \). However, there is an additional complication that arises from the components \( \ell_{2ij} \) and the elements \( \ell_{ij} \) of its inverse. We have
\[ \ell_{2ij} = 2n^{-1} \sum_1^n \delta(u_{t_i}) x_{t_i}^j x_{t_i}^i \to_p 2f(0) q_{ij} = \ell_{2ij}, \text{ say.} \]

Similarly, define \( \ell_{ij} = (1/2) f(0) q_{ij} \). Then (14) may be written as
\[ q_n = \ell_{ij} + \left( \ell_{ij} - \ell_{ij} \right) + n^{-1/2} \ell_{2ij} \left( \ell_{im} \ell_{1m} \right) + O_p(n^{-1/2}). \]

The order of magnitude of the second term of (15) depends on that of
\[ \ell_{2ij} - \ell_{2ij} = 2n^{-1} \sum_1^n (\delta(u_{t_i}) - f(0)) x_{t_i} x_{t_i}' + o(1). \]

This term is much more difficult to analyze and is larger than \( O_p(n^{-1/2}) \).

To see what is involved, set \( x_{t_i} = 1 \) and consider
\[ n^{-1} \sum_1^n (\delta(u_{t_i}) - f(0)) = \int_R \delta(u) d(F_n(u) - F(u)), \]
where \( F_n \) is the empirical distribution function of \( u_{t_i} \) and \( F \) is the cdf of \( u_{t_i} \). In general, we have the weak convergence (Billingsley, 1968, p. 141)
\[ \sqrt{n}(F_n(u) - F(u)) \to_d Y(u), \]
where \( Y(\cdot) \) is a Gaussian process with covariance kernel \( F(u)(1 - F(u)) \), \( u \leq v \). However, we cannot employ (17) in (16) because the implied limit variate, viz.
\[ \int_{\mathbb{R}} \delta(u) dY(u) \], does not exist. For instance, if \( u_t \) were uniformly distributed then \( Y(u) \) would be a Brownian bridge process and \( \int_{\mathbb{R}} \delta(u) dY(u) \) would be its "derivative," which, like the derivative of Brownian motion, does not exist as an ordinary random variable.

There is another way to proceed. Note that (16) is the derivative at \( s = 0 \) of

\[
\begin{align*}
&n^{-1} \sum_1^n ((1/2) \text{sgn}(u_t + s) - F(s)) - n^{-1} \sum_1^n ((1/2) \text{sgn}(u_t) - F(0)) \\
&= \{F_n(0) - F_n(-s)\} - f_0^s dF.
\end{align*}
\]

For \( s = -tn^{-1/3} \) this expression is

\[
-\{F_n(tn^{-1/3}) - F_n(0) - f(0)tn^{-1/3} - (1/2)f'(0)tn^{-2/3}\} = o(n^{-2/3})
\]

say.

Now Kim and Pollard (1990, theorem 4.7 and example 6.5, p. 216) show that

\[
n^{2/3}\{F_n(tn^{-1/3}) - F_n(0) - f(0)tn^{-1/3}\} \to_d (1/2)t^{2}f'(0) + f(0)^{1/2}W(t)
\]

with \( W(t) \) a two-sided Brownian motion. Thus,

\[
n^{2/3}X_n(t) \to_d f(0)^{1/2}W(t).
\]

Define \( \hat{f}_n(0) \) to be the left derivative of the concave majorant of \( F_n \) (i.e. the smallest concave function on \( [0,\omega) \) that is everywhere greater than or equal to \( F_n \)). Kim and Pollard further show that \( n^{1/3}(\hat{f}_n(0) - f(0)) \) has a limit distribution given by that of the slope at the origin of the concave majorant of Brownian motion with quadratic drift, a result that is originally due to Prakasa Rao (1969). Treating \( \hat{f}_n(0) = n^{-1} \sum_1^n \delta(u_t) \) as an estimate of \( f(0), \) these results suggest that (16) and, hence, \( \ell_{2ij} - \ell_{2ij} \) are both \( O_p(n^{-1/3}) \). But the second term in the expansion (15) involves the product \( (\ell_{2}^{ij} - \ell_{2}^{ij})\ell_{1j}, \) where \( \ell_{1} \equiv_d N(0,Q). \) Because \( \ell_{1} \) has zero mean this term will contribute to the asymptotic expansion of the distribution of \( q_n \) only through the variance of \( \ell_{1} \) and, hence, will produce an adjustment of \( O(n^{-2/3}) \). It follows that only the first and third members on the right side of (15) contribute to the expansion up to \( O(n^{-1/2}). \) In
particular, setting \( \xi = \xi_d = N(0, Q) \) we have

