MARGINAL VERSUS AVERAGE COST PRICING IN THE
PRESENCE OF A PUBLIC MONOPOLY

by

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The Arrow-Debreu analysis of decentralized resource allocation in a Walrasian economy assumes constant or decreasing returns to scale in production (12). Recently, several authors have extended this analysis to economies with a public monopoly, i.e., a firm with increasing returns to scale.

In this literature, the salient feature is the characterization of increasing returns to scale technologies as nonconvex production sets, so that under this definition both single and multi-product firms may exhibit increasing returns.

Our intended model here is an economy with a competitive sector consisting of households and firms with convex technologies and a public sector consisting of firms with nonconvex technologies. A special case is a single multi-product firm which produces products for regulated markets (with a non-convex technology) and produces products for unregulated markets (with a convex technology), e.g., A.T. & T.

We consider for such an economy two of the general equilibrium concepts that have been investigated in this literature. One is Hotelling's notion of a marginal cost pricing (MCP) equilibrium (15), and the other is Boiteaux's notion of an average cost pricing (ACP) equilibrium (6).

A marginal cost pricing equilibrium is a family of consumption plans, production plans, prices and lump sum taxes such that households are maximizing utility subject to after-tax income; firms with constant or decreasing

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returns are maximizing profits; the public monopoly is pricing at marginal cost, where potential losses are covered by the lump-sum taxes; and all markets clear.

Mantel (17), Beato (4), Brown-Heal (9), and Cornet (11) have demonstrated the existence of a socially efficient MCP equilibrium, under varying technical assumptions on the economy's aggregate technology.

Guesnerie (13), Mantel (17), and Brown-Heal (8) have constructed examples with increasing returns, where all of the socially efficient MCP equilibria are Pareto inefficient. Indeed, Beato and MasColell (5) have constructed an example of an economy with increasing returns where all of the MCP equilibria are in the interior of the social production possibility set.

In all these models, the income distribution is fixed.

Brown-Heal (8) have given conditions for some or all of the MCP equilibria to be Pareto efficient; in addition, they have shown that any Pareto optimal allocation in an economy with increasing returns can be supported as a MCP equilibrium subject to an appropriate redistribution of endowments and share holdings in firms. For a discussion of related results, see Beato (3).

An average cost pricing equilibrium is a family of consumption plans, production plans and prices such that households are maximizing utility subject to their budget constraint; firms with constant or decreasing returns are maximizing profits; the public monopoly is pricing at average cost, i.e., breaking even or making zero profits; and all markets clear.

The optimality properties of an average cost pricing equilibrium have been well studied in the literature of the second best; in fact, Boiteaux's original article is the seminal general equilibrium contribution to that
literature. For a recent treatment of the optimality issues, we suggest Guesnerie (14).

Surprisingly, the existence question, in a general equilibrium model with increasing returns, has received little attention, with the exception of Scarf (18) and Brown-Heal (10). Unfortunately, all of the extant proofs of existence of a MCP or an ACP equilibrium are somewhat technical in nature and lack the transparency of counting equations and unknowns which many economists accept as an intuitive, if not formally correct, proof of existence.

One purpose of this paper is to demonstrate the existence of a MCP and an ACP equilibrium in a simple economy with increasing returns, where the equilibrium notions are characterized by systems of behavioral equations and market clearing conditions.

We give both an intuitive proof of existence by counting equations and unknowns; and a formal argument that these systems of equations have a solution by use of a simple fixed-point argument. This argument depends only on the obvious geometrical fact that a continuous function on the interval [0, 1] which assumes both a negative and a positive value on the interval must cross the x-axis. Our formal proof contains all the essential ideas which underlie the existence proofs in the literature.

In addition, we review several of the standard partial equilibrium prescriptions for the regulation of a public monopoly and show that in a general equilibrium model they can be interpreted as MCP or ACP equilibria.

