STOCHASTIC GAMES II: THE MINMAX THEOREM

by

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1. **INTRODUCTION**

A two-person, zero-sum stochastic game consists of a (finite) set $S$ of states; each state $S$ is a (finite) matrix game. The entries of these matrices consist of

1) a payoff (from the column-chooser, $B$, to the row-chooser, $A$) and

2) a lottery on $S$, determining which state will be played next.

Shapley [1953] introduced this concept, studying stochastic games which terminate with probability 1 after finitely many steps; equivalently, these games could be thought of as infinite in duration, but with a non-zero discount rate. In this case the min-max theorem is straightforward (Shapley [1953], Monash [1979, 1981]). Gillette [1957] studied stochastic games with zero stop probabilities, establishing the min-max theorem in a couple of special cases. In these cases, the optimal strategies are stationary (i.e., dependent only upon the current state, rather than the history); thus the game "should" go into a Markov chain. The payoff can be defined either as the Cesàro limit $\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} d_i$ or the Abel limit $\lim_{r \to 0} \sum_{i=1}^{\infty} d_i (1-r)^{i-1}$, where $d_i$ is the payoff on the $i^{th}$ play, since, with best play, these limits exist and are equal (compare Royden [1963]).

In The Big Match, Blackwell and Ferguson [1968] considered a more difficult example. Although this game still has a value, it cannot be guaranteed by stationary strategies; furthermore, no strategy is better than $\epsilon$-optimal. Extending these methods, Bewley and Kohlberg [1976] showed that the Cesàro limit of the values of the $N$-stage games exists, and equals the Abel limit of the values of the $r$-discounted games;
furthermore, no strategy for either player can guarantee an average payoff (in any sense) better than this number \( v_\infty \). Thus \( v_\infty \) is the only candidate for min-max value. Finally, the min-max theorem for stochastic games was proved by Monash [1979] and independently by Mertens and Neyman [1980]. This paper is a revision of Monash [1979].

2. DEFINITIONS

Without loss of generality, a stochastic game can be described by finite sets \( S, A, B, C \) and measurable functions

\[
d : S \times B \times B \times C \rightarrow [-\bar{N}, \bar{N}],
\]

\[
s : S \times A \times B \times C \rightarrow S, \quad \text{and}
\]

\[
q : [0,1] \rightarrow C
\]

such that:

1) \( S \) is the state space;

2) Player A (resp. B) chooses a move from his choice set A (resp. B);

3) \( s \), composed with \( q \), reproduces the lottery in each entry of each state matrix; and

4) \( d \) is the payoff function.

A state \( s^* \in S \) is absorbing if \( \Delta(s^*, a, b, c) = s^* \) for all \( a, b, c \) and \( d(s^*, a, b, c) = v(s^*) \), a constant. \( S^* \subseteq S \) is the set of absorbing states \( S_\infty = S - S^* \).

A play of the game is just a sequence \( s_0, a_1, b_1, c_1, s_1, a_2, b_2, c_2, s_2, \ldots \), where \( s_i = \Delta(s_{i-1}, a_i, b_i, c_i) \), for all \( i \): let \( d_i = d(s_{i-1}, a_i, b_i, c_i) \), the payoff on the \( i \)th turn. Writing \( t_i = (s_{i-1}, a_i, b_i, c_i) \in T = S \times A \times B \times C \), we denote a play by
\( t = (t_1, t_2, t_3, \ldots) \); thus

\[
T^\infty = S \times A \times B \times C \times S \times A \times B \times C \times S \times \ldots
\]

\[= \{\text{all possible plays}\}.\]

The subsequence \((t_1, \ldots, t_n)\) is denoted by \(t(n)\); we use this notation even if we are thinking of this subsequence as belonging to many different possible plays.

Strategies for \(A\) will always be denoted by \(\sigma\), and strategies for \(B\) by \(\tau\). These strategies will always be of the form

\[
\text{Prob}(a \in A \text{ (resp. } b \in B) \text{ on turn } k) = \text{function}(t_1, \ldots, t_{k-1}).
\]

Thus, by the Kolmogorov Extension Theorem (see Kolmogorov [1950] or Monash [1981]), a pair \((\sigma, \tau)\) determines a probability measure \(\mu(\sigma, \tau)\) on \(T^\infty\). Unless otherwise noted, all expectations below are with respect to this measure. Let

\[
T^x = \{t \in T : s_i \in S^x \text{ for some } i\},
\]

and \(T^\infty\) the complement. In the next section we write \(P(*) = \mu(T^x)\).

Following Bewley and Kohlberg [1976] or Monash [1981], recall that for all \(s \in S\), for all \(r \in (0,1)\), \(V_s(r)\) = the value of the \(r\)-discount game, starting in \(s\), satisfies

\[
V_s(r) = \text{val}(\exp(d(s,a,b,c) + (1-r)\sum_{s \in S} P(s)V_s(r))), \tag{2.1s}
\]

where \(P(s)\) = the probability that \(\delta(s,a,b,c) = s\), and \text{val} is the ordinary min-max value. For some \(r > 0\), all the \(V_s(r)\) are algebraic, as are the optimal strategies in the games (2.1s). Thus, on \((0, r)\),
\[ V_s(r) = V_\infty(s) \cdot r^{-\frac{1}{n}} + (\ldots)r^{-\frac{1}{n}} + \ldots \]
\[ = V_\infty(s) \cdot u^{-n} + (\ldots)u^{-n+1} + \ldots , \]

where \( u = r^{1/n} \). Let \( 0 < u < r^{1/n} \); on \((0,u)\), we write \( W_s(u) = V_s(u^n) \), so that \( \lim_{u \to 0^+} u^n W_s(u) = \lim_{r \to 0^+} r V_s(r) = V_\infty(s) \).

In Sections 4 through 6, we assume \( v_\infty(s) = 0 \) for all \( s \in S_{\infty} \).

In that case we have \( \lim_{u \to 0^+} u^{n-1} W_s(u) < \infty \) for all \( s \); thus, writing

\[ \overline{W}(u) = \max_{s \in S_{\infty}} \left| W_s(u) \right| , \]
we have \( \lim_{u \to 0^+} u^{n-1} \overline{W}(u) < \infty \), also.