\[
q_{ni} = (1/2 f(0))q^{ij}\xi_j - n^{-1/2}(1/2 f(0))^3f'(0)q^{ij}_{ijk}\zeta_m\zeta_m(q^{im}\xi_m) + o_p(n^{-1/2}).
\]

Observe that when \( f'(0) = 0 \), which will be the case for symmetric error distributions, the term of \( O_p(n^{-1/2}) \) itself drops out, leaving only the first order asymptotic term. In the general case, we can derive an asymptotic expansion of the density of some linear combination such as \( r = c'q_n = c_iq_{ni} \) of the standardized error vector \( q_n \). The formulae for the expansion of the density up to \( O(n^{-1/2}) \) may then be deduced from those in the literature (e.g. Sargan (1976) and Phillips (1977, 1982), where the last reference puts them in a form that is especially simple to interpret). With a little algebra, we obtain the following explicit expansion to \( O(n^{-1/2}) \):

\[
\text{pdf}(r) = (1/\omega)\varphi(r/\omega) \left[ 1 + n^{-1/2}\{a_1(r/\omega) + a_2(r/\omega)^3\} \right] + o(n^{-1/2}),
\]

where \( \varphi(x) = (2\pi)^{-1/2}\exp(-(1/2)x^2) \) and the parameters are:

\[
\omega^2 = (1/2 f(0))^2c_1c_jc_j,
\]

\[
a_1 = (1/2 \omega)q_{st}f_{st} - 3a_3,
\]

\[
a_3 = (1/2 \omega^3)(1/2 f(0))^2f_{st}c_sc_t,
\]

\[
f_{st} = 2q_{st}f(0)\zeta_m(q^{ij}\zeta_m)q_{st}.
\]

5. ESTIMATING THE COVARIANCE MATRIX

In conventional Taylor expansions of the first order conditions, the Hessian is often used to produce an estimate of the asymptotic covariance matrix of the estimator. The same idea may be applied here. Taking the dominant member of the "Hessian," \( 2A_n \), in the expansion (12) to be \( 2n^{-1}\sum_1^n\delta(u_t)x_tx_t' \) we simply replace the delta function in this expression by a delta sequence \( \delta_m(\cdot) \) for which

\[
\lim_{m \to \infty} \int_\infty^\infty \delta_m(x)\varphi(x)dx = \varphi(0)
\]
for all $\varphi \in S$. For example, the delta sequence $\delta_m(x) = (m/\pi)^{1/2}e^{-mx^2}$ satisfies (20) and corresponds to a density estimate based on a normal kernel. The covariance matrix estimate arising from this delta sequence takes the form

$$2n^{-1}\sum_1^n \delta_m(u_t)x_t x_t' .$$

The errors, $u_t$, in this expression can be replaced by residuals to produce a feasible covariance matrix estimate. The parameter $m$ is like a bandwidth (for the normal kernel given above we would have $h = 1/m^{1/2}$) and must be chosen so that $m \to \infty$, $nm^{-1/2} \to \infty$ as $n \to \infty$ (i.e. $h \to 0$, $nh \to \infty$) for consistency (e.g. Silverman, 1986, p. 71).

Since delta sequences encompass most density estimates (e.g. Walter and Blum, 1979), this approach to the estimation of the covariance matrix is really quite general.

