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EXISTENCE OF EQUILIBRIUM IN A HYPERFINITE EXCHANGE ECONOMY: I and II

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EXISTENCE OF EQUILIBRIUM IN A HYPERFINITE
EXCHANGE ECONOMY:  I*

by

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I. Introduction

The purpose of this paper is to propose a model of economic exchange, a hyperfinite exchange economy, which allows a systematic and comparative investigation of economies where there may be a finite, or measure space of traders and a finite or denumerable number of commodities.

Our main result is a metatheorem on existence theorems in several different types of standard exchange economies.

We shall show, using nonstandard analysis, that the theorems of Gale-Mas-Colell (4), (5), Peleg-Yaari (6), and Bewley (2) on the existence of a competitive equilibrium in exchange economies with convex preferences, although differing in their assumptions regarding the number of agents and commodities, are formally equivalent to the existence theorem for a hyperfinite exchange economy.

The paper is divided into several sections. First, we define hyperfinite exchange economies in Section II. The essential feature of a hyperfinite exchange economy is that it has all of the formal properties of an Arrow-Debreu exchange

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economy, even though it may have an infinite, i.e., non-
standard, number of commodities or agents.

In Section III, we give a nonstandard proof of the
Peleg-Yaari existence theorem for an economy with a denum-
erable number of commodities and a finite number of agents.
Bewley's existence theorem is an immediate corollary of the
Peleg-Yaari result.

Our proof consists of defining an appropriate hyper-
finite exchange economy and asserting that a competitive
equilibrium exists by transfer of the Gale-Mas-Colell exis-
tence theorem for exchange economies with a finite number
of commodities and agents. The final step in the proof is
to take the standard part of the equilibrium allocation and
prices in the hyperfinite exchange economy and show that they
constitute an equilibrium in the given standard economy.

Since in the standard model, we are dealing with an
infinite dimensional commodity space, i.e., the space of real
valued sequences, the choice of topology is crucial to our
analysis. We shall use the product topology and give an eco-
nomic justification for it, in terms of its myopic properties,
in the final section of the paper.

In a subsequent paper, we shall show that a similar
metatheorem holds for the existence theorems of Anderson (1) and
Schmeidler (7), i.e., these theorems are formally equivalent to
the existence theorem for a hyperfinite exchange economy, where
preferences need not be convex, but there are "many more" agents
than commodities.
II. Hyperfinite Exchange Economies

Let \( R_w \) denote the space of real valued sequences and \( *R_w \) the nonstandard extension of \( R_w \). \( R_w \) is a linear metric space, hence \( *R_w \) is also a linear metric space.

If \( d \) is a hyperfinite integer, then \( *R_d \) will denote a \( d \)-dimensional internal linear subspace of \( *R_w \). Formally \( *R_d \) has all the properties of a finite dimensional subspace of \( R_w \). In particular, the relative topology on \( *R_d \) is given by the nonstandard extension of the Euclidean metric on the finite dimensional subspaces of \( R_w \).

Let \( T \) denote an internal initial segment of \( *\mathbb{N} \), the nonstandard integers, and \( |T| = m \).

Let \( P \) denote the internal set of binary relations on \( *R_d^+ \), the nonnegative orthant of \( *R_d \).

A hyperfinite exchange economy is an internal map \( \mathcal{E}: T \rightarrow P \times *R_d^+ \).

For each agent \( t \), the projection of \( \mathcal{E}(t) \) on \( P \), denoted \( \succ_t \), is to be interpreted as the preference relation of agent \( t \). The projection of \( \mathcal{E}(t) \) on \( *R_d^+ \), denoted \( e(t) \), is the endowment of agent \( t \).

An allocation is an internal map \( g: T \rightarrow *R_d^+ \) such that
\[
\sum_{t \in T} g(t) = \sum_{t \in T} e(t).
\]

The positive price simplex, \( \Delta_d^+ \), consists of \( p \in *R_d^{++} \) such that \( \sum_{i=1}^{d} p_i = 1 \), where \( *R_d^{++} \) is the positive orthant of \( *R_d \).
If \( p \in \Lambda_d^0 \), then the demand set of the \( t^{th} \) trader is \( D(p, t) = \{ x \in \mathbb{R}_d^+ \mid p \cdot x \leq p \cdot e(t) \text{ and } p \cdot y \leq p \cdot e(t), y \in \mathbb{R}_d^* \Rightarrow y \not\succ_t x \} \).

A competitive equilibrium for a hyperfinite exchange economy \( \mathcal{E} \) consists of a price \( p \in \Lambda_d^0 \), an allocation \( g \), and an internal set \( S \) that \( g(t) \in D(p, t) \), for all \( t \in S \), where \( |S|/m \sim 1 \).

**Theorem (1)** If \( \mathcal{E} \) is a hyperfinite exchange economy satisfying the following assumptions, for all \( t \in T \):

(i) irreflexivity: \( x \not\sim_t x \)

(ii) monotonicity: \( x \succeq y, x \neq y \Rightarrow x \succ_t y \)

(iii) continuity: \( \{ (x, y) : x \succ_t y \} \) is relatively open in \( \mathbb{R}_d^+ \)

(iv) convexity: \( \{ x \in \mathbb{R}_d^+ : x \succ_t y \} \) is convex for all \( y \in \mathbb{R}_d^+ \)

and (v) \( \sum_{t \in T} e(t) > 0 \).

Then \( \mathcal{E} \) has a competitive equilibrium.

**Proof:** By transfer of the Gale-Mas-Colell existence theorem (4), (5).

**III. Markets with Countably Many Commodities**

Peleg and Yaari proved the existence of a competitive equilibrium in a market with countably many commodities and a finite number of traders. In this section we give a non-standard proof of their result, where we drop their assumptions that preferences are complete or transitive.