3. STATEMENT OF THEOREM

Our main result is

**Theorem I:** For any starting state \( s_0 \in S \), for any \( \epsilon > 0 \), there exists a strategy \( \sigma \) for \( A \) such that, for any strategy \( \tau \) for \( B \),

\[ \lim_{N \to \infty} \inf \exp \left\{ \frac{1}{N} \sum_{i=1}^{N} d_i \right\} > v_\infty(s) - \epsilon . \]

Theorem I clearly follows from the following two propositions:

**Proposition 3.1:** Suppose, for all \( s \notin S^* \), \( v_\infty(s) = 0 \). Then the conclusion of Theorem I holds.

**Proposition 3.2:** Proposition 3.1 \( \iff \) Theorem I.

In this section, we prove Proposition 3.2; the remainder of the paper is devoted to Proposition 3.1.

The proof of Proposition 3.2 depends upon
Lemma 3.3: Let \( G \) be a stochastic game, with state set \( S \). Let \( H \) be another stochastic game, identical to \( G \) except for the following modification: Replace a single state \( x \in S \) by an absorbing state \( y \) such that \( v(y) = v_\infty(x) \). Then, for all \( s \in S \),

\[
v_{\infty, H}(s) = v_{\infty, G}(s),
\]

where \( v_{\infty, G}(s) \) (resp. \( v_{\infty, H}(s) \)) is simply \( v_\infty(s) \) in the game \( G \) (resp. \( H \)).

Proof: Let \( V_{G, s}(r) \) (resp. \( V_{H, s}(r) \)) be \( V_s(r) \) in the game \( G \) (resp. \( H \)). Define

\[
\hat{V}(r) = r^{-1}v_\infty(x) - V_{G, x}(r)
\]

\[
\overline{V}(r) = \min_{s \in S - \{x\}} (V_{H, s}(r) - V_{G, s}(r)).
\]

Then, for any \( s \in S - \{x\} \), (2.1s) gives

\[
V_{H, s}(r) = \operatorname{val}(\operatorname{Exp}(d(s, a, b, c))) + (1-r) \sum_{s \in S - \{y\}} p(s) \cdot V_{H, s}(r)
\]

\[
+ r^{-1} \cdot (1-r)p(y) \cdot v(y)
\]

\[
\geq \operatorname{val}(\operatorname{Exp}(d(s, a, b, c))) + (1-r) \sum_{s \in S} p(s) \cdot V_{G, s}(r)
\]

\[
+ (1-r) \min_{P \in [0,1]} ((1-p) \cdot \overline{V}(r) + p \cdot \hat{V}(r)),
\]

where \( P \) corresponds to \( P(x: a, b, s) \),

\[
v_{G, s}(r) + (1-r)((1-P^*) \cdot \overline{V}(r) + P^* \cdot \hat{V}(r))
\]

for some \( P^* \in [0,1] \).

Picking \( \hat{s} \) now so that
\[ V_{H,s}(r) - V_{G,s}(r) = \overline{V}(r) \text{ in some interval } [0, \overline{r}) , \]

\[ \overline{V}(r) = V_{H,s}(r) - V_{G,s}(r) \geq (1-r)((1-p*)\overline{V}(r) + p*\overline{\hat{V}}(r)) \]

\[ r\overline{V}(r) \geq (1-r)p*(\overline{\hat{V}}(r) - \overline{V}(r)) . \]

So either \( \overline{V}(r) \geq 0 \), or \( \overline{\hat{V}}(r) - \overline{V}(r) < 0 \). In either case,

\[ \min_{s \in S - \{x\}} (v_{\infty,s}^H(s) - v_{\infty,s}^G(s)) \]

\[ = \min_{s \in S - \{x\}} \lim_{r \to 0^+} (rV_{H,s}(r) - rV_{G,s}(r)) \]

\[ = \lim_{r \to 0^+} r \min_{s \in S - \{x\}} (V_{H,s}(r) - V_{G,s}(r)) \]

\[ = \lim_{r \to 0^+} r\overline{V}(r) \]

\[ \geq 0 \text{ or } \geq \lim_{r \to 0^+} r\overline{\hat{V}}(r) = 0 . \]

So, for all \( s \in S - \{x\} \), \( v_{\infty,s}^H(s) - v_{\infty,s}^G(s) \geq 0 \); that is,

\[ v_{\infty,s}^H(s) \geq v_{\infty,s}^G(s) . \]

But, by symmetry (i.e., interchanging the names A and B ),

\[ -v_{\infty,s}^H(s) \geq -v_{\infty,s}^G(s) . \]

Hence \( v_{\infty,s}^H(s) = v_{\infty,s}^G(s) \), for all \( s \in S - \{x\} \).

Since Lemma 3.3 is clearly true for state \( x \), we are done. \( \Box \)
Proof of Proposition 3.2:

We now proceed by induction on \(|S_\infty|\), the number of non-absorbing states.

\(|S_\infty| = 0\). Trivially true.

So assume for \(|S_\infty| - 1\), and prove for \(|S_\infty|\).

To every state \(s\), associate a number \(\alpha(s)\) such that

\[
\nu_\infty(s) - \alpha(s) = \sup_\sigma \inf_\tau \lim_{N \to \infty} \inf_{i=1}^{N} \exp(d_i).
\]

By the Bewley-Kohlberg result, \(\alpha(s) \geq 0\) for all \(s \in S\).

Want to show: \(\alpha(s) = 0\) for all \(s\). If so, done.

So suppose otherwise.

**Definition:** We will call a strategy \(\sigma\), starting in state \(s\), \(\varepsilon\)-optimal (for \(s\)) if

\[
\inf_\tau \lim \inf_{N} \sum_{i=1}^{N} \exp(d_i) \geq \nu_\infty(s) - \varepsilon.
\]

**Case 1:** There exist states \(s_1\), \(s_2\) such that \(\alpha(s_1) > \alpha(s_2) > 0\).

Let \(\varepsilon = \frac{1}{3}(\alpha(s_1) - \alpha(s_2))\).

Consider the modified game \(H\), where \(s_2\) is replaced by an absorbing state \(y\) such that \(\nu(y) = \nu_\infty(s_2)\). Then \(H\), by induction, has an \(\varepsilon\)-optimal strategy, for any initial state.

Consider, then, the following strategy, for the game \(G\) starting in state \(s_1\): Play the \(\varepsilon\)-optimal strategy for \(H\), until "absorbed" in "y"; this is meaningful because \(G\) and \(H\) are identical outside
of state \( s_2 \). Once in \( s_2 \), play an \((\alpha(s_2) + \epsilon)\)-optimal strategy, which exists by the definition of \( \alpha(s_2) \). Then this strategy is clearly

\[(\alpha(s_2) + 2\epsilon)\)-optimal for \( s_1 \).

But

\[\alpha(s_2) + 2\epsilon = \frac{2}{3}\alpha(s_1) + \frac{1}{3}\alpha(s_2) < \alpha(s_1),\]

contradicting the definition of \( \alpha(s_1) \).