6. MODEL EXTENSIONS

It would appear that the approach suggested here remains valid for a large class of models with weakly dependent, rather than iid, errors. All that is required is that $\ell_1 = n^{-1/2}\sum_1^n \text{sgn}(u_t)x_t$ satisfy some central limit theorem for weakly dependent errors. For example, suppose that $u_t$ satisfies

(A2) The sequence $(u_t)$ is strictly stationary and strong mixing with mixing coefficients $\beta_k$ that satisfy $\sum_k \beta_k < \infty$, and $u_t$ has zero median and probability density $f(\cdot)$ that is positive and analytic at zero.

Then $\text{sgn}(u_t)$ is also strong mixing with mixing coefficients that satisfy the same summability condition and we have the central limit theorem

$$n^{-1/2}\sum_1^n \text{sgn}(u_t)x_t \to_d N(0, V) ,$$

where

$$V = \sum_{k=-\infty}^{\infty} R_k E\{\text{sgn}(u_0)\text{sgn}(u_k)\} , \text{ with } R_k = \lim_{n \to \infty} n^{-1}\sum_1^n x_t x_t' + k .$$

When $x_t = 1$ we have
\[ V = \mathbb{E}(\text{sgn}(u_0))^2 + 2\sum_{k=1}^{m} \mathbb{E}(\text{sgn}(u_0)\text{sgn}(u_k)) \]
\[ = 1 + 2\sum_{k=1}^{m}\{P(u_0 < 0, u_k < 0) + P(u_0 > 0, u_k > 0)\} \]
\[ - \{P(u_0 > 0, u_k < 0) + P(u_0 < 0, u_k > 0)\}. \]

Using the same expansion (5) as before, we deduce the following limit theory for the LAD estimator

\[ n^{1/2}(\hat{\beta}_n - \beta^0) \xrightarrow{d} N(0, (1/2 f(0))^2 Q^{-1} Q^{-1}). \]

For inference \( V \), as well as \( f(0) \), now needs to be estimated. The situation is entirely analogous to the estimation of autocorrelation consistent covariance matrix estimation as it arises in conventional regression contexts (cf. White (1984), Newey and West (1987)).

Models in which the carrier variables \( x_t \) are random may also be accommodated. The details are close to those given in Pollard (1989a) so we shall only touch on them briefly here. Suppose the \( x_t \) are strictly stationary, ergodic, square integrable and \( \mathcal{F}_{t-1} \) measurable for some increasing sequence of \( \sigma \)-fields \( \mathcal{F}_t \), suppose \( u_t \) is independent of \( \mathcal{F}_{t-1} \) and satisfies (4.1) and let \( V_n = \sum_{i=1}^{n} \mathbb{E}(z_t' z_t') = n\mathbb{E}(x_t x_t') = n \Omega \), where \( z_t = \text{sgn}(u_t) x_t \). Then the following two conditions hold:

\[ \max_{t \leq n} \| V_n^{-1/2} z_t \| = \max_{t \leq n} \| \Omega^{-1/2} (n^{-1/2} z_t) \| \xrightarrow{p} 0, \]
\[ V^{-1/2} (\sum_{i=1}^{n} z_t' z_t') V^{-1/2} = \Omega^{-1/2} (n^{-1/2} \sum_{i=1}^{n} z_t' z_t') \Omega^{-1/2} \xrightarrow{p} I_k \]

and we may use a martingale central limit theorem (e.g. Hall and Heyde (1980), theorem 3.2, p. 58) to establish that

\[ V_n^{-1/2} \sum_{i=1}^{n} \text{sgn}(u_t) x_t' \xrightarrow{d} N(0, I). \]