I. The Model

Our model will be the neoclassical two-input, two-output, two-household, two-firm economy where inputs are inelastically supplied. The inputs are
capital \((K)\) and labor \((L)\). The outputs are corn \((C)\) and electricity \((E)\). Each household has a utility function which we shall denote as \(U_x\) and \(U_y\). Endowments and shareholdings in firms are given by \((K_x, L_x), (K_y, L_y); (\theta_xC, \theta_xE), (\theta_yC, \theta_yE)\). Each firm has a production function which we denote as \(F_C\) and \(F_E\) or equivalently cost functions \(e_C\) and \(e_E\). Let \(K = K_x + K_y\) and \(L = L_x + L_y\).

We make the same assumptions regarding firms and households as does Bator in his classic expository piece on welfare economics (1), with one exception: we do not assume constant returns to scale, although firms are assumed to exhibit diminishing marginal rate of substitution along any isoquant, i.e., the markets for inputs are competitive.

Under these assumptions, we construct the Edgeworth-Bowley box for production and the social production possibility frontier, \(eff\), (see Figures 1 and 2). In general, the social production possibility frontier is non-concave.

![Figure 1](image1.png)

![Figure 2](image2.png)

Let \(P_C\) and \(P_E\) denote the prices of corn and electricity, and \(w\)
and \( r \) denote the prices of labor and capital. The marginal rate of transformation (MRT) at a point \((\tilde{C}, \tilde{E})\) on the social production possibility frontier is simply the slope of the frontier at that point and will be denoted \( P_{E}/P_{C} \).

A point \((\tilde{C}, \tilde{E})\) is said to be socially efficient if it lies on this frontier.

Each point \((\tilde{C}, \tilde{E})\) on the frontier also determines a unique point in the Edgeworth-Bowley box for production, i.e., the point on the efficiency locus corresponding to the tangency of the isoquants defined by
\[
F_{E}(L_{E}, K_{E}) = \tilde{E} \quad \text{and} \quad F_{C}(L_{C}, K_{C}) = \tilde{C}.
\]
The slope of their common tangent line at this point will be denoted as \( w/r \) and is the marginal rate of technical substitution (MRTS) at this point.

We shall use repeatedly that the MRT at a point \((\tilde{C}, \tilde{E})\) is the ratio of the marginal costs. That is,
\[
-P_{E}/P_{C} = \frac{\partial E_{E}(w/r, \tilde{E})}{\partial E} / \frac{\partial E_{C}(w/r, \tilde{C})}{\partial C}.
\]

II. Marginal Cost Pricing

As is common in this literature, we assume that each household's income at the prevailing prices is a fixed proportion of GNP. This assumption, which is called a fixed structure of revenues, guarantees that households have positive after-tax income.

Formally, a fixed structure of revenues is defined as follows: there are fixed \( \alpha_{x} \) and \( \alpha_{y} \) where \( \alpha_{x}, \alpha_{y} > 0 \) and \( \alpha_{x} + \alpha_{y} = 1 \) such that
\[
(K_{x}, L_{x}) = \alpha_{x}(k, l) \quad \alpha_{x} = \Theta_{x}C = \Theta_{x}E \quad ; \quad (K_{y}, L_{y}) = \alpha_{y}(k, l) \quad \alpha_{y} = \Theta_{y}C = \Theta_{y}E.
\]
Hence, the income of household $x$, $I_x = wL_x + rK_x + \Theta_{xC} (p_C - wL_C - rK_C) + \Theta_{xE} (p_E - wL_E - rK_E) = \alpha_x (wL + rK) + \alpha_x (p_C + p_E - w(L_C + L_E)) - r(K_C + K_E) = \alpha_x (wL + rK) + \alpha_x (p_C + p_E - wL - rK) = \alpha_x (p_C + p_E)$, if production is socially efficient. Similarly, the income of household $y$, $I_y = \alpha_y (p_C + p_E)$. Consequently, a household's income depends only on the relative product prices and outputs of firms.

Note that each agent's income is positive and is net of the lump sum taxes necessary to cover the losses of firms producing with increasing returns. Hence, if electricity is produced with increasing returns then $\Theta_{xC} (p_E - wL_E - rK_C)$ is the lump-sum tax imposed on household $x$. Another interpretation of the lump-sum taxes is that the shareholdings carry unlimited liability.