Note that for \( m \) finite, this is the "standard" definition of a competitive equilibrium in an exchange economy.
In the (P-Y) model, there are \( m \) traders whose consumption sets are \( \mathbb{R}^+_w \), with the product topology.

Each trader, \( t \), has an endowment vector \( i(t) \in \mathbb{R}^+_w \) and a utility function \( U_t: \mathbb{R}^+_w \to \mathbb{R} \), where \( U_t \) is assumed to be quasi-concave, strongly monotonic, and continuous with respect to the product topology.

If \( T = \{1, 2, \ldots, m\} \), then an allocation is a function \( g: T \to \mathbb{R}^+_w \) such that \( \sum_{t \in T} g(t) = \sum_{t \in T} i(t) \).

Let the social endowment be denoted as \( 0 \), then they assume that \( \sum_{t \in T} i(t) > 0 \).

Peleg and Yaari restrict their attention to price sequences \( p \in \mathbb{R}^{++}_w \) having the property that \( p \cdot 0 < \infty \) or equivalently \( p \cdot 0 = 1 \). Demand sets are defined in the obvious fashion for such price systems and a competitive equilibrium is then a price \( p \) and an allocation \( g \) such that \( g(t) \) is in the demand set of each agent \( t \).

It is important to note that a (P-Y) competitive price system need not give a finite value to arbitrary consumption vectors \( x \in \mathbb{R}^+_d \) and may not be summable. Despite these limitations, every competitive equilibrium, in the sense of Peleg-Yaari, is Pareto optimal.

Let \( \mathbb{d} \) be an infinite integer and \( \mathbb{R}^+_d \) be the internal set of sequences \( x \), such that \( x_i = 0 \) for all \( i > \mathbb{d} \). We define the hyperfinite exchange economy \( \mathcal{E} \) where:
(i) $T = \{1, 2, \ldots, m\}$

(ii) $e_j(t) = \begin{cases} i_j(t) & \text{for } j \leq d \\ 0 & \text{for } j > d \end{cases}$

(iii) $x \succ_t y$ iff $U_t(x) > U_t(y)$

and $i(t)$, $U_t$ denote the nonstandard extensions of $i(t)$ and $U_t$ to $*R_w$.

It is immediate that $E$ satisfies the conditions of Theorem 1. Hence $E$ has a competitive equilibrium $\langle p, g \rangle$.

Let $q = \frac{1}{\alpha} p$ where $\alpha = \sum_{t \in T} e(t)$.

**Theorem (2)** $\langle 0q, 0g \rangle$ exists and is a competitive equilibrium for the given (P-Y) economy.

**Proof:** Let $e = \sum_{t \in T} e(t)$, since $|T|$ is finite we see that $e$ is standard. $\sum_{t \in T} g(t) = \sum_{t \in T} e(t) = e$, hence $g(t) \preceq e$ for all $t$. Therefore, each $g(t)$ is near standard.*

$q \cdot e = 1$ implies that $q$ is also near standard.

Consequently, $\langle 0q, 0g \rangle$ exists.

$q \cdot e(t_0) \not\leq 0$ for some $t_0 \in T$, since $q \cdot e = \sum_{t \in T} q \cdot e(t) = 1$.

Suppose $q_i \not\leq 0$ for all finite $i$. Let $y = 0g(t_0)$ and $x = y + (1, 0, 0, \ldots)$, then $x \succ_{t_0} y$ by monotonicity of $\succ_{t_0}$.

Since $\succ_{t_0}$ is continuous in the product topology, which is

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*Throughout this paper, near standard will be defined with respect to the product topology.*
strongly myopic (see Section IV), there is some \( n \) such that \\
\( \bar{x}_n \succ_{t_0} y \), \\

where \( (\bar{x}_n)_i = \begin{cases} x_i & \text{for } i \leq n \\ 0 & \text{for } i > n \end{cases} \)

Again by continuity of \( \succ_{t_0} \) in the product topology, \\
\( \bar{x}_n \succ_{t_0} g(t_0) \), but \( q \cdot \bar{x}_n \not\succeq 0 \), hence is in the budget set of 
trader \( t_0 \), which contradicts the maximality of \( g(t_0) \). Therefore, for some finite \( i \), say \( i = 1 \), \( q_i \not\succeq 0 \).

Since \( e_1 \not\succeq 0 \), some trader, say \( t_1 \), has a noninfinitesimal amount of the first good.

Therefore, \( q \cdot g(t_1) = q \cdot e(t_1) \not\succeq 0 \). Suppose for some finite \( j \neq 1 \), that \( q_j \not\succeq 0 \), say \( j = 2 \). Let \( z = 0^g(t_1) \) and \( v = z + (0, 1, \ldots, 0, 0, \ldots) \), then \( \bar{v}_m \succ_{t_1} z \) by the same argument as above. But \( q \cdot \bar{v}_m \not\succeq q \cdot e(t_1) \), hence there is a \( u \in R^+_w \) such that \( q \cdot u \not\succeq q \cdot e(t_1) \) and \( \bar{u}_m \succ_{t_1} g(t_1) \), by the continuity of \( \succ_{t_1} \), in the product topology. This contradicts the maximality of \( g(t_1) \).

Hence \( q_i \not\succeq 0 \) for all finite \( i \).