Moreover,

\[ n^{-1} \sum_{i=1}^{n} e^{-iu_t \sigma} x_t x_t' \xrightarrow{p} \mathbb{E}(e^{-iu_t \sigma} x_t x_t') = c_{f_\sigma}(-\sigma) \Omega \]

and so

\[ n^{-1} \sum_{i=1}^{n} \delta(u_t) x_t x_t' \xrightarrow{p} f(0) \Omega, \]
where \( f(\cdot) \) is the probability density of \( u_t \). These results enable us to deduce the limit law
\[
n^{1/2}(\hat{\beta}_n - \beta^0) \rightarrow_d N(0, (1/2 f(0))^2 \Omega^{-1})
\]
in the same way as before. This covers the case of strictly exogenous stationary regressors \( x_t \) with \( \Omega = E(x_t x_t') \). It also covers the case of stable autoregressions with \( x_t = (y_{t-1}, \ldots, y_{t-k}) \) with \( (u_t) = \text{iid}(0, \sigma^2) \) (the finite variance \( \sigma^2 \) ensures that the elements of \( x_t \) are square integrable) and \( \Omega = (\omega_{ij}) \) with \( \omega_{ij} = E(y_{t-i} y_{t-j}) \).

7. AUTOREGRESSIONS WITH INFINITE VARIANCE ERRORS

An autoregression with infinite variance errors is quite difficult to analyze by conventional methods and has so far resisted a complete study. An and Chen (1982) and Pollard (1989a) considered the prototypical case of an AR(1) with \( \text{iid} \) Cauchy errors. Pollard showed that the LAD estimator \( \hat{\beta}_n \) lies within \( O_p(n^{-1}) \) of \( \beta^0 \), improving slightly on the \( O_p(n^{-1+\epsilon}) \), \( \epsilon > 0 \), result of An and Chen. Pollard was unable to find the limit distribution of \( n(\hat{\beta}_n - \beta^0) \) and left this as an unsolved problem. Davis, Knight and Liu (1990) provide a solution to this problem as an optimization functional in a recent paper that came to the author's attention after the present work was in its second draft. The heuristic development that follows provides an alternative solution to the problem.

We shall consider the stable AR(1)
\[
y_t = \beta^0 y_{t-1} + u_t, \text{ with } |\beta^0| < 1
\]
and \( \text{iid} \) errors with infinite variance. To develop a limit theory we shall assume that \( u_t \) is in the domain of attraction of a stable law with characteristic exponent \( \alpha \). Specifically we say that \( u \in \mathcal{D}(\alpha) \) if
\[
P(u > x) = c_1 x^{-\alpha} L(x)(1 + c_2(x)), \ x > 0, c_1 > 0
\]
and
\[ P(u < -x) = c_2 x^{-\alpha} L(x)(1 + \alpha_2(x)), \quad x > 0, \quad c_2 \geq 0 \]

with \( 0 < \alpha < 2 \), \( L(x) \) a slowly varying function at \( \infty \) and \( \alpha_2(x) \to 0 \) as \( |x| \to \infty \). If \( L(x) = 1 \) in (22) and (23), then \( u \) is in the normal domain of attraction of a stable law with parameter \( \alpha \) and we write \( u \in \mathcal{D}(\alpha) \). With this terminology in hand, we assume

(\( A_3 \)) The sequence \( (u_t) \) is iid with \( u_t \in \mathcal{D}(\alpha) \), where \( 0 < \alpha < 2 \), with zero median and with probability density \( f(\cdot) \) that is positive and analytic at zero. If \( \alpha > 1 \), \( E(u_t) = 0 \) and if \( \alpha = 1 \) then \( u_t = -u_t \) (i.e. \( u_t \) is symmetrically distributed).

Define the normalizing sequence

\[ a_n = \inf \{ x : P(|u| > x) < n^{-1} \}. \]

For \( u \in \mathcal{D}(\alpha) \) we have \( a_n = n^{-1/\alpha} L(n) \) for some slowly varying function \( L(n) \). For \( u \in \mathcal{N}(\alpha) \) we have \( a_n = cn^{1/\alpha} \) for some constant \( c > 0 \). With this construction we have the following weak convergence results for \( 0 < \alpha < 2 \):

\[
\begin{align*}
a_n^{-1} \Sigma_1^{nr} u_t & \xrightarrow{d} U_\alpha(1), \\
a_n^{-1} \Sigma_1^{nr} u_t & \xrightarrow{d} U_\alpha(r), \\
(a_n^{-1} \Sigma_1^{nr} u_t, a_n^{-2} \Sigma_1^{nr} u_t^2) & \xrightarrow{d} (U_\alpha(r), f_0^r(dU_\alpha)^2).
\end{align*}
\]

