WEAKLY DEMOCRATIC REGULAR TAX EQUILIBRIA

IN A LOCAL PUBLIC GOODS ECONOMY WITH PERFECT CONSUMER MOBILITY

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Donald K. Richter
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1. Introduction

In this paper we study the existence of equilibria in a local public goods economy which has the following structure. There are a finite number of geographical regions. Each region has a government which provides public goods locally by buying private goods inputs on competitive markets and transforming these inputs into outputs of public goods by using a convex technology. The local public goods are pure in the sense that all the local residents consume the total produced, with no spillovers to other regions. Each regional government raises revenue from the residents of its region to cover the costs of the public goods it provides. The local governments are assumed to be weakly democratic in the sense that no public sector proposal (that is, a vector of public good provisions and a tax scheme to pay for it) will be enacted for which there exists an alternative proposal which is unanimously preferred by the local residents. (Note that it is not being assumed that unanimity is a necessary condition for a public sector proposal to be enacted.)

Private goods are produced by price-taking, profit maximizing producers using convex technologies, and are traded across regions. A continuum of price-taking, utility maximizing consumers, who are perfectly

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mobile across regions, treat parametrically the public sector proposals of the various local governments in deciding where to live. There are a finite number of different types of consumers, with type defined by a consumer's preferences, initial endowments, and share of private sector profits.

A feasible allocation is a specification of a partition of the consumers among the regions, a vector of technically feasible public good provisions (and an associated vector of private good inputs used in its production) for each region, a technically feasible aggregate production vector for the private sector, and a vector of private goods consumption for each consumer such that excess demand for each private good is zero. An equilibrium is defined by a vector of private good prices, a feasible allocation, and a vector of regional tax rates (on wealth), which satisfy the following conditions: (i) the private sector production vector is profit maximizing at the given price vector; (ii) each consumer's private goods bundle is utility maximizing subject to the after-tax budget constraint faced in the region in which he lives, and taking as given the vector of public good provisions in the region; (iii) no consumer desires to move to another region; (iv) the local public sectors are in equilibrium in the following sense: (a) each local government has a balanced budget (i.e., the revenue raised equals the cost of the private goods used to produce the region's public goods vector); (b) a regular tax scheme (specifically a proportional wealth tax) is used in each region (with the tax rate allowed to vary across regions); (c) in each region there is no alternative public sector proposal (involving any scheme of taxation) which the residents would unanimously prefer to the given public sector proposal.
This characterization of public sector equilibrium dates back to Foley [5], who dealt with a single region model in which mobility considerations played no role. The basic idea is to avoid a detailed examination of the (quite possibly very complex) political mechanisms of the local governments while still requiring that the equilibrium public sector proposals have certain realistic features. Whatever the local political mechanism is, it seems reasonable to assume that it is weakly democratic and that the tax scheme which is used to balance the budget reflects some sort of regular incidence of taxation over the residents of the region.

We say an allocation is locally Pareto efficient if there is no other feasible allocation involving the same partition as the given allocation which can make everyone better off. The assumption that the local governments are weakly democratic, along with the profit maximization equilibrium condition, insure that an equilibrium allocation as defined above is efficient in this restricted sense.

In light of this property, asking whether an equilibrium exists amounts to asking the following question. Among the locally Pareto efficient allocations corresponding to all possible partitions, are there any which are decentralizable (in the sense described by conditions (i)-(iii) of equilibrium) for a given initial distribution of endowments, and correspond to regular regional tax structures in which all regions have balanced budgets?

An allocation is said to be globally Pareto efficient if there is no other feasible allocation which can make everyone better off. Unlike the definition of local Pareto efficiency, there is no requirement that the partition involved in the alternative feasible allocation be the same as the one in the given allocation.
We have motivated interest in locally Pareto efficient allocations on positive grounds. But normative considerations would seem to motivate interest in the set of globally Pareto efficient allocations. In particular we can ask the analogous decentralization question for them—will there always be a globally Pareto efficient allocation which is decentralizable and corresponds to a regular regional tax structure in which all regions have balanced budgets. Under conventional economic assumptions in economies where the local public goods are pure, the answer is no.

1 See, for example, Bewley [2] and Wooders [14]. The possibility of decentralizing globally Pareto efficient allocations is generally referred to as the Tiebout hypothesis.