We now show that \( \langle 0^g, 0^g \rangle \) is a \((P-V)\) equilibrium.

\[
0(\sum_{t \in T} g(t)) = 0(\sum_{t \in T} e(t)) \text{, and } 0(\sum_{t \in T} g(t)) = \sum_{t \in T} 0^g(t),
\]

\[
0(\sum_{t \in T} e(t)) = \sum_{t \in T} 0^g(t) \text{ since } |T| \text{ is finite. Hence }
\]

\[
\sum_{t \in T} 0^g(t) = \sum_{t \in T} i(t). \text{ Suppose for some } t_0, 0^g(t_0) <
\]
\(0^q i(t)\), then by monotonicity, continuity of \(\succ_t^0\) in product topology, and the strong myopia of the product topology there is for some \(n\), and \(x \in R^+_w\), \(x^\nu_n \succ^t_0 g(t_0)\) where \(0^q x^\nu_n < 0^q i(t)\).

Hence \(q^\nu x^\nu_n < q^\nu e(t_0)\), a contradiction. Hence \(0^q 0^g(t) \geq 0^q i(t)\) for all \(t\). Suppose for some \(t_1\) that \(0^q 0^g(t) > 0^q i(t)\).

Then \(\sum_{t \in T} 0^q 0^g(t) > \sum_{t \in T} 0^q i(t)\), which contradicts \(\sum_{t \in T} 0^g(t) = \sum_{t \in T} i(t)\).

Therefore \(0^q 0^g(t) = 0^q i(t)\), for all \(t \in T\). Suppose for some \(t_1\), that \(0^q y \leq 0^q i(t_1)\), \(y \in R^+_w\), and \(y \succ^t_{t_1} 0^g(t_1)\).

Again invoking the continuity of \(\succ^t_{t_1}\) in the product topology and the strong myopia of the product topology, there is an \(m\) such that \(y^\nu m \succ^t_{t_1} g(t_1)\). Hence there is a \(u \in R^+_w\) such that \(u^\nu m \succ^t_{t_1} g(t_1)\) and \(0^q u^\nu_m < 0^q e(t_1)\), by continuity of \(\succ^t_{t_1}\) in product topology. That is, \(q^\nu u^\nu_m < q^\nu e(t_1)\) and \(u^\nu_m \succ^t_{t_1} g(t_1)\), a contradiction. Therefore, \(0^q g(t)\) is in the demand set of trader \(t\).

It is doubtful if the original argument of (P-Y) can be extended to unordered preferences, since their proof used the equal treatment property of core allocations in a replicated economy and Scarf's existence theorem for the core of an N-person game, where in both instances agents are assumed to have quasi-concave utility functions.
Bewley's existence theorem for an exchange economy with a finite number of agents, where the commodity space is $\ell_\infty$, the space of bounded real valued sequences, with the Mackey topology for the pairing $\langle \ell_\infty, \ell_1 \rangle$ and $\ell_1$ is the space of summable real valued sequences, is an immediate consequence of the (P-Y) existence theorem.

Bewley assumes that each agent has $\ell_\infty^+$, the space of nonnegative bounded real valued sequences, as his consumption set; that the social endowment $i$ is uniformly bounded away from $0$ and is in $\ell_\infty^+$; that preferences are given by quasi-concave, strongly monotonic, and Mackey continuous utility functions. He then demonstrates the existence of a competitive equilibrium with strictly positive prices in $\ell_1$.

Since $i \in \ell_\infty^+$, we can assume that each agent's consumption set is norm bounded, but on norm bounded subsets of $\ell_\infty$, the Mackey topology agrees with the product topology. Hence by the (P-Y) existence theorem there exists an allocation $g$ and a price $p \in R_W^+$ such that $g(t)$ is in the demand set of each agent $t$ and $p \cdot i = 1$. The allocation $g$ takes values in $\ell_\infty^+$, since $g(t) \leq i$ for each $t$. Let $u = (1, 1, \ldots 1, \ldots)$, then $i \geq \beta u$ where $\beta > 0$. Hence $\beta q \cdot u \leq q \cdot i = 1$, and $q \cdot u \leq 1/\beta$. That is, $q \in \ell_1$. 
III. A Myopic Characterization of the Product Topology on $R^\infty$

A topology $\mathcal{J}$ on $R^\infty$ is said to be strongly myopic if for all $x \in R^w$, $x_n$ converges to $x$ with respect to $\mathcal{J}$. The fact that the product topology, $\mathcal{J}_p$, is strongly myopic played a crucial role in our proof of the (P-Y) existence theorem. It is conceivable that there are other topologies on $R^w$, e.g., the uniform topology or the box topology, which are finer than the product topology and are strongly myopic, in which case, those topologies would admit a larger family of strongly myopic preferences than the product topology.

In this section, we shall show that if these topologies are required to be locally convex (linear) topologies on $R^w$, then this is not the case. That is, the product topology is the finest locally convex (linear) topology on $R^w$ which is strongly myopic.

If $R_\infty$ is the set of real valued sequences which are nonzero in at most a finite number of places, then the product topology is the Mackey topology for the duality $\langle R^w, R_\infty \rangle$. (See example 3 on page 249 in (g)). Suppose $\mathcal{J}$ is a strongly myopic topology on $R^w$, then every $\mathcal{J}$-continuous linear functional $L$ can be represented as a sequence $p$. This can be seen by considering $u^i = (0, 0, \ldots, 0, 1, 0, \ldots)$ and letting $p_i = L(u^i)$, for $i = 1, 2, \ldots$. Then $L(x_n) = \sum_{i=1}^{\infty} p_i x_i$, but $x_n$ converges to $x$ with respect to $\mathcal{J}$ and $L$ is $\mathcal{J}$-continuous. Therefore, $L(x) = \sum_{i=1}^{\infty} p_i x_i = p \cdot x$. Hence, $p \in R_\infty$. 