Hence the only possibility is:

\textbf{Case 2:} There exists \( \bar{\alpha} > 0 \) such that, for all \( s \in S_\infty \), \( \alpha(s) = \bar{\alpha} \).

Now, let \( v_0 = \min_{s \in S_\infty} v_\infty(s) \).

Let \( S_0 \subseteq S_\infty \) be \( \{ s \in S_\infty : v_\infty(s) = v_0 \} \); let \( \hat{S} \) be the complement.

\textbf{Case 2a:} \( \hat{S} \) is non-empty. Then let \( v_1 = \min_{s \in \hat{S}} (v_\infty(s)) \); \( v_1 > v_0 \). Let

\[\beta = \frac{v_1 - v_0}{2(v_1 + N)} \leq \frac{1}{2} .\]

By repeated applications of Lemma 3.3, replace the states in \( \hat{S} \) by absorbing states with the same \( v_\infty \). Then the states in \( S_0 \) still have value \( v_0 \). Assuming Proposition 3.1, this new game has an \( \epsilon \)-optimal strategy, where

\[\epsilon = \min \left\{ \frac{v_1 - v_0}{4}, \frac{\beta \bar{\alpha}}{4} \right\} .\]

Play this strategy until "absorbed," and an \((\alpha + \epsilon)\)-optimal strategy thereafter (unless the "absorption" is genuine). Fixing (any) \( \tau \), we have two cases:
Case 1: Expected value if "absorbed" $\geq \frac{v_0 + v_1}{2}$ or $P(\tau) = 0$. Then

$$
\liminf_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \exp(d_i) \geq P(\tau) \left( \frac{v_0 + v_1}{2} - (\bar{\alpha} + \varepsilon) \right) + (1 - P(\tau))(v_0 - \varepsilon) \\
= v_0 - \bar{\alpha} + P(\tau) \left( \frac{v_1 - v_0}{2} - \varepsilon \right) + (1 - P(\tau))(\bar{\alpha} - \varepsilon) \\
\geq v_0 - \bar{\alpha} + \min \left\{ \frac{v_1 - v_0}{4}, \frac{7\bar{\alpha}}{8} \right\};
$$

since $\tau$ was arbitrary, this contradicts the definition of $\bar{\alpha}$.

Case 2: Expected value if "absorbed" $< \frac{v_0 + v_1}{2}$ and $P(\tau) > 0$.

Let $\gamma = \frac{\text{prob(genuine absorption)}}{P(\tau)}$. Then

$$
\frac{v_0 + v_1}{2} > \text{Expected value if "absorbed"} \\
\geq \gamma(-\tilde{N}) + (1-\gamma)v_1; \text{ i.e.,} \\
v_1 - \frac{v_1 - v_0}{2} > v_1 - \gamma(v_1 + \tilde{N}) \\
\gamma > \frac{(v_1 - v_0)}{2(v_1 + \tilde{N})} = \beta.
$$

Hence

$$
\liminf_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \exp(d_i) \geq v_0 - \varepsilon - P(\tau)(1-\gamma)(\bar{\alpha} + \varepsilon) \\
\geq v_0 - \varepsilon - (1-\beta)(\bar{\alpha} + \varepsilon) \\
> v_0 - \bar{\alpha} + (\beta \bar{\alpha} - 2\varepsilon);
$$

since $\varepsilon \leq \frac{\beta \bar{\alpha}}{4}$, this is a contradiction.
Case 2b: $\mathcal{S} = \emptyset$.

Deducting $v_0$ from all payoffs, this is exactly the case of Proposition 3.1. Hence there exists an $\frac{a}{2}$-optimal strategy, for our final contradiction.

So $\alpha(s) = 0$ for all $s \in S^*$.

But this is exactly what we wanted to prove.

\[\square\]

4. **Preliminary Computations**

For the rest of this paper, we will assume $v_\infty(s) = 0$ for all $s \in S_\infty$. We will always choose A's strategy $\sigma$ to be $\text{Prob}(a) = f(a, s, u) = \text{the optimal (stationary) strategy in the } u^n\text{-discount game, for some } u$, in the current state $s$. Without loss of generality (see Monash [1979] or [1981]), B's strategy $\tau$ is pure: $b_k = \text{function}(\ell(k-1))$.

Let us now focus on one move of the game. Fix $s \in S_\infty$, $u \in (0, u)$, and $b \in B$, with A playing strategy $\{f(a, s, u)\}$. Let $p_\ast(u) = \text{Prob}(s, a, b, c) \in S^*)$, given the probability distributions $f(a, s, u)$ on A and $q$ on C. In Sections 5 and 6, if a play $\ell$ is understood along with a sequence of $u$'s, we will let

$$p_\ast(i) = \begin{cases} p_\ast(u) \text{ on the } i^{\text{th}} \text{ turn, if } s_{i-1} \in S_\infty \\ 0 \text{, if } s_{i-1} \in S^* \end{cases}$$

Meanwhile, let $\mathcal{S} = \delta(s, a, b, c)$.

We distinguish three cases:

1) $p_\ast(u) \equiv 0$ on $(0, u)$;

2) Not 1, and
\[ \lim_{u \to 0^+} \exp(v_{\infty}(s) : \bar{s} \in S^*) = 0 ; \]

3) Not 1 or 2.

We further distinguish between:

A. Either Case 1, or \( \text{order}(P_*(u)) \geq n ; \)

B. Not Case 1, and \( \text{order}(P_*(u)) \leq n-1 . \)

Observe that \( P_*(u) \) is a rational function of \( u \), and thus has finitely many zeroes; without loss of generality, none of them occur on \((0, \bar{u}) \). Define \( \delta(u) \) as follows (where we suppress the dependence upon \( s \) and \( b \)):

If Case A, then

\[ \delta(u) = -\exp(v_{\infty}(s) : \bar{s} \in S^*) \cdot P_*(u) \cdot u^{-n} ; \]

if Case B, then

\[ \delta(u) = -\exp(v_{\infty}(s) : \bar{s} \in S^*) \cdot P_*(u) + (1 - (1 - u^n)(1 - P_*(u))W_s(u) \cdot u^{-n} . \]

The point of this definition may be found in the following propositions (where we write \( \exp(d : S_\infty) \) for \( \exp(d : \bar{s} \in S_\infty) \), and so forth):

**Proposition 4.1:** \( \exp(d : S_\infty)(1 - P_*(u)) \geq \delta(u) - \exp(W_s - W_s(u) : S_\infty)(1 - P_*(u)) + \eta(u) \), for \( u \in (0, \bar{u}) \), where \( \lim_{u \to 0^+} \eta(u) = 0 \).