2 Suppose we impose a very weak regularity requirement on the local tax systems, namely that any two individuals that are identical except for tastes are taxed the same. Thus we rule out Lindahl taxation. Then consider an economy having 2 regions, 1 type of private good and 1 type of public good, and a continuum of consumers represented by the interval [0,1]. Half the consumers (type 1) have the utility function (applicable to either region) $u^1(g,y) = g + (y/8)$, where $g$ is the amount of the local public good provided in the region in which the consumer lives, and $y$ denotes the consumption of the private good. The other half of the consumers (type 2) have the utility function $u^2(g,y) = g + 2y$. There is an aggregate endowment of 1 unit of the private good, divided evenly among all consumers. Each local government uses a constant returns to scale technology in producing the public good in its region, with 1 unit of input of the private good yielding 1 unit of output of the public good. There is no private good production.

Clearly we can confine our search for a decentralizable globally Pareto efficient (GPE) allocation to GPE allocations which assign all consumers of the same type the same positive utility level. Given a GPE allocation with this property, at least half of the consumers living in one of the regions must be of type 1 (where these type 1 consumers represent at least $r \geq 1/4$ of all the consumers in the economy), and each consumer of type 1 must be consuming the same bundle $(g^*, y^*)$. Suppose $y^1 > 0$. Then by reducing $y^1$ to 0, the resulting $ry^1$ units of the private good could be used to produce $ry^1$ additional units of the public good, raising the utility level of type 1 consumers by $ry^1(8y^1/8) > 0$, and increasing the utility level of any type 2 consumers living in this region. Hence the given allocation could not have been GPE. Thus $y^1 = 0$, and hence $g > 0$ (since the given allocation yields a positive utility level to each type). But since both utility functions are monotonic in $g$, $g > 0$ implies everyone must live in this region if the allocation is GPE. (There are no congestion effects nor any characteristics specific
Richter [11] and Greenberg [6] have studied equilibria of the type described earlier, with one important exception. They both assumed that the partition of consumers was exogenous, thus ignoring probably the most crucial aspect of local public goods economies—the mobility of consumers. Another way of phrasing the question we are asking in this paper is: as we sift through the proportional wealth tax equilibria of Richter and Greenberg corresponding to all possible fixed partitions, will we always be able to find a fixed partition equilibrium which satisfies one additional condition—nobody wants to move?

In [12] an attempt was made to explicitly treat consumer mobility in the context of a regional model with pure local public goods. That model was only a partial success. While public goods were consumed locally, it was assumed that a weakly democratic central government levied a proportional wealth tax on all the consumers in the economy in order to raise revenue to pay for all the local public goods provided in all the regions. Furthermore, local governments did not actually produce their public goods. Instead it was assumed that public goods were produced by an aggregated, profit maximizing production sector. Thus local governments played no real role. In the present paper we have given the local governments a real role to play, requiring them to produce their own public goods, making

\[ y_1^1 = 0 \]

But since, all income of type 1 consumers must be taxed away for the given allocation to qualify as a decentralized equilibrium. But then from our regularity assumption on the tax system, all income of type 2 consumers must also be taxed away. Hence the economy's total endowment of 1 unit of the private good must be used to produce 1 unit of the public good, yielding a utility level of 1 for type 2 consumers. But by moving to the empty region (where the tax rate would be zero), type 2 consumers could achieve a utility level of 2 by consuming their endowments of the private good. Thus no GPE allocations in this economy can be decentralized if a regular tax system is used.
them fiscally autonomous, and permitting them to compete for residents on the basis of tax rates as well as public goods menus. These changes have led us to a radically different approach to the existence question from that taken in [12].

The paper is organized as follows. In Section 2 the basic assumptions of the model are presented and the equilibrium notion is rigorously defined. Section 3 contains a heuristic overview of the existence proof and Section 4 contains the rigorous existence proof.

2. The Model

\( R^n_+ (R^n_-) \) denotes the nonnegative (nonpositive) orthant of \( n \)-dimensional Euclidean space \( R^n \). The inequality convention used for vectors is \( \gg \), \( > \), \( \geq \).

Superscripts are primarily used to refer to consumer type, while subscripts are generally reserved for regions or goods. The index \( i \) or \( l \) will mainly be used to refer to consumer type, and the index \( j \) for regions. If \( y^i_j \in R^n \), where \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, \gamma\} \), then the vector \( y = (y^1_1, y^2_1, \ldots, y^m_1, y^1_2, y^2_2, \ldots, y^m_2, \ldots, y^1_\gamma, y^2_\gamma, \ldots, y^m_\gamma) \in R^{nm\gamma} \) will be denoted by \( (y^i_j) \). \( \sum_{ij} y^i_j \) denotes the summation \( \sum_{i=1}^m \sum_{j=1}^\gamma y^i_j \).