A household's demand for products derive utility maximization subject to its budget constraint:

(1) $\frac{\partial U_x}{\partial E_x} / \frac{\partial U_x}{\partial C_x} = \frac{p_E}{p_C}$

(2) $p_E x_x + p_C C_x = \alpha_x (p_E E + p_C C)$

(3) $\frac{\partial U_y}{\partial E_y} / \frac{\partial U_y}{\partial C_y} = \frac{p_E}{p_C}$

(4) $p_E y_y + p_C C_y = \alpha_y (p_E E + p_C C)$

A firm's demand for factors derive from cost minimization subject to its output constraint:

(5) $\frac{\partial F_E}{\partial L_E} / \frac{\partial F_E}{\partial K_E} = w/r$

(6) $F_E (L_E, K_E) = E$
(7) \[
\frac{\partial F_C}{\partial L_C} / \frac{\partial F_C}{\partial K_C} = \frac{w}{r}
\]

(8) \[
F_C(L_C, K_C) = C
\]

Equilibrium is defined as a set of relative prices \( P_E / P_C \) and \( w/r \); produce demands \( E_x, C_x \) and \( E_y, C_y \); factor demands \( L_E, K_E \) and \( L_C, K_C \); and output levels \( E \) and \( C \), such that all market clear. That is,

Product Markets: \( (9) \quad E_x + E_y = E \) \( (10) \quad C_x + C_y = C \)

Factor Markets: \( (11) \quad L_E + L_C = L \) \( (12) \quad K_E + K_C = K \)

We only need two rather than three relative prices because of the fixed income distribution assumption.

Hence, a MCP equilibrium in this economy is characterized by a system of twelve equations in twelve unknowns. We now use a fixed-point argument to demonstrate the existence of a solution to this system of equations.

First, a lemma.

**Lemma:** A continuous map, \( f(x) \), of a compact interval of the real line into itself has a fixed point.

**Proof:** Let the interval be \([-1, 1]\) and \( f: [-1, 1] \rightarrow [-1, 1] \).

Let \( g(x) = f(x) - x \), then \( g(-1) > 0 \) and \( g(1) < 0 \). Hence, there is some \( x \in [-1, 1] \) such that \( g(x) = 0 \), i.e., \( f(x) = x \).

The geometry of this proof is given in Figure 3.
The social production possibility frontier, eff, can be "stretched" onto a compact interval of the real line without tearing it. More formally, eff is homeomorphic to a compact interval of the real line. Therefore, by the lemma, any continuous map, \( \Phi \), of eff into itself will have a fixed point (see (9) for details).

Consider the following continuous map, \( \Phi \), of eff into eff:

\[
(E_1, C_1) \rightarrow P_E / P_C \rightarrow (E_2, C_2) \rightarrow (E_3, C_3),
\]

where

(i) \( (E_1, C_1) \) is an arbitrary point on eff

(ii) \( P_{E_1} / P_{C_1} \) is the MRT at \( (E_1, C_1) \)
(iii) \((E_2, C_2)\) is the aggregate demand at relative product prices \(P_{E_1}/P_{C_1}\), given production outputs \(E_1\) and \(C_1\).

(iv) \((E_3, C_3)\) is the intersection of the ray from the origin through \((E_2, C_2)\) and eff—under our assumptions on the technology, this intersection is unique.

We illustrate this construction in Figure 4:

![Figure 4](image)

Note that \((E_2, C_2)\) lies on the line through the point \((E_1, C_1)\) with slope \(P_{E_1}/P_{C_1}\), by Walras' law. We can now prove the following theorem.
Theorem (1): A socially efficient MCP equilibrium exists, i.e., equations (1) through (12) have a solution, which is socially efficient.