and
Proposition 4.2:

1. If Case 1 (above), then \( \delta(u) = 0 \) and \( P_\star(u) \cdot \exp(v_\infty(s)) : S_\star = 0 \);

2. If Case 2, then

\[
\left| \exp(v_\infty(s) : S_\star) \right| < o(u^0)
\]

and

\[
\left| \frac{\delta(u) \cdot u^n}{P_\star(u)} \right| < o(u^0) .
\]

3. If Case 3,

\[
\frac{-P_\star(u) \cdot \exp(v_\infty(s) : S_\star) \cdot u^{-n}}{\delta(u)} = 1 + o(u^0) .
\]

From Equation (2.1s), we have

\[
W_S(u) \leq (1 - P_\star(u)) \cdot (\exp(d : S_\infty) + (1 - u^n) \cdot \exp(W_S(u) : S_\infty))
\]

\[
+ P_\star(u) \cdot \exp(v_\infty(s) : S_\star) u^{-n} .
\]

(4.3)

Proof of Proposition 4.1:

Rearranging (4.3), we have

\[
(1 - P_\star(u)) \exp(d : S_\infty) \geq (1 - (1 - P_\star(u))(1 - u^n)) W_S(u)
\]

\[
- (1 - P_\star(u))(1 - u^n) \exp(W_S(u) - W_S(u) : S_\infty)
\]

\[
- P_\star(u) \exp(v_\infty(s) : S_\star)
\]

(4.4)

If Case A holds, then

\[
(4.4) = \delta(u) - (1 - P_\star(u)) \exp(W_S(u) - W_S(u) : S_\infty)
\]

\[
+ (P_\star(u) + u^n - u^nP_\star(u)) W_S(u) + u^n(1 - P_\star(u))
\]

\[
\cdot \exp(W_S(u) - W_S(u) : S_\infty) .
\]

(4.5)
If Case B holds, then (4.4) equals

\[ \delta(u) - (1 - P_\ast(u)) \cdot \exp(W_s(u) - W_s(u) : S_\infty) \]
\[ + u^n (1 - P_\ast(u)) \cdot \exp(W_s(u) - W_s(u) : S_\infty) . \quad (4.6) \]

Let \( \overline{F} > 0 \) be such that \( |P_\ast(u)| \leq \overline{F} u^n \) whenever Case A holds. Writing

\[ n(u) = -(\overline{P} + 4) u^n W(u) , \]

and observing that

\[ (4.5) \geq \delta(u) - (1 - P_\ast(u)) \cdot \exp(W_s - W_s(u) : S_\infty) + n(u) , \]
\[ (4.6) \geq \delta(u) - (1 - P_\ast(u)) \cdot \exp(W_s - W_s(u) : S_\infty) + n(u) \]

and \( \lim_{u \to 0^+} n(u) = 0 \),

we are done. \( \square \)

Proof of Proposition 4.2:

1. Suppose Case 1 holds: \( P_\ast(u) = 0 \). Then so does Case A, and

\[ \delta(u) = -\exp(v_\infty(s) : s \in S^\ast) \cdot P_\ast(u) \cdot u^{-n} \]
\[ = 0 \text{ for all } u , \]

and so done.

2. Suppose, then, Case 2 holds. Since \( \exp(v_\infty(s) : S^\ast) \) is a power series in \( u \), with limit 0 as \( u \to 0 \), it is indeed \( o(u^0) \). Now, if Case A, then
\[
\frac{\delta(u) \cdot u^n}{P_* (u)} = \left| -\exp(v_\infty(\bar{s}) : S*) \right| \< o(u^0); \\
\]

while, if Case B, then

\[
\frac{\delta(u) \cdot u^n}{P_* (u)} = \left| -\exp(v_\infty(s) : S*) + \frac{P_* (u) \cdot W_s(u) \cdot u^n}{P_* (u)} + \text{higher order terms} \right| \< o(u^0) + |u^n W_s(u)| + \text{higher order terms} \< o(u^0).
\]

3. Suppose Case 3 holds. If Case A, then

\[
-\frac{P_* (u) \cdot \exp(v_\infty(\bar{s}) : S*) \cdot u^{-n}}{\delta(u)} \equiv 1, \text{ by definition.}
\]

So suppose Case B: order \( P_* (u) \leq n-1 \). It is clearly enough to check

\[
\left| \frac{(P_* (u) + u^n - u^n P_* (u)) W_s(u)}{-P_* (u) \cdot \exp(v_\infty(\bar{s}) : S*) u^{-n}} \right| \< o(u^0).
\]

Then order \((P_* (u) + u^n - u^n P_* (u)) = \text{order } (P_* (u)) \)

order \((\exp(v_\infty(\bar{s}) : S*)) = 0\),

and so the order of the left-hand-side is

\[
\geq \text{order } (P_* (u)) + \text{order } (W_s(u)) - \text{order } (P_* (u)) - 0 - \text{order } (u^{-n}) \\
= \text{order } (u^n W_s(u)) \\
> 1.
\]

Hence done.
5. THE ABSORBING CASE

Recall that a fixed strategy pair \((\sigma, \tau)\) induces a probability measure \(\mu(\sigma, \tau)\) on \(T^\infty\), the space of all possible plays. If \(s_0 \in S^*\), Proposition 3.1 is trivial; thus it follows immediately from

**Proposition 5.1:** For any starting state \(s \in S^\infty\), for any \(\varepsilon > 0\), there exists a strategy \(\sigma\) for \(A\) such that

\[
\inf \liminf \int \frac{1}{N} \sum_{i=1}^{N} d_1 d \mu(\sigma, \tau) > -(6N + 3)\varepsilon.
\]

**Proof of Proposition 5.1:**

As remarked earlier, the strategy \(\sigma\) will be the form \(\text{Prob}(a) = f(a, s, u)\), the optimal strategy in the \(u^N\)-discount game, for \(u\) cleverly chosen. Specifically, writing \(u_0\) for the \(u\) prevailing on the \((N + 1)^{st}\) move, we set \(u_0 = u_0(1 - \frac{1}{2}\varepsilon)^{\nu(N)}\), for \(u_0\) sufficiently small and \(\nu(N)\) a non-negative integer depending upon the history of the first \(N - 1\) moves.

Write \(q = 1 - \frac{1}{2}\varepsilon\). Recalling Proposition 4.2, choose \(R > 0\) and \(\nu\) sufficiently small so that each \(\sigma(u^0)\) is \(< Ru\). Assume \(\varepsilon < 1\).