A. Regions, Public Goods, and Private Goods

There are \( \gamma \geq 2 \) non-overlapping geographical regions. \( g^j \in R^n_+ \) denotes the vector of local public goods provided in region \( j \).

There are \( n \) private goods. Some of these goods, such as land, may be associated with a particular region. The total endowment of private goods in the economy is denoted by the \( n \)-vector \( w \).
B. Production

The government of region \( j \) produces public goods for its residents using the production set \( Y_j \subset R_+^{\sigma_j} \times R_+^n \), where \((g_j, x_j) \in Y_j\) means the vector of public goods \( g_j \) is produced from the vector of private good inputs \( x_j \). The aggregate public goods production set \( Y_G \equiv \bigotimes_{j=1}^{\gamma} Y_j \).

We write \((g, x) \in Y_G\) when \( g = (g_1, \ldots, g_{\gamma})\), \( x = (x_1, \ldots, x_{\gamma})\), and \((g_j, x_j) \in Y_j \forall j\).

For simplicity of exposition we treat only an aggregated private production sector. \( Y \subset R_+^n \) summarizes the production possibilities for the private sector, with outputs measured positively and inputs negatively. No public goods are used as inputs or produced as outputs by the private sector.

(B.1) \( Y \) and \( Y_j \forall j \) are closed and convex.

(B.2) \( Y \cap R_+^n = \{0\} \) and \( Y_j \cap R_+^{\sigma_j+n} = \{0\} \forall j \).

(B.3) If \((g_j, x_j) \in Y_j\) and \(\tilde{x}_j < x_j\), \( \exists \tilde{g}_j > g_j \in (\tilde{g}_j, \tilde{x}_j) \in Y_j \).

C. Consumption

There are a continuum of consumers of \( m \) different types. All consumers of the same type have the same tastes and initial endowments, and receive the same share of total profits. \( r^i \in (0, 1] \) denotes the fraction of all consumers who are of type \( i \), with \( \sum_{i=1}^{m} r^i = 1 \). The preferences of a consumer of type \( i \) \((i = 1, \ldots, m)\), if he resides

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\(^3\) (B.3) is a very strong assumption. We will assume later that preferences are monotonic in public goods, but for technical reasons we cannot assume monotonicity in private goods over all of \( R_+^n \). (B.3) essentially substitutes for the latter assumption, because it says that each private good can be used to increase the output of some public good.
in region \( j \) (\( j = 1, \ldots, \gamma \)), are summarized by a utility function

\[ u_j^i(g_j, y_j^i) \]

defined on the consumption set \( R_j^{\sigma_j+n} \), where \( y_j^i \) represents the private goods consumption bundle. We summarize all these bundles with the \( n\gamma \)-dimensional vector \( y = (y_j^i) \).

Each consumer of type \( i \) has a semi-positive initial endowment \( n \)-vector \( w^i \) of private goods, with \( w = \frac{\sum r_i^1 w^i}{1} \). There are no initial endowments of public goods. Each consumer of type \( i \) receives a non-negative share \( \theta^i \) of the profits of the private sector, where \( \frac{\sum r_i^1 \theta^i}{1} = 1 \).

A partition of the consumers among the \( \gamma \) regions is denoted by a \( m \)-vector \( r = (r_j^1) \), where \( r_j^1 \in [0, r^1] \) denotes the fraction of all consumers who are of type \( i \) and live in region \( j \). The set of all possible partitions is \( P = \{ r \in R_+^{m} | \sum_{j=1}^{\gamma} r_j^1 = r^1 \forall i = 1, \ldots, m \} \).

\( u_j^i \) is continuous on \( R_+^{\sigma_j+n} \)

\( u_j^i \) is concave on \( R_+^{\sigma_j+n} \).

\( u_j^i \) is increasing in every component of \( g_j \).

There exists a parameter \( \tilde{\zeta} > 0 \) such that \( u_j^i \) is non-increasing in each private good coordinate above the level \( \tilde{\zeta} \).

\( w^i > 0 \) and \( w >> 0 \).