Let $\mathcal{J}_{sm}$ be the finest locally convex (linear) topology on $\mathbb{R}_w$ which is strongly myopic. $\mathcal{J}_{sm}$ exists since it is generated by the family of seminorms $||v||_a$, where for each $x \in \mathbb{R}_w$, $||x_n - x||_a$ converges to 0. Note that every $p \in \mathbb{R}_\infty$ defines such a seminorm, $|p \cdot x|$. Hence every linear functional defined by a $p \in \mathbb{R}_\infty$ is $\mathcal{J}_{sm}$ continuous. Therefore $\mathcal{J}_{sm} \subseteq \mathcal{J}_p$. But $\mathcal{J}_p$ is strongly myopic, hence $\mathcal{J}_p \subseteq \mathcal{J}_{sm}$.

This result complements our work in (3), where we showed that the strict topology is the finest locally convex (linear) topology on $\ell_\infty$ which is strongly myopic.

In (3), we explored the economic interpretation of myopia, when the $x \in \ell_\infty$ were interpreted as state-contingent claims. We argued that myopia resolved the St. Petersburg paradox in the sense of Buffon, i.e., agents "neglect" sufficiently unlikely states. This resolution has been justly criticized as assuming away the essential feature of the St. Petersburg paradox. That is, $x \in \mathbb{R}_w$ and $x \not\in \ell_\infty$ is the interesting case. The myopic characterization of the product topology, given above, is a response to this criticism.*

*We owe this observation to John Geanakoplos.
REFERENCES


EXISTENCE OF EQUILIBRIUM IN A HYPERFINITE
EXCHANGE ECONOMY: II*

by

D. J. Brown and L. M. Lewis

I. Introduction

Hyperfinite exchange economies were defined in (1), where
it was shown that the existence theorems of Gale-Mas-Colell (4) and
Peleg-Yaari (7) are formally equivalent to the existence of a com-
petitive equilibrium in a hyperfinite exchange economy with convex
preferences.

Here we extend that analysis to the nonconvex case. Our
first result is a new proof of the existence of a near standard
competitive equilibrium in a nonconvex hyperfinite exchange economy,
where there are "many more" agents than commodities, first established
by Lewis in (5).

With this proof in hand, it is relatively easy to demonstrate
the formal equivalence of the existence theorems of Anderson (2) and
Schmeidler (9).

Our proof consists of first proving the existence of a com-
petitive equilibrium in a hyperfinite exchange economy with noncon-
convex preferences, where the ratio of the number of commodities to the
number of agents is infinitesimal, by transferring Anderson's exist-
ence theorem for approximate equilibria in a finite economy with

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nonconvex preferences. In theorem (2), we establish the existence of a near standard competitive equilibrium where the initial endowment is integrable and agent's preferences lie in the nonstandard extension of a compact set of preferences, each of which is continuous in the product topology. Then, following Rashid (8), we construct a measure-theoretic exchange economy from the hyperfinite exchange economy and demonstrate that Schmeidler's theorem holds by taking the standard part of a near standard competitive equilibrium and showing that it is a measure-theoretic competitive equilibrium.

Finally, using Rashid's construction, we prove our major result, the existence of a competitive equilibrium in an exchange economy with a measure space of agents and a countable number of commodities, where agents have nonconvex and incomplete preferences, which are continuous in the product topology. This result extends the Peleg-Yaari theorem for a finite number of agents to a measure space of agents.

II. Hyperfinite Exchange Economies

Let \( R_w \) denote the space of real valued sequences and \( ^*R_w \) the nonstandard extension of \( R_w \). \( R_w \) is a linear metric space, hence \( ^*R_w \) is also a linear metric space.

If \( d \) is a hyperfinite integer, then \( ^*R_d \) will denote a \( d \)-dimensional internal linear subspace of \( ^*R_w \). Formally \( ^*R_d \) has all the properties of a finite dimensional subspace of \( R_w \). In particular, the relative topology on \( ^*R_d \) is given by the nonstandard extension of the Euclidean metric on the finite dimensional subspaces of \( R_w \).
Let $T$ denote an internal initial segment of $^*\mathbb{N}$, the nonstandard integers, and $|T| = m$.

Let $P$ denote the internal set of binary relations on $^*\mathbb{R}_d^+$, the nonnegative orthant of $^*\mathbb{R}_d$.

A hyperfinite exchange economy is an internal map $\mathcal{E}: T \rightarrow P \times ^*\mathbb{R}_d^+$.

For each agent $t$, the projection of $\mathcal{E}(t)$ on $P$, denoted $>_t$, is to be interpreted as the preference relation of agent $t$. The projection of $\mathcal{E}(t)$ on $^*\mathbb{R}_d^+$, denoted $e(t)$, is the endowment of agent $t$.

An allocation is an internal map $g: T \rightarrow ^*\mathbb{R}_d^+$ such that

$$\sum_{t \in T} g(t) = \sum_{t \in T} e(t).$$

The positive price simplex $^0_\Delta_d$, consists of $p \in ^*\mathbb{R}_d^{++}$ such that $\sum_{i=1}^{d} p_i = 1$, where $^*\mathbb{R}_d^{++}$ is the positive orthant of $^*\mathbb{R}_d$.

If $p \in ^0_\Delta_d$, then the demand set of the $t^{th}$ trader is

$$D(p, t) = \{x \in ^*\mathbb{R}_d^+ | p \cdot x \leq p \cdot e(t) \text{ and } p \cdot y \leq p \cdot e(t), y \in ^*\mathbb{R}_d^+ \Rightarrow y \succ_t x\}.$$  

A competitive equilibrium for a hyperfinite exchange economy $\mathcal{E}$ consists of a price $p \in ^0_\Delta_d$, an allocation $g$, and an internal set $S$ such that $g(t) \in D(p, t)$, for all $t \in S$, where $|S|/m \approx 1$.