Then \(u_0 \in (0, \nu) \subseteq (0, 1)\) must satisfy the following four conditions:

1. For every \(u \in (0, u^0], n(0) > -\varepsilon\)
2. For every \(u \in (0, u^0], w(u) < \frac{\varepsilon}{4} u\)
3. \(Ru_0 < \varepsilon\)
4. \((1 + \varepsilon)^3 \cdot \frac{u_0^{1/2}}{1 - q^{1/2}} < \varepsilon\).

To define \(\nu(N)\), we first define a set of benchmarks \(\tilde{\nu}\) on \(1, 0, 1, 2, ...\) by:
\[ m(-1) = -\infty \]
\[ \tilde{m}(0) = 0 \]
\[ \tilde{m}(i) = \tilde{m}(i-1) + (u_0q^{i-1})^{-n+\frac{1}{2}}, \text{ for } i = 1, 2, 3, \ldots \]

Next, define sequences \( \overline{m}_0, \overline{m}_1, \overline{m}_2, \ldots \) and \( \mathcal{L} = (\xi_0, \xi_1, \xi_2, \ldots) \), \( \mathcal{L} \) increasing, in conjunction with the sequences \( u_0, u_1, u_1, \ldots \) and \( \nu(0), \nu(1), \nu(2), \ldots \) by:

1) \( \overline{m}_0 = 0 \)

2) \( \nu(0) = 0 \)

3) \( u_N = u_0q^{\nu(N)} \) for \( N = 1, 2, 3, \ldots \)

4) If \( \overline{m}_{N-1} + \delta_N(u_{N-1}) > \tilde{m}(\nu(N-1) + 1) \), then \( \nu(N) = \nu(N-1) + 1 \) and \( N \in \mathcal{L} \); if \( \overline{m}_{N-1} + \delta_N(u_{N-1}) < \tilde{m}(\nu(N-1) - 1) \), then \( \nu(N) = \nu(N-1) - 1 \) and \( N \in \mathcal{L} \); otherwise \( \nu(N) = \nu(N-1) \) and \( N \notin \mathcal{L} \).

5) If \( N \notin \mathcal{L} \), then \( \overline{m}_N = \overline{m}_{N-1} + \delta_N(u_{N-1}) \).

6) If \( N = \xi_i \in \mathcal{L} \), then \( \overline{m}_N = \overline{m}_{N-1} + \delta_N(u_{N-1}) + W_{\xi_i} (u_{N-1})^{\xi_i-1} \).

Fix \( \sigma \) as above, and any (pure) \( \tau \). Proposition 5.1 follows instantly (by redefining \( \epsilon \)) from:

**Proposition 5.2:** \( \lim_{N \to \infty} \int_{T^*} \frac{1}{N} \sum_{i=1}^{N} d_i \, d\mu > -\epsilon \)

and

**Proposition 5.3:** \( \lim_{N \to \infty} \inf_{T^*} \frac{1}{N} \sum_{i=1}^{N} d_i \, d\mu > -\epsilon \).

We now prove Proposition 5.2, deferring Proposition 5.3 to the next section.
Proof of Proposition 5.2:

Let \( T_k = \{ t = (t_1, t_2, \ldots) \in T^*: s(t_k) \in S^* \) but \( s(t_{k-1}) \notin S^* \} \); thus \( T^* = T_1 \cup T_2 \cup T_3 \cup \ldots \).

So

\[
\lim_{N \to \infty} \int_{T^*} \frac{1}{N} \sum_{i=1}^{N} d_i \, du
\]

\[
= \lim_{N \to \infty} \sum_{k=1}^{\infty} \int_{T_k} \frac{1}{N} \sum_{i=1}^{N} d_i \, du
\]

\[
= \sum_{k=1}^{\infty} \lim_{N \to \infty} \int_{T_k} \frac{1}{N} \sum_{i=1}^{N} d_i \, du
\]

by the Lebesgue Dominated Convergence Theorem (Royden [1963]),

\[
= \sum_{k=1}^{\infty} \int_{T_k} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} d_i \, du
\]

\[
= \sum_{k=1}^{\infty} \int_{T_k} P_* (k) \cdot \exp(\nu_\infty(s_{t_k} \in T_k)) \, du.
\]

(5.4)

The following is a special case of Proposition 4.1 of Monash [1981]

(identifying \( Z^* = \delta^{-1}(S^*) \) for all \( i \)).

Proposition 5.5: There exists a probability measure \( \hat{\nu} \) on \( T \) such

that, for all \( N \), for all \( f_N : T^{N-1} \to \mathbb{R} \) such that \( s(t_{N-1}) \in S^* \)

implies \( f_N(t_1, \ldots, t_{N-1}) = 0 \),

\[
\int_{T^*} f_N(t(N-1)) \, d\hat{\nu}(t) = \int_{T^*} \frac{1}{N} \sum_{i=1}^{N-1} (1 - P_* (i)) \, d\hat{\nu}(t)
\]
Now, assume temporarily,

**Proposition 5.6:** For all $N$, for all $t$,

$$\sum_{k=1}^{N} (P^*(k) \cdot \exp(v_\infty(s_k)) : t \in T_k \cdot \prod_{i=1}^{k-1} (1 - P^*(i))) > -\varepsilon$$

Let

$$f_k(t(k-1)) = \begin{cases} P^*(k) \cdot \exp(v_\infty(s) : t \in T_k) & \text{if } \delta(t_{k-1}) \notin S^* \\ 0 & \text{if } \delta(t_{k-1}) \in S^* \end{cases}$$

Then $f_k$ satisfies the hypothesis of Proposition 5.5. Thus, for all $N$,

$$\sum_{k=1}^{N} P^*(k) \cdot \exp(v_\infty(s_k)) : t \in T_k$$

$$= \sum_{k=1}^{N} \int_{T^\infty} f_k(t(k-1)) d\nu$$

$$= \sum_{k=1}^{N} \int_{T^\infty} f_k(t(k-1)) \cdot \prod_{i=1}^{k-1} (1 - P^*(i)) d\nu$$

$$> \int_{T^\infty} ( - \varepsilon ) d\nu$$

as these are the partial sums of equation (5.4), this establishes Proposition 5.1.