Here \( U_\alpha(r) \) is a Lévy \( \alpha \)-stable process and \( f_0^r(dU_\alpha)^2 = [U_\alpha]_r \) is its quadratic variation process. The first result of (24) is classical (e.g. Ibragimov and Linnik (1971), Ch. 2); the second is its functional version; and (25) is a joint functional limit law for first and second sample moments that is proved in Resnick (1986, pp. 94–95).

To find the limit distribution of the LAD estimator we work from expansion (5), setting \( x_t = y_{t-1} \). We scale (5) by \( na_n^{-1} \), giving

\[
\begin{align*}
a_n^{-1} \Sigma_1^{nr} \text{sgn}(u_t) y_{t-1} &= 2[a_n^{-2} \Sigma_1^{nr} \delta(u_t)] a_n(\beta_n - \beta^0) \\
&\quad - 2\sum_{k=2}^{\infty} (-1)^k (1/k!)[a_n^{-k} \Sigma_1^{nr} \delta^{(k-1)}(u_t)] y_{t-1}^{k+1} a_n(\beta_n - \beta^0)^k
\end{align*}
\]

Observe that \( \text{sgn}(u_t) \) and \( y_{t-1} \) are independent, \( \text{sgn}(u_t) \) is bounded and the product variate \( X_t = \text{sgn}(u_t) y_{t-1} \in \mathcal{D}(\alpha) \) — see Appendix A of Phillips (1990). Note also that
\( u_t, y_{t-1} \) and \( X_t \) all have the same normalizing sequence \( a_n \), so that by (24) we have
\[
a_n^{-1} \sum_{i=1}^{n} \text{sgn}(u_i) y_{t-1} \rightarrow_d S_{\alpha}(1),
\]
where \( S_{\alpha}(r) \) is an \( \alpha \)-stable process. The right side of (26) may be treated term by term. First observe that \( y_{t-1}^2 = X_t^2 \in \mathcal{P}(\alpha/2) \) and by (25)
\[
a_n^{-2} \sum_{i=1}^{n} y_{t-1}^2 \rightarrow_d f_0^1(\text{d}S_{\alpha})^2.
\]
However, our interest is in the factor in square brackets in the first term on the right side of (26), and this involves the generalized random variable \( \delta(u_t) \). Since \( u_t \) is independent of \( y_{t-1}^2 \), the effect of \( \delta(u_t) \) in the summation is to randomly select elements that will remain in the summation according to subsequent realizations of the independent draw \( u_t \). The remaining elements in the sum are weighted according to \( E(\delta(u_t)) = E(\delta(u_t) | \mathcal{F}_{t-1}) = f(0) \) and we obtain
\[
a_n^{-2} \sum_{i=1}^{n} \delta(u_i) y_{t-1}^2 \rightarrow_d f(0) f_0^1(\text{d}R_{\alpha})^2,
\]
where \( R_{\alpha}(r) \) is a Lévy \( \alpha \)-stable process independent of \( S_{\alpha}(r) \). Similarly, for the higher order terms on the right side of (26) we have
\[
E(\delta^{(k-1)}(u_t)) = E(\delta^{(k-1)}(u_t) | \mathcal{F}_{t-1}) = (-1)^{k-1} f^{(k-1)}(0)
\]
and
\[
a_n^{-k-1} \sum_{i=1}^{n} \delta^{(k-1)}(u_i) y_{t-1}^{k+1} \rightarrow_d (-1)^{k-1} f^{(k-1)}(0) f_0^1(\text{d}R_{\alpha})^{k+1}.
\]
This leads to the following limit theory for the LAD estimator of \( \beta^0 \) in (21) under errors that satisfy \( (A_3) \)
\[
(27) \quad a_n(\hat{\beta}_n - \beta^0) \rightarrow_d \xi
\]
where the limit variate \( \xi \) is a solution of the equation
\[
S_{\alpha}(1) = 2\{f(0) f_0^1(\text{d}R_{\alpha})^2\} \xi + 2 \sum_{k=2}^{\infty} \{(1/k!) f^{(k-1)}(0) f_0^1(\text{d}R_{\alpha})^{k+1}\} \xi^k.
\]
In the above expression \( f_0^1(\text{d}R_{\alpha})^s \) is the \( s \)-variation process of \( R_{\alpha} \) \( s \geq 2 \). Observe that for distributions \( f(\cdot) \) for which \( f^{(j)}(0) = 0, \forall j \geq 1 \), we have the simple result that the
limit variate is