Proof: Let \((E^*, C^*)\) be a fixed-point of the map \(\Phi\). At such a point \((E_1, C_1) = (E_2, C_2) = (E_3, C_3) = (E^*, C^*)\). Hence, demand, \((E_2, C_2) = (E_1, C_1)\), supply, at the relative product prices \(p^*_E/p^*_C\), the MRT at \((E^*, C^*)\). The equilibrium relative prices in the factor markets, \(w^*/r^*\), is the MRTS at the tangency of \(F_E(L^*_E, K^*_E) = E^*\) and \(F_C(L^*_C, K^*_C) = C^*\) in the Edgeworth Bowley box for production. This point of tangency gives us the equilibrium values of \(L^*_E, K^*_E\) and \(L^*_C, K^*_C\); where \(L^*_E + L^*_C = L\) and \(K^*_E + K^*_C = K\). Finally, the household demands \(C^*_x, E^*_x\) and \(C^*_y, E^*_y\) total to the aggregate demand \(C^*\) and \(E^*\). This completes the proof.

Note that the existence of a socially inefficient MCP equilibrium, in this model, is precluded by the assumption that inputs are inelastically supplied (see (5)).

III. Average Cost Pricing

Initially, we assume that only electricity is produced with increasing returns and is priced at average costs; while corn is produced with constant or decreasing returns and is priced at marginal cost. Consequently, the income of household \(x\), \(I_x = wL_x + rK_x + \Theta_{xc}(p_C - wL_C - rK_C)\).

Similarly, the income of household \(y\), \(I_y = wL_y + rK_y + \Theta_{yc}(p_C - wL_C - rK_C)\).

Note that we do not assume a fixed schedule of revenues in this section of the paper.

The system of equations describing an ACP equilibrium differ from
the MCP equilibrium equations in the following manner: Equations (1), (3), (6), (7), (8), (9), (10), and (12) remain the same. We shall denote the new equations with primes.

\[(2') \quad P_{E} x + P_{C} x = wL_x + rK_x + \Theta xC (P_{C} - wL_C - rK_C)\]

\[(4') \quad P_{E} y + P_{C} y = wL_y + rK_y + \Theta yC (P_{C} - wL_C - rK_C)\]

\[(5') \quad P_{E} E = wL_E + rK_E\]

Equilibrium is defined as a set of relative prices $P_{E}/r$, $P_{C}/r$, and $w/r$; product demands $E_x$, $C_x$ and $E_y$, $C_y$; factor demands $L_E$, $K_E$ and $L_C$, $K_C$; and output levels $E$ and $C$, such that all markets clear. Hence, an ACP equilibrium in this model, is characterized by a system of 12 equations in 13 unknowns. In general, such a system has a one parameter family of solutions—more formally, a one dimensional manifold of solutions. The natural parameterization is with respect to the price of output in the regulated market.

The problem of the second best is to maximize the social welfare function over this manifold of ACP equilibria. Relative prices in a ACP equilibrium which satisfy the first order conditions of this maximization problem are the so-called Ramsey prices (see (2)).

Again, we shall invoke a fixed point argument to prove the existence of an ACP equilibrium. First, we add another equation to give us a system of thirteen equations in thirteen unknowns.

\[(13') \quad (E, C) \in \text{eff}\]
Consider the following continuous map, \( \Psi \), of eff into eff:

\[(E_1, C_1) \rightarrow \frac{(w/r)}{P_{E_1} / r, P_{C_1} / r} \rightarrow (E_2, C_2) \rightarrow (E_3, C_3) \], where

(i) \( (E_1, C_1) \) is an arbitrary point on eff

(ii) \( w/r \) is the MRTS at the tangency of \( F_E(L_{E_1}, K_{E_1}) = E_1 \) and \( F_C(L_{C_1}, K_{C_1}) = C_1 \) in the Edgeworth-Bowley box for production.

(iii) \( P_{E_1} / r = e_E(w/r, E_1) / E_1 \)

(iv) \( P_{C_1} / r = \frac{\partial e_C(w/r, C_1)}{\partial C} \)

(v) \( (E_2, C_2) \) is the aggregate demand at the relative prices \( w/r, P_{E_1} / r, P_{C_1} / r \) given production outputs \( E_1 \) and \( C_1 \).