So we pass to the
Proof of Proposition 5.6:

Fix \( t \) and \( N \). Recalling Proposition 4.2, we make the simplifying assumption that Cases 1 or 3 hold everywhere (for fullest detail see Monash [1979]); thus, for \( k = 1, \ldots, N \),

\[
either \quad \delta(u_{k-1}^{\top}) = P_*(k) \cdot \exp(v_{\infty}(s_k^\top)) \cdot T_k = 0
\]

or

\[
\left| -P_*(k) \cdot \exp(v_{\infty}(s_k^\top)) \cdot u_{k-1}^n \right| \cdot \frac{\delta(u_{k-1}^\top)}{\epsilon (1 - R_{k-1}^\top, 1 + R_{k-1}^\top)} \subseteq (1 - \epsilon, 1 + \epsilon) .
\]

(5.8)

Writing \( F_k = P_*(k) \cdot \exp(v_{\infty}(s_k^\top)) \cdot T_k \cdot \prod_{i=1}^{k-1} (1 - P_*(i)) \), our task is to bound \( \sum_{k=1}^{N} F_k \) below. We spread out this sum as the integral of a step function by defining \( A(z) \) on \([0, N] : A(z) = F([z] + 1) \), where \([z] \) is the usual greatest integer function. Thus \( \int_{0}^{N} A(z)dz = \sum_{k=1}^{N} F_k \).

First, observe that, for \( \ell_j \),

\[
\sum_{k=\ell_j+1}^{\ell_{j+1}-1} \frac{\delta_k(u_{j}^\top)}{\epsilon \left( 1 - \epsilon, 1 + \epsilon \right)} = 1 + \frac{\sum_{k=\ell_j+1}^{\ell_{j+1}} \delta_k(u_{j}^\top)}{\epsilon \left( 1 - \epsilon, 1 + \epsilon \right)} .
\]

since

\[
|W_{s_k^\top}(u_{j}^\top) - W_{s_k^\top}(u_{j}^\top)| \leq 2\overline{W}(u_{j}^\top) < \frac{\epsilon}{2} u_{j}^\top ,
\]

while

\[
\sum_{k=\ell_j+1}^{\ell_{j+1}} u_{j}^\top \approx u_{j}^\top , \text{ by definition.}
\]

Thus we can define a function \( m(z) \) on \([0, N] \) such that
Figure 1. Illustration of Lemma 5.7
1) \( m \) is linear on \([k, k+1]\), for \( k = 0, 1, \ldots, N-1 \).

2) \( m(\epsilon_j) = \bar{m}_j \), for \( \epsilon_j \in \mathcal{L} \).

3) \( \frac{m(k+1) - m(k)}{\epsilon_{k+1}(u_k)} \in (1-\epsilon, 1+\epsilon) \), for \( k = 0, 1, \ldots, N-1 \). (5.9)

We now want a finite, increasing sequence \( J = \{0 = j_0, j_1, j_2, \ldots, N\} \), containing every integer \( 0, 1, \ldots, N \); \( J \) should be partitioned into five sets, with the following properties:

1) For \( j_1 \in J_1 \), \( m \) is constant on \([j_1, j_{i+1}]\).

2) For \( j_1 \in J_2 \) or \( J_3 \), \( m \) is increasing on \([j_1, j_{i+1}]\).

3) For \( j_1 \in J_4 \) or \( J_5 \), \( m \) is decreasing on \([j_1, j_{i+1}]\).

4) There exists a bijection \( \phi : J_2 \rightarrow J_4 \) such that if \( \phi(j_h) = j_1 \),
   1) \( i < h \)
   2) \( m(j_h) = m(j_{i+1}) \)
   3) \( m(j_{h+1}) = m(j_{i}) \).

5) For any \( j_h, j_1 \in J_3 \) with \( h < i \), \( m(j_h) < m(j_1) \).

Let \( J_i = \bigcup_{j_h, j_{h+1} \in J_i} \{j_h, j_{h+1}\}, i = 1, 2, 3, 4, 5 \).

6) The sets \( J_i \) cover \([0, N]\).

Lemma 5.7. There exists such a sequence \( J \).

Proof (See Figure 1):

We will prove this lemma by induction on the number \( H \) of maxima attained by \( m \) on the interval \([0, N]\) (this number is \( \leq N+1 \), and hence finite).

Clearly it will be enough to construct the sets \( J_1, \ldots, J_5 \).

\( H = 1 \): Then \( m \) is either monotonic non-increasing or monotonic non-decreasing. In the first case, let \( J_5 \) be the set on which \( m \) is
strictly decreasing, and \( J_1 \) the balance; in the second let \( J_3 \) be the set where \( m \) is strictly increasing, and \( J_1 \) the balance.

Inductive step: Assume true for \( H \).

So suppose the maxima occur at \( y_1, \ldots, y_{H+1} \), and the minima at \( (x_0), x_1, x_2, \ldots, x_H, (x_{H+1}) \), so that \( x_{i-1} < y_i < x_i \), for \( i = 1, \ldots, H+1 \) (\( x_0 \) or \( x_{H+1} \) may not exist). Apply induction to \([0, x_H]\) (recall that \( m \) changes direction only at integral arguments), and construct a tentative \( J_1, \ldots, J_5 \). Then, for every point \( j \) in \([x_H, y_{H+1}]\), either there exists \( i \in J_5 \subseteq [0, x_H] \) such that \( m(i) > m(j) \), or else not. In the first case, put \( j \) into \( J_2 \) and move \( i \) into \( J_4 \); in the second, put \( j \) into \( J_3 \). Finally, put \([y_{H+1}, N) \) into \( J_5 \).

It is clear that this is the desired partition.

\[ \square \]

Our result is now clear (again redefining \( \epsilon \)) from the following four lemmas:

**Lemma 5.10a:** \( \int_{J_1} A(z)dz = 0 \).

**Lemma 5.10b:** \( \int_{J_5} A(z)dz \geq 0 \).

**Lemma 5.10c:** \( \int_{J_3} A(z)dz > -\epsilon \).

**Lemma 5.10d:** \( \int_{J_2 \cup J_4} > -6M\epsilon \).

Lemma 5.10a is obvious, since \( A(z) \equiv 0 \) on \( J_1 \) by construction.

Similarly, \( A(z) \) is increasing on \( J_5 \), and so Lemma 5.10b holds.
Proof of Lemma 5.10c:

\[
\int \frac{A(z)dz}{J_3} = \sum_{j_1 \in J_3} A([j_1]) \cdot (j_{i+1} - j_1)
\geq -\epsilon (n+1) \sum_{j_1 \in J_3} u_j^n (m(j_{i+1}) - m(j_1)) \cdot \prod_{i=1}^{\infty} (1 - P_*(i))
\]

by (5.8) and (5.9),

\[
\geq -\epsilon (n+1) \sum_{j_1 \in J_3} u_j^n (\hat{m}(j_{i+1}) - \hat{m}(j_1))
\]

\[
= -\epsilon (n+1) \sum_{j_1 \in J_3} u_0^{1/2}
\]

\[
= -\epsilon (n+1) \frac{u_0^{1/2}}{1 - q^{1/2}}
\]

> -\epsilon .