$$\xi = S_{\alpha}(1)/2f(0)\int_0^1(dR_\alpha)^2.$$  

For example, if \((v_1, v_2)\) are independent uniform on \([-a, a]\) for some fixed \(a > 0\), then \(u = v_1/v_2\) has density

$$f(u) = \begin{cases} 
1/4 & |u| \leq 1 \\
1/4u^2 & |u| > 1 
\end{cases}.$$  

Such a density obviously satisfies \((A_3)\) and \(f^{(j)}(0) = 0, \forall j \geq 1\).

It is of some interest to compare the LAD estimator \(\hat{\beta}_n\) with the OLS estimator \(\bar{\beta}_n\). Bloomfield and Steiger (1983, p. 105) speculate that \(\hat{\beta}_n\) is asymptotically more efficient than \(\bar{\beta}_n\) and conjecture that \(\hat{\beta}_n\) converges at a faster rate for the case where \(0 < \alpha < 2\). This can, in fact, be demonstrated by considering the limit distribution of \(\hat{\beta}_n\). Let us suppose that \(u \in \mathcal{N}(\alpha)\), as it is in the cases studied experimentally by Bloomfield and Steiger. Then, \(u_t y_{t-1} \in \mathcal{D}(\alpha)\) rather than \(\mathcal{N}(\alpha)\) and the norming sequence for sums of \(u_t y_{t-1}\) has the form

$$b_n = c(n \log n)^{1/\alpha}, \quad c = \text{const.}$$  

(see Appendix A of Phillips (1990)). Since \(\bar{\beta}_n - \beta^0 = \Sigma_{t=1}^n u_t / \Sigma_{t=1}^n y_{t-1}^2\), it follows that

$$(28) \quad (n/\log n)^{1/\alpha}(\hat{\beta}_n - \beta^0) = \left[\frac{a_{1,2}^2 \gamma_{t-1}^2}{a_{1,1}^2 y_{t-1}^2}\right]^{-1}\left[(n \log n)^{-1/\alpha} \Sigma_{t=1}^n y_{t-1} u_t\right]^{-d} P_{\alpha/2}^{1/2} Q_{\alpha},$$

where \(Q_{\alpha}\) is a stable variate of exponent \(\alpha\), independent of the positive stable variate \(P_{\alpha/2} = \int_0^1(dS_{\alpha})^2\). Comparing (28) with (27) we see that the LAD estimator has a factor of \((\log n)^{1/\alpha}\) in its favor, confirming the conjecture of Bloomfield and Steiger concerning the higher relative efficiency of LAD. The difference arises because, \(u_t y_{t-1} \in \mathcal{D}(\alpha)\) and sample covariances of \(u_t\) and \(y_{t-1}\) form the numerator of \(\hat{\beta}_n - \beta^0\), whereas \(\text{sgn}(u_t)y_{t-1} \in \mathcal{N}(\alpha)\) because \(\text{sgn}(u_t)\) is bounded and the LAD estimator depends on sample covariances of \(\text{sgn}(u_t)\) and \(y_{t-1}\). The difference in rates of convergence of \(\hat{\beta}_n\)
and \( \hat{\beta}_n \) to \( \beta^0 \) is then due to the differences in the tail thickness of the distributions of \( u_t y_{t-1} \) and \( \text{sgn}(u_t) y_{t-1} \). By bounding the effect of the error through the presence of the \( \text{sgn}(u_t) \) factor in \( \text{sgn}(u_t) y_{t-1} \), the LAD estimator successfully achieves a faster rate of convergence. This helps to explain the simulation results of Bloomfield and Steiger.