(vi) \( (E_3, C_3) \) is the intersection of the ray from the origin through \( (E_2, C_2) \) and eff.

**Theorem (2):** A socially efficient ACP equilibrium exists.

**Proof:** Let \( (E^*, C^*) \) be a fixed point of the map \( \Psi \), then use the argument in the proof of Theorem (1) to complete the proof.

Finally, if both firms produce with increasing returns to scale, then replacing equation (7) by equation (7'); \( P_C = wL_C + rK_C \); and making
an obvious change in the definition of $\Psi$, i.e., (iv) is now
\[ p_{c1} / r = e_c(w/r, C), \]
we can prove the following theorem.

**Theorem (3):** A socially efficient ACP equilibrium exists, where each firm breaks even.

**IV. Regulation**

In this last section of the paper, we review several of the standard partial equilibrium policy prescriptions for regulating a public monopoly. In the present general equilibrium model, they are simply decentralized interpretations of a MCP or an ACP equilibrium, and their consistency and social efficiency are therefore assured by Theorems 1 and 2.

We consider first the policies of Lange and Lerner. The interested reader may consult (16) for a more detailed discussion. Lange proposed that the public monopoly, electricity (E), be given the desired output $E^\star$ which it should produce at minimum cost and sell at average cost, subject to the prevailing relative factor prices $w^\star/r^\star$. Although this policy was put forward by Lange as an application of the marginal cost pricing principal, it is clearly only consistent with the notion of an ACP equilibrium, if electricity is produced with increasing returns to scale.

Later, Lerner modified Lange's proposal by suggesting that the desired output $E^\star$ must be sold at marginal cost, if the pricing rule is to satisfy the necessary conditions for Pareto optimality. The appropriate notion of equilibrium in this case is that of a MCP equilibrium.

Turning to more current policy prescriptions, we note that the primary activity of a public utility's regulatory commission is the setting of
rates, i.e., prices. A common prescription is to fix the rate of return for the public monopoly; require it to meet all demand; and have it produce efficiently.

In our model, this corresponds to an ACP equilibrium where the regulator sets the price \( P^* \); requires the public monopoly to meet all demand; and to produce the demand, \( E^* \), at minimum cost. In this case, the public monopoly makes normal economic profits, i.e., breaks even.

Another regulatory policy, which one sees in real world markets, e.g., British Rail, is that the public monopoly, electricity (E), is first given a subsidy, \( S \), and then required to price at marginal cost subject to a break-even constraint. That is, given the prevailing factor price ratio \( w/r \) find an output \( E \) which satisfies the following equation:

\[
(15) \quad e_E(w/r, E) - E \frac{\partial e_E(w/r, E)}{\partial E} = S
\]

and sell \( E \) at marginal cost.

The efficacy of this regulatory policy in a general equilibrium model reduces to a question of the existence of a subsidy \( S \) such that the output and price of the public monopoly are consistent with the utility maximizing behavior of households; the profit maximizing behavior of competitive firms; and the clearing of product and factor markets.

If \( E \) is produced with decreasing marginal costs, then (15) has at most one solution. Note that if \( F_E(L_E, K_E) \), the production function for electricity, is homogeneous of degree \( r \), where \( r > 1 \), then \( e_E(w/r, E) \) is concave, e.g., \( F_E \) is Cobb-Douglas with increasing returns to scale.
Clearly, any MCP equilibrium implicitly defines a subsidy $S^*$ with the desired properties. Simply let

$$S^* = e^*_E(w^*/r^*, E^*) - E^* \frac{\partial e^*_E(w^*/r^*, E^*)}{\partial E},$$

where $w^*/r^*$ and $E^*$ are the MCP equilibrium values of $w/r$ and $E$. Note that $S^*$ is raised through lump-sum taxation.

It should be noted that all these regulatory policies lack behavioral incentives, both in the MCP or ACP interpretations, and that the development of incentive compatible variants is an important and difficult task.

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