Proof of 5.10d: Let \( \gamma : J_2 \to J_2 \) be \( \phi^{-1} \).

\[
\int \left( \frac{A(z)dz}{J_2} \right) = \sum_{j_1 \in J_2} A([j_1]) \cdot (j_{i+1} - j_1) + \sum_{j_1 \in J_2} A([j_1]) \cdot (j_{i+1} - j_1)
\geq -\sum_{j_1 \in J_2} (\epsilon n)^2 (m(j_{i+1}) - m(j_1)) \cdot u_j^n \left( \prod_{i=1}^{\infty} (1 - P_*(i)) \right)
\]

\[
- \sum_{j_1 \in J_4} (\epsilon n)^2 (m(j_{i+1}) - m(j_1)) \cdot u_j^n \left( \prod_{i=1}^{\infty} (1 - P_*(i)) \right)
\geq -\sum_{j_1 \in J_4} ((\epsilon n)^2 u_j^n - (\epsilon n)^2 \gamma(j_1)) (m(j_{i+1}) - m(j_1)) \cdot \prod_{i=1}^{\infty} (1 - P_*(i))
\]

(5.11)

by the defining properties of \( \phi \).
Now, \( u_{\gamma(j_1)} = q^\lambda u_{\gamma(j_1)} \), where \( \lambda = -1, 0, 1 \) (to see this, observe that if \( \frac{m(i)}{N} \leq \frac{m(i)}{N-1} \leq \frac{m(i+1)}{N} \), \( v(N) \) must = i or i+1). Thus (5.11)

\[
\begin{align*}
&\geq ((1-\varepsilon)^2 - q^{-n}(1+\varepsilon)^2) \cdot \sum_{j_1 \in J_{\Lambda, j_1}} u_{\gamma(j_1)} \left( m(j_{i+1}) - m(j_i) \right) \cdot \prod_{i=1}^{k-1} (1 - \prod_{i=1}^{k-1} (1 - \prod_{i=1}^{k-1} (1 - P_k(i)))
\end{align*}
\]

by the properties of \( m \) and the fact that \( |v_m(s^*)| \leq \hat{N} \) for all \( s^* \in S^* \),

\[
\geq -6\hat{N}\varepsilon .
\]

This completes the proof of Lemma 5.10d, hence of Proposition 5.6, and hence of Proposition 5.2.

\[
\square
\]

6. THE NON-ABSORBING CASE

We now prove

Proposition 5.3:

\[
\liminf_{T \to \infty} \int_{T_\infty}^{T} \frac{1}{N} \sum_{i=1}^{N} d_i d \mu > -\varepsilon .
\]

Proof:

Lemma 6.1:

Let \( \{ e_{ik} \}_{k=1}^{\infty} \) converge uniformly.

Let \( E = \liminf_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (\lim_{k \to \infty} e_{ik}) \)

Then \( E = \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} e_{jj} . \)
Proof: Easy.

Identifying \( e_{ik} \) as \( \int_{(T_1 \cup \ldots \cup T_k)^c} d_i du \), we observe that

\[
\left| \int_{T_\infty} d_i du - \int_{(T_1 \cup \ldots \cup T_k)^c} d_i du \right| \leq \hat{N} \sum_{k=1}^{\infty} u(T_k) + 0 \text{ as } k \to \infty.
\]

Thus Lemma 6.1 gives

\[
\lim \inf_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \int_{T_\infty} d_i du = \lim \inf_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \int_{(T_1 \cup \ldots \cup T_1)^c} d_i du.
\]

Let \( \omega_i(u) = W_{S_{i-1}}(u) - W_{S_{i-1}}(u) \). Then, by Proposition 4.1,

\[
\int_{(T_1 \cup \ldots \cup T_1)^c} d_i du \geq \int_{(T_1 \cup \ldots \cup T_1)^c} (\delta_i(u_{i-1}) - \omega_i(u_{i-1}) + n(u_{i-1})) du
\]

\[
\geq \int_{(T_1 \cup \ldots \cup T_1)^c} (\delta_i(u_{i-1}) - \omega_i(u_{i-1})) du - \epsilon \cdot u((T_1 \cup \ldots \cup T_c)^c).
\]

Of course, also,

\[
\int_{(T_1 \cup \ldots \cup T_1)^c} d_i du \geq -\hat{N} \mu((T_1 \cup \ldots \cup T_c)^c).
\]

Thus, setting

\[
f_k(t_{k-1}) = \begin{cases} \max(-\hat{N}, \delta_k(u_{k-1}) - \omega_k(u_{k-1})) & \text{if } \delta(t_{k-1}) \notin S^* \\ 0 & \text{otherwise,} \end{cases}
\]

we see that
\[
\int_{(T_1 \cup \ldots \cup T_1)^c} f_i^*(\tau(i-1))d\nu - \epsilon \\
= \int_{T^\infty} f_i^*(\tau(i-1))d\nu - \epsilon.
\]

Hence, to establish Proposition 5.3, it is

Enough to show:

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_{T^\infty} f_k^*(\tau(k-1))d\nu \geq 0. \tag{6.2}
\]

Applying Proposition 5.5, for each \( N \),

\[
\int_{T^\infty} \sum_{k=1}^{N} f_k^*(\tau(k-1))d\nu = \int_{T^\infty} \sum_{k=1}^{N} f_k^* \cdot \prod_{i=1}^{k-1} (1 - P_k(i))d\nu.
\]

Thus

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_{T^\infty} f_k d\nu = \liminf_{N \to \infty} \int_{T^\infty} \frac{1}{N} \sum_{k=1}^{N} f_k^* \cdot \prod_{i=1}^{k-1} (1 - P_k(i))d\nu \\
\geq \int_{T^\infty} \liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k-1} (1 - P_k(i))d\nu,
\]

by Fatou's Lemma (Roynen [1963]).