**Theorem (1)** If $\mathcal{E}$ is a hyperfinite exchange economy satisfying the following assumptions, for all $t \in T$:

(i) irreflexivity: $x \not\succ_t x$

(ii) monotonicity: $x \succ y$, $x \neq y \Rightarrow x \succ_t y$
(iii) continuity: \((x, y): x \geq_t y\) is relatively open in \(*R_d^+\)

(iv) transivity: \(x \geq_t y, y \geq_t z \Rightarrow x \geq_t z\)

and (v) \(\Sigma e(t) > 0\).

\(t \in T\)

(vi) \(d/m \sim 0\)

Then \(E\) has a competitive equilibrium.

Proof: By transfer of the Anderson existence theorem (2).

A vector \(x \in *R_d\) is said to be near standard if for all finite \(i\), \(x_i\) is finite, i.e., \(x\) is near standard in the product topology. \(*x > \not\sim 0\) means \(x_i > \not\sim 0\) for all finite \(i\).

If \(d\) is finite, then a family of preference relations on \(*R_d^+\) is said to be near standard if they lie in the nonstandard extension of a family of preference relations on \(R_d^+\), denoted \(\mathcal{P}\), which is compact in the topology of closed convergence.

If \(d\) is infinite, then a family of preference relations on \(*R_d^+\) is said to be near standard if they lie in the nonstandard extension of a family of preference relations on \(R_w^+\), denoted \(\mathcal{P}\), which is compact with respect to the Hausdorff uniformity on closed subsets of \(R_w^+ \times R_w^+\), where \(R_w^+\) has the product topology.

A near standard competitive equilibrium for a hyperfinite exchange economy \(E\) is a competitive equilibrium \(\langle p, g, S \rangle\) such that \(p\) is near standard and "almost all" traders in \(S\) have near

*If \(x\) is a near standard vector in \(*R_d\), then \(0_x\), the standard part of \(x\), is the sequence \(0_{x_i}\), the standard part of \(x_i\), for all finite \(i\).
standard allocations. That is, for every \( \varepsilon > 0 \), there exists an internal set of traders, \( V_\varepsilon \), such that \{t \in S | g(t) \text{ is not near standard}\} \subseteq V_\varepsilon \text{ and } |V_\varepsilon|/m < \varepsilon.

If \( h(t) \) is an internal map from \( T \) into \( \mathbb{R}_d^* \), then \( h(t) \) is said to be integrable if for all finite \( i \), \( \left( \frac{1}{m} \sum_{t \in T} h(t) \right)_i \) is integrable, i.e., \( \left( \frac{1}{m} \sum_{t \in T} h(t) \right)_i \) is finite and \( \left( \frac{1}{m} \sum_{t \in V} h(t) \right)_i \sim 0 \), if \( |V|/m \sim 0 \), for all finite \( i \).

For the remainder of this paper, we shall assume that preferences are near standard, i.e., lie in \( \mathbb{P} \), where each preference relation in \( \mathbb{P} \) is irreflexive, strongly monotonic, convex, and continuous in the product topology.

**Theorem (2):** If \( \mathcal{E} \) is a hyperfinite exchange economy satisfying the following assumptions:

(i) preferences, \( >_t \), are near standard

(ii) \( \sum_{t \in T} e(t) > 0 \)

(iii) \( e(t) \) is integrable

(iv) \( \frac{1}{m} \sum_{t \in T} e(t) \wedge > 0 \),

then \( \mathcal{E} \) has a near standard competitive equilibrium \( \langle q, g, s \rangle \), where \( q \wedge > 0 \).

**Proof:** Assumptions (i) and (ii) guarantee that \( \mathcal{E} \) satisfies the conditions of Theorem (1), hence \( \mathcal{E} \) has a competitive equilibrium \( \langle p, g, s \rangle \). Let \( q = \frac{1}{\alpha} p \), where \( \alpha = p \cdot \frac{1}{m} \sum_{t \in T} e(t) \). We shall show that \( \langle q, g, s \rangle \) is a near standard competitive equilibrium and that \( q \wedge > 0 \).
\[ q \sum_{t \in T} e(t) = 1 \], hence by assumption (iv) \( q \) is near standard. "Almost all" traders have near standard allocations, since \( e(t) \) is integrable and we can apply lemma (1) of Brown-Khan (3), which states: If \( \mathcal{A} \) is an integrable function from \( T \) to \( \mathbb{R} \), then \( (\forall \varepsilon > 0) (\exists n \in \mathbb{N}) \{ |W_n|/m \geq 1 - \varepsilon \} \) where \( W_n = \{ t \in T : |\mathcal{A}(t)| \leq n \} \).

We simply use this lemma to construct a standard sequence, which for the given \( \varepsilon \) bounds the allocations, on the finite indices, of all but \( \varepsilon|S| \) traders. Hence \( \langle q, q, S \rangle \) is a near standard competitive equilibrium.

Suppose \( q_i \) is infinitesimal for all finite \( i \). Since
\[ q \sum_{t \in T} e(t) = 1 \], there is a noninfinitesimal fraction of traders in \( S \) who have a noninfinitesimal income at prices \( q \). In addition, a noninfinitesimal fraction of these traders have near standard allocations.

Let \( t_1 \) be one such trader, i.e., \( t_1 \in S \), \( q \cdot e(t_1) > 0 \), and \( g(t_1) \) is near standard. Let \( y = g(t_1) + (1, 0, 0, \ldots) \), then
\[ y >_{t_1} g(t_1) \]. Hence \( y >_{t_1} g(t_1) \) for some finite \( n \). This follows from the assumptions that each preference relation in \( \mathcal{P} \) is strongly monotonic, continuous in the product topology—therefore strongly myopic—, and that \( g(t_1) \) is near standard. Hence, for any standard preference \( > \) in \( *\mathcal{P} \), \( y >_{t_1} g(t_1) \) for some finite \( n \). Since \( >_{t_1} \) is near standard in the topology of closed convergence, (or near standard with respect to the Hausdorff uniformity, see appendix), the same conclusion holds for \( >_t \). But \( q \cdot y_n > 0 \), hence \( y_n \) is in the budget set of agent \( t_1 \), contradicting the assumption that \( t_1 \in S \), i.e., \( g(t_1) \in D(q, t_1) \), the demand set of agent \( t_1 \).
Hence, for some finite \( i \), say \( i = 1 \), \( q_1 \neq 0 \).