8. AUTOREGRESSIONS WITH INFINITE VARIANCE ERRORS AND A UNIT ROOT

The case \( \beta^0 = 1 \) in (21) has recently been studied by Knight (1989). It may also be treated by our approach here. When \( \beta^0 = 1 \), the output \( y_t \) of (21) is an integrated process. In consequence, we have the weak convergence

\[
(29) \quad a_n^{-1} y_{[n]} \xrightarrow{d} S_{\alpha}(\cdot),
\]

where \( S_{\alpha}(\cdot) \) is an \( \alpha \)-stable process. Further we have

\[
n^{-1/2} \Sigma_1^{[n]} \text{sgn}(u_t) \xrightarrow{d} W(\cdot)
\]

and

\[
(30) \quad n^{-1/2} a_n^{-1} \Sigma_1^n \text{sgn}(u_t) y_{t-1} = n^{-1/2} \Sigma_1^n (a_n^{-1} y_{t-1}) \text{sgn}(u_t) \xrightarrow{d} f(0) S_\alpha^{-d} w,
\]

where \( W(\cdot) \) is standard Brownian motion and \( S_{\alpha}^{-d}(r) \) signifies the left limit of the process \( S_{\alpha}(\cdot) \) at \( r \). Next observe that

\[
n^{-1} a_n^{-2} \Sigma_1^n y_{t-1}^2 = n^{-1} \Sigma_1^n \left( a_n^{-1} y_{t-1} \right)^2 \xrightarrow{d} f(0) S_\alpha^2
\]

and

\[
n^{-1} \Sigma_1^n \delta(u_t - f(0)) \left( a_n^{-1} y_{t-1} \right)^2 \xrightarrow{p} 0
\]

so that

\[
(31) \quad n^{-1} \Sigma_1^n \delta(u_t) \left( a_n^{-1} y_{t-1} \right)^2 \xrightarrow{d} f(0) S_\alpha^2.
\]

Combining (30) and (31) we have
\begin{equation}
\frac{n^{1/2}a_n(\hat{\beta}_n - 1)}{\sum_{i=1}^n \text{sgn}(u_t)y_{t-1}^2} \approx \frac{n^{-1/2}a_n^{-1}\sum_{i=1}^n \text{sgn}(u_t)y_{t-1}^2}{2n^{-1}a_n^{-2}\sum_{i=1}^n \delta(u_t)y_{t-1}^2} \frac{f_0^1S^{-d}W}{2f(0)f_0^1S^2},
\end{equation}

as given in Theorem 3 of Knight (1989). Since \( a_n = n^{1/\alpha}L(n) \) we have the remarkable result, due to Knight, that the LAD estimator converges at a faster rate in the unit root model for \( 0 < \alpha < 2 \) than the OLS estimator. As remarked by Knight, robust estimators such as LAD retain the advantages of the strong signal from \( y_{t-1} \) (due to integration and thick tailed errors) but alternate the effects of outliers in the error \( u_t \) to the extent that they occur in the sample covariance between \( \text{sgn}(u_t) \) and \( y_{t-1} \) (in the LAD case).

9. CONCLUSION

The methods outlined here seem to offer some promise as tools for the analysis of regression asymptotics in non-regular cases like that of the LAD estimator. Our approach has not always been rigorous and in Sections 7 and 8, in particular, is quite heuristic. Nevertheless, it is hoped that the results obtained point to the usefulness of the approach. One would like to hope that the approach can be made entirely rigorous by providing a tight probabilistic framework for the use of the generalized random variables that appear here. In the meantime it seems reasonable to conclude that the classical approach warrants more attention than it has yet received. It may indeed offer some advantages that empirical process methods do not seem to presently enjoy, viz. the capacity to develop higher order asymptotics.
10. REFERENCES