So, if we establish

**Lemma 6.3:** For all \( \tau \in T^\infty \),

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k^*(\tau(k-1)) \cdot \prod_{i=1}^{k-1} (1 - P_k(i)) \geq 0,
\]

we are done.
Proof: Suppose we know

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k(t(k-1)) \cdot \prod_{i=1}^{k} (1 - P_{\ast}(i)) > 0.
\]

Then either there exists \( N \) such that \( k > N \) implies that \( 1 - P_{\ast}(k) > \frac{1}{2} \), in which case we are done immediately, or else

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k(t(k-1)) \cdot \prod_{i=1}^{k-1} (1 - P_{\ast}(i)) \geq \liminf_{N \to \infty} \left( - \sum_{k=1}^{N} \prod_{i=1}^{k} (1 - P_{\ast}(i)) \right) = 0.
\]

But we can in fact show the stronger

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f'_k(t(k-1)) \cdot \prod_{i=1}^{k} (1 - P_{\ast}(\text{abs}: t, i-1)) > 0 \quad (6.3')
\]

where \( f'_k = \delta_k - \omega_k < f_k \).

Letting \( P_i = P_{\ast}(i) \), for all \( i \), we have

\[
\sum_{k=1}^{N} f'_k \cdot \prod_{i=1}^{N} (1 - P_i) = \sum_{k=1}^{N} f'_k \cdot \prod_{i=1}^{k} (1 - P_i) + \sum_{j=1}^{N-1} \left( \prod_{k=1}^{j} f'_k \cdot \prod_{i=1}^{j} (1 - P_i) \cdot P_j \right). \quad (6.4)
\]

Lemma 6.5: There exists a number \( \Omega_0 \) such that for all \( t \), for all \( N \), for all \( N_0 \), \( \sum_{k=1}^{N_0} f'_k < 0 \) implies that

\[
\sum_{k=N_0+1}^{N} f'_k \cdot \prod_{i=N_0+1}^{N} (1 - P_{\ast}(i)) > -\Omega_0.
\]
Proof: \( \sum_{k=1}^{N_0} f'_k < 0 \) implies that \( u_{N_0} = u_0 \) or \( u_1 \). Write \( \sum_{k=N_0+1}^{N} f'_k = -M - 2\overline{w}(u_1) \); assume \( M \geq 0 \) (else \( \sum_{k=N_0+1}^{N} f'_k \cdot \prod_{i=N_0+1}^{N} (1-P_i) \geq -2\overline{w}(u_1) \)).

Then \( \sum_{k=N_0+1}^{N} \delta_k \leq \sum_{k=N_0+1}^{N} (\delta_k - \omega_k) + 2\overline{w}(u_1) \)

\[ = -M - 2\overline{w}(u_1) + 2\overline{w}(u_1) \]

\[ = -M. \quad (6.6) \]

Recalling Proposition 4.2, and noting that

\[ |v_\infty(s)| \leq M \text{ for all } s \in S, \]

(6.6) implies that

\[ \sum_{k=N_0+1}^{N} p_\lambda(k) \geq \frac{(1-\varepsilon)}{N} \cdot \frac{M u_1^n}{2N} \]

\[ \geq \frac{M u_1^n}{2N}; \]

hence

\[ \sum_{k=N_0+1}^{N-1} (1-p_\lambda(k)) \leq \prod_{k=N_0+1}^{N-1} (1-p_\lambda(k)) \]

\[ \leq \frac{M u_1^n}{2N} \leq e \frac{M u_1^n}{2N}, \]
by a well-known inequality (which can be derived immediately from the
observation $\ln(1-P) < -P$ for $0 < P < 1$). Thus

$$
\sum_{k=N_0+1}^{N} f_k' \prod_{i=N_0+1}^{N} (1-P_i) > (-M - 2\overline{W}(u_1)) e^{-\frac{M u_1^n}{2N}}
- \frac{2N u_1^n}{u_1^2} - 2\overline{W}(u_1)
.$$ 

So if we set $\Omega_0 = 3M u_1^{-n}$, we are done.

Returning now to the proof of (6.3'), we distinguish two cases:

**Case 1:** $\sum_{k=1}^{\infty} P_k < \infty$.

Then

$$\lim \inf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k' \prod_{i=1}^{k} (1-P_i) \geq \lim \inf_{N \to \infty} \frac{1}{N} \Omega_0 + \frac{N-1}{\sum_{j=1}^{N-1} (-\Omega_0) \cdot P_{j+1}}.$$ 

by (6.4) and Lemma 6.5,

$$\geq \lim \inf_{N \to \infty} \left( -\frac{\Omega_0}{N} \right) \cdot \left( 1 + \sum_{k=2}^{\infty} P_k \right) = 0.$$

**Case 2:** $\sum_{k=1}^{\infty} P_k = \infty$. 

Case 2a: There exists $N_1$ such that $N > N_1$ implies $\sum_{k=1}^{N} f_k > 0$.

Then

$$\sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i) = \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i) + \sum_{j=1}^{N_1} \left( \sum_{k=1}^{j} f_k \cdot \prod_{i=1}^{j} (1 - P_i) \cdot P_{j+1} \right) + \sum_{j=N_1+1}^{N} \left( \sum_{k=1}^{j} f_k \cdot \prod_{i=1}^{k} (1 - P_i) \cdot P_{j+1} \right) \geq -\Omega_0 + \sum_{j=1}^{N} (-\Omega_0) \cdot P_{j+1} + \text{positive terms};$$

hence

$$\liminf_{N \to \infty} \frac{\sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i)}{N} \geq \liminf_{N \to \infty} \frac{\text{constant}}{N} = 0.$$ 

Case 2b: There exists no such $N_1$. Then for arbitrary $\epsilon^* > 0$, there exists $N^*$ such that

$$\sum_{i=1}^{N*} (1 - P_i) < \epsilon^* \quad \text{and} \quad \sum_{k=1}^{N*} f_k < 0.$$

Let $N > N^*$.

Then
\[
\sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i)
\]
\[
= \sum_{k=1}^{N^*} f_k \cdot \prod_{i=1}^{k} (1 - P_i) + \sum_{k=N^*+1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i) \cdot \prod_{i=1}^{N - k} (1 - P_i)
\]
\[
\geq -\Omega_0 - \sum_{j=1}^{N^*-1} \Omega_0 \cdot P_{j+1} - \epsilon^* = \Omega_0 - \sum_{j=N^*+1}^{N-1} \Omega_0 \cdot P_{j+1}
\]

by Lemma 6.5 and a slight extension of (6.4),

\[
\geq -\text{constant} - N\epsilon^* \Omega_0 ;
\]

hence
\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i) \geq -\Omega_0 \epsilon^* .
\]

But \(\epsilon^*\) was arbitrary:

hence
\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i) \geq 0 .
\]

This completes the proof of equation (6.3'), hence of Lemma 6.3, hence of Proposition 5.3, hence of Proposition 5.1, hence of Proposition 3.1, and hence of Theorem I.

Q.E.D.
BIBLIOGRAPHY


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