Suppose for some finite \( i_0 \neq 1 \), that \( q_{i_0} > 0 \). By assumption (iv) and an argument similar to that above, we can find a trader \( t_2 \) such that \( g_1(t_2) > 0 \), \( g(t_2) \) near standard, and \( t_2 \in S \). Again using strong monotonicity, strong myopia, and the fact that preferences are near standard, we can find a \( v \in R^+ \) such that \( \forall n \geq t_2 \), for some finite \( n \), \( q \cdot \forall n \leq q \cdot e(t_2) \), a contradiction.

Therefore, \( q_i \neq 0 \) for all finite \( i \).

III. Markets with a Measure Space of Agents

Under assumptions (i), (iii), and (iv) of theorem (2), Rashid (8) has given a method for constructing a measure-theoretic exchange economy \( 0E \) from a hyperfinite exchange economy \( E \), for finite \( d \). The measure space is simply the Loeb space over \( T \), generated by the algebra of internal subsets of \( T \) with counting measure. The measurable assignments of traders to preferences and endowments is just the standard parts of \( \succ_{t} \) and \( e(t) \), denoted \( 0\succ_{t} \) and \( 0e(t) \).

Theorem (3): If \( E \) satisfies the assumptions of theorem (2) and \( d \) is finite, then \( 0E \) has a competitive equilibrium.

Proof: By theorem (2), \( E \) has a near standard competitive equilibrium \( \langle q, g, S \rangle \), where \( q \succ 0 \). We shall show that \( \langle 0q, 0g \rangle \),

\[ ^{+} \text{Note that } 0E \text{ satisfies the conditions of Schmeidler's existence theorem, i.e., preferences are irreflexive, transitive, strongly monotonic, and continuous. Moreover, the assignment of agents to preferences is measurable and the initial endowment is integrable.} \]
the standard parts of $q$ and $g(t)$ constitute an equilibrium in $\mathcal{E}$.

First, we show that $g(t)$ is Loeb integrable:

(i) \[ \frac{1}{m} \sum_{t \in T} q(t) = \frac{1}{m} \sum_{t \in T} e(t) \] and \[ \frac{1}{m} \sum_{t \in T} e(t) \] is finite, hence \[ \frac{1}{m} \sum_{t \in T} g(t) \] is finite.

(ii) let $V$ be an internal set of traders such that $|V|/m \sim 0$ and consider \[ \frac{1}{m} \sum_{t \in V} g(t) \]. Then

\[ q \cdot \frac{1}{m} \sum_{t \in T} q(t) = \frac{1}{m} \sum_{t \in V} q \cdot g(t) = \frac{1}{m} \sum_{t \in V} q \cdot e(t) \]

= \[ q \cdot \frac{1}{m} \sum_{t \in V} e(t) \sim 0 \]. Since $e(t)$ is integrable and $q$ is near standard. But each component of $q$ is noninfinitesimal, hence \[ \frac{1}{m} \sum_{t \in V} g(t) \sim 0 \].

Consequently, $\int_{0}^{g}$ exists and $\int_{0}^{g} = \int_{0}^{e}$. 

If $t \in S$, then $q \cdot g(t) = q \cdot e(t)$. Hence, \[ 0 \cdot q \cdot 0 \cdot g(t) = 0 \cdot q \cdot 0 \cdot e(t) \] on a set of Loeb measure one, i.e., $|S|/m \sim 1$.

Suppose for some $t \in S$, there is a standard $x$ such that $0 \cdot x \leq 0 \cdot 0 \cdot e(t)$ and $x^{0} \succ_{t} 0 \cdot g(t)$. Then by continuity and the fact that preferences in $\mathcal{E}$ are near standard in the topology of closed convergence, there is some $y$ such that $y \succ_{t} g(t)$ and $q \cdot y \leq q \cdot e(t)$, a contradiction. This completes the proof.

We now drop the assumption that $d$ is finite.
Theorem (4): If $E$ satisfies the assumptions of theorem (2), then $0E$ has a competitive equilibrium.

Proof: By theorem (2), $E$ has a near standard competitive equilibrium $\langle q, s \rangle$, where $q \succ s > 0$. Let $q_c$ be the nonstandard extension of $0q$, the standard part of $q$, and define $q_\infty = q - q_c$.

Let $h_i(t) = g_i(t) - \frac{(q_i)_{i} [e_{i}(t) - q_{i}(t)]}{(q)_{i}}$ for all $i$.

Then for almost all $t$ and for all finite $i$, $h_i(t) \sim g_i(t)$.

$q_c \cdot h(t) = q_c \cdot g(t) - q_\infty \cdot [e(t) - g(t)] = q \cdot g(t) - q_\infty \cdot e(t) = q \cdot e(t) - q_\infty \cdot e(t) = q_c \cdot e(t)$.

We shall show that $\langle 0q_c, 0h(t) \rangle$ is a competitive equilibrium for $0E$.

$R_w$ with the product topology is a metric space and Loeb (6) has shown that the standard part of an internal near standard function from an internal measure space into the nonstandard extension of a metric space is measurable. Hence, $0e(t)$ is measurable.

Anderson (1) has extended this theorem of Loeb to uniform spaces. Hence, $0x_t$ is measurable.

If $f: T \rightarrow R_w$, where $f_i(t)$ is integrable for every $i$, then we define the integral of $f$ as $\int f_i$ for $i = 1, 2, \ldots$, and denote it as $\int f$.

Now applying Rashid's construction to $E$, we see that $0E$ is a standard measure-theoretic exchange economy.
0e is integrable by assumption (iii). Suppose for some internal set of traders \( V \) that \( |V|/m \sim 0 \). Consider

\[
\frac{1}{m} \sum_{t \in V} h(t). \quad \text{Then } q_c \cdot \frac{1}{m} \sum_{t \in V} h(t) = \frac{1}{m} \sum_{t \in V} q_c \cdot h(t) = \frac{1}{m} \sum_{t \in V} q_c \cdot e(t)
\]

\[
= q_c \cdot \frac{1}{m} \sum_{t \in V} e(t) \sim 0, \quad \text{since } e(t) \text{ is integrable and } q_c \text{ is standard. But } (q_c)_i \text{ is noninfinitesimal for each finite } i, \quad \text{hence } (\frac{1}{m} \sum_{t \in V} h(t))_i \sim 0 \text{ for all finite } i. \quad \text{Therefore, } 0h \text{ is integrable and } \int 0h = \int 0e. \]

We have also shown that \( q_c \cdot h(t) \) and \( q_c \cdot t(t) \) are integrable, hence for almost all \( t \), \( q_c \cdot h(t) \) and \( q_c \cdot e(t) \) are near standard. It is clear that if \( 0(q_c \cdot h(t)) \) exists, then

\[
0q_c \cdot 0h(t) \leq 0(q_c \cdot h(t)). \quad \text{If } 0q_c \cdot 0h(t) < 0(q_c \cdot h(t)), \text{ then using strong monotonicity, strong myopia, and the fact that preferences are near standard, we can find a } y \in \mathbb{R}^+_w \text{ such that } q \cdot y \leq q \cdot e(t) \text{ and } y \succ_t g(t) \text{ for some } t \in S, \text{ a contradiction. Hence, for almost all } t, 0q_c \cdot 0h(t) = 0(q_c \cdot h(t)). \]

Let \( S \) be an internal set of full measure, where for all \( t \in S \):

(a) \( g(t) \) is near standard

(b) \( q \cdot g(t) \) is near standard

(c) \( g(t) \) is in the demand set of trader \( t \).

Suppose \( t \in S \) and for some infinite \( n_0 \) and \( n_1 \), that

\[
a_t = \sum_{j=n_0+1}^{n_1} q_j \cdot g_j(t) \neq 0. \quad \text{Let } y = \frac{\bar{Y}}{q_{n_0}}(t) + \frac{a_t}{q_{n_1}} (1, 0, 0, \ldots). \]

Then \( y >_t \frac{\bar{Y}}{q_{n_0}}(t) \) and \( \frac{\bar{Y}}{q_{n_0}}(t) \sim g(t) \), hence \( y >_t g(t) \); but
\( q \cdot y \leq q \cdot g(t) = q \cdot e(t) \) which contradicts (c). Therefore,

\[
\frac{1}{m} \sum_{t \in S} \alpha_t = \frac{1}{m} \sum_{t \in S} \sum_{j = n_0 + 1}^{n_1} q_j \cdot g_j(t) = \sum_{j = n_0 + 1}^{n_1} \frac{1}{m} \sum_{t \in S} q_j \cdot g_j(t)
\]

\[
= \frac{1}{m} \sum_{j = n_0 + 1}^{n_1} q_j \cdot \left( \frac{1}{m} \sum_{t \in S} g_j(t) \right) \not\geq 0 . \quad \text{But} \quad \sum_{j = 1}^{n_0} q_j \left( \frac{1}{m} \sum_{t \in T} g_j(t) \right)
\]

\[
= \frac{1}{m} \sum_{t \in T} \sum_{j = 1}^{n_0} q_j \cdot g_j(t) \not\geq 0 \quad \text{and} \quad \sum_{t \in S} q_j \cdot e_j(t) = \frac{1}{m} \sum_{t \in S} q_j \cdot e_j(t) \not\geq 0 , \quad \text{since} \quad \sum_{t \in T} q_j(t) = \frac{1}{m} \sum_{t \in T} e_j(t) \not\geq 0 .
\]

\[
\begin{align*}
\sum_{t \in T} \left( \frac{1}{m} \sum_{t \in T} e_j(t) \right) + \frac{d}{m} \sum_{j = n_0 + 1}^{n_0} q_j \cdot e_j(t) \not\geq 0 .
\end{align*}
\]

Hence \( q \cdot e(t) \not\geq 0 \), since \( q \cdot e(t) = q \cdot e_j(t) \). Therefore, \( \sum_{j = n_0 + 1}^{n_0} q_j \cdot e_j(t) \not\geq 0 \), which implies for almost all \( t \in S \),

\[
\frac{1}{m} \sum_{t \in S} \sum_{j = n_0 + 1}^{n_0} q_j \cdot e_j(t) \not\geq 0 .
\]

Consequently \( q \cdot e(t) \not\geq q \cdot e(t) \) for almost all \( t \in S \). Hence,

\[
0 q \cdot e(t) = 0 (q \cdot e(t)) , \quad \text{and therefore} \quad 0 q \cdot h(t) = 0 q \cdot e(t) \quad \text{for}
\]

almost all \( t \in S \).
The argument that \( h(t) \) is maximal in the budget set
\[ \{ y \in \mathbb{R}^+_w \mid 0^q_c \cdot y \leq 0^q_c \cdot 0^e(t) \} \]
is now the familiar one given above, employing strong myopia and the fact that preferences are near standard.

This completes the proof.

**Corollary 4.1:** If (i) \( \mathcal{E} \) satisfies the assumptions of theorem (4)
(ii) there exists finite noninfiniteimal positive \( c_0 \) and \( c_1 \) such that for all \( t \in T \),
\[ c_0^u \leq e(t) \leq c_1^u \]
where \( u = (1, 1, \ldots, 1) \in \mathbb{R}^+_d \).

Then \( 0^\mathcal{E} \) has a competitive equilibrium, with prices in \( l_1 \).

**Proof:** By assumption (i), \( 0^\mathcal{E} \) has a competitive equilibrium
\[ \langle 0^q_c, 0^h(t) \rangle. \]

By assumption (ii), \( \int^0 e(t) \) is uniformly bounded away from zero. Hence \( 0^q_c \) is in \( l_1 \).

If one wishes to consider \( l^+_\infty \) as the commodity space with the Mackey topology for the pairing \( \langle l_\infty, l_1 \rangle \); assumes that preferences are Mackey continuous; and assumes that consumption sets are norm bounded. Then corollary 4.1 implies Bewley's (unpublished) theorem on the existence of a competitive equilibrium in an economy with a measure space of agents and a countable number of commodities, where we have dropped his assumptions that preferences are complete or convex.

This result follows from the observation that the Mackey topology reduces to the product topology on norm bounded subsets of \( l_\infty \).
REFERENCES


APPENDIX

The Compact Topology on the Space of Preferences

In "Neighboring Economic Agents," Debreu remarks that an appropriate way to topologize the space of preferences over a commodity space, whose topology is metrizable, is to define a uniform structure over the space of preferences. This uniformity is called the Hausdorff uniformity by Bourbaki and is derived from the uniformity, generated by the metric, on the underlying commodity space.

For the case at hand, the metric space is $\mathbb{R}_w \times \mathbb{R}_w$ with the product topology and metric $d$. Let $U$ be the uniformity generated by the bounded metric $d_1 = d/1 + d$. As is well known, $d$ and $d_1$ generate the same uniformity on $\mathbb{R}_w \times \mathbb{R}_w$. Since $\mathbb{R}_w \times \mathbb{R}_w$ is a bounded metric space under $d_1$, the Hausdorff metric on the closed subsets of $\mathbb{R}_w \times \mathbb{R}_w$, denoted $\mathbb{R}_w \times \mathbb{R}_w$, generates the Hausdorff uniformity on $\mathbb{R}_w \times \mathbb{R}_w$ (see E. Michael, "Topologies on Spaces of Subsets").

Narens, in "Topologies of Closed Subsets," defines a topology of seminal importance in the nonstandard analysis of exchange economies. This is his so-called compact topology. The properties of the compact topology necessary for this paper are given in the next three propositions of Narens.
Proposition (1) Let \((X, \Gamma)\) be a topological space. \(B \subseteq \Gamma\) can be considered in two ways: as a closed subset of \(*X\) and as a point in the space \((\Gamma', \mathcal{C})\) where \(C = \{\psi \mid \psi \subseteq \Gamma, A \in \mathcal{C} \Rightarrow 0A \in \mathcal{C}\}\). Consider \(B\) as a closed subset of \(*X\) and let \(A = 0B\). Then in the space \((\Gamma', \mathcal{C})\), \(B\) is near standard to \(*A\), and \(0(*A) = A\).

Proposition (2) \((\Gamma', \mathcal{C})\) is a compact space.

Proposition (3) If \((X, \Gamma)\) is a bounded metric space, \(C\) is the compact topology on \(\Gamma\), and \(\mathcal{H}\) the Hausdorff topology on \(\Gamma\), then \(C \subseteq \mathcal{H}\).

From (2) and (3) we see that a subset of \(\Gamma\) which is compact with respect to \(\mathcal{H}\) is also compact with respect to \(C\). For our purposes, \(X = R_w \times R_w\) with the bounded metric \(d_1\).

If \(\succ\) is a binary relation on \(R_w^+\) which has open graph, i.e., continuous in the product topology, then \(\succ\), the complement of \(\succ\) in \(R_w^+ \times R_w^+\) is relatively closed. We assume that our preferences, i.e., \(\succ\), belong to a set \(P\) which is compact with respect to the topology induced by the Hausdorff uniformity, hence compact in Narens' compact topology. If \(\succ \in P\), then \(\succ\) denotes its standard part with respect to \(\mathcal{H}\). Since \(\mathcal{H}\) is finer than \(C\), the standard part of \(\succ\) with respect to \(\mathcal{H}\) equals the standard part of \(\succ\) with respect to \(C\). The following theorem is essential for the analysis of this paper.

Theorem If \(x\) and \(y\) are near standard, then \(0_x \succ 0_y \iff \mu(x) > \mu(y)\), where \(\mu(x)\) denotes the monad of \(x\) with respect to the product topology on \(R_w^+\).
Proof: Immediate consequence of proposition (1).

Note that if the underlying space $X$ is compact then Narens shows that $C = \mathcal{N}$. Hence, when the commodity space is a finite dimensional locally convex (linear) topological vector space, i.e., $R^n$, the topology of closed convergence on the space of closed subsets is identical to the compact topology of Narens, i.e., we compactify $R^n$ by taking its Alexandroff compactification.

If $X$ is an arbitrary uniform topological space, then we do now know the relationship between the compact topology and the topology induced by the Hausdorff uniformity on the space of closed subsets of $X$. To extend the analysis of this paper to commodity spaces which are nonmetrizable infinite dimensional locally convex (linear) topological vector spaces—hence uniform topological spaces—we would have to show that the topology induced by the Hausdorff uniformity is finer than the compact topology of Narens.