Let \( y^i = (y_1^i, \ldots, y_\gamma^i) \). For fixed \( i \) and \( j \) and any given \( \rho^i \in (0,1) \), define \( \phi_j^i(\rho^i) = \sup[u_j^i](r_1^i, \ldots, r_\gamma^i) \geq 0 \) with \( r_j^1 > 0 \) and \( \sum_{j=1}^{\gamma} r_j^1 = \rho^i r^1 \), and \( (g, x, y^i, s) \in Y_G \times R_+^{n Y} \times Y \equiv u_j^i(g_j, y_j^i) = \bar{u} \) if

\( \bar{u} \)

This satiation assumption is motivated by technical convenience. Since \( \tilde{\zeta} \) can be chosen arbitrarily large, it is not very restrictive.
\( r^i_j > 0 \) and \( \left\{ \sum_{j \in J} y^i_j - \sum x^i_j - s - w \leq 0 \right\} \). (Since (8.2) implies the set \( \{(g, x, s) \in Y_G \times Y| s + \sum x^i_j + w \geq 0\} \) is bounded, \( \phi^i_j(\rho^i) \) clearly exists.)

If all the resources in the economy are devoted to pleasing a fraction \( \rho^i \) of the type 1 consumers, \( \phi^i_j(\rho^i) \) is the largest common utility level they could obtain, provided that some of them live in the given region \( j \). (Obviously \( \phi^i_j \) is non-increasing in \( \rho^i \).)

\[(C.6) \quad \exists \rho^i \in (0, 1) \text{ and } x^i_j > 0 \text{ such that}
\]

(a) \( \max_{y^i_j \in R^n_+} u^i_j(0, y^i_j) > \max_{j} \phi^i_j(\rho^i) \)

(b) If \( (g_j, x_j) \in Y_j \) with \( x_j < 0 \), and \( u^i_j(g_j, y^i_j) < \phi^i_j(\rho^i) \),

\[ \exists y^i_j < y^i_j - k^i_j x_j \text{ s.t. } u^i_j(0, y^i_j) > u^i_j(g_j, y^i_j). \]

The left-hand side maximum in (a) exists because of (C.4). (C.6a) could be satisfied if, for example, utilities were increasing over a sufficiently large range in some private good whose marginal utility diminished sufficiently slowly. If this good was a necessary input in the production of public goods, (b) could also be satisfied. (The closer \( \rho^i \) is chosen to 1, the lower will be the values of \( \phi^i_j \), and hence the more likely the conditions are to be satisfied.)

D. Allocations and the Equilibrium Concept

**Definition 2.1.** An allocation \( (r, g, x, y, s) \) is an element of the set \( A = P \times Y_G \times \mathbb{R}^{n \gamma} \times Y \).

An allocation specifies a partition \( r \), public good provisions in all regions \( g = (g_1, \ldots, g_\gamma) \), private good inputs to public goods
production $x = (x_1, \ldots, x_\gamma)$, private good consumption bundles for all consumer types in all regions $y = (y_j^i)$, and the private sector production vector $s$. (Implicit in this definition is the property that all consumers of the same type living in the same region receive the same private goods bundle.)

Definition 2.2. The excess demand function $z : A \to \mathbb{R}^n$ is defined by

$$z(r, g, x, y, s) = \sum_{ij} r_j^i y_j^i - \sum_{i} x_j^i - s - w.$$ 

Definition 2.3. An allocation $(r, g, x, y, s)$ is feasible if $z(r, g, x, y, s) = 0$.

Definition 2.4. A feasible allocation $(r, g, x, y, s)$ is locally Pareto efficient if there does not exist a feasible allocation $(r, \tilde{g}, \tilde{x}, \tilde{y}, \tilde{s})$ with $u_j^i(g_j, y_j) > u_j^i(\tilde{g}_j, \tilde{y}_j)$ for all $i$, $j$ with $r_j^i > 0$.

This efficiency notion was first introduced by Ellickson [4], who called it Pareto efficiency with respect to a partition. As mentioned in the Introduction, the important feature is that only alternative feasible allocations involving the given partition $r$ are considered.

Definition 2.5. A proportional wealth tax local public competitive equilibrium is a price vector $p^* > 0$, a feasible allocation $(r^*, g^*, x^*, y^*, s^*)$, and a vector of tax rates $(t_1^*, \ldots, t_\gamma^*) \in \mathbb{R}^{\gamma}_{[0,1]}$ such that

(a) (private sector profit maximization) $p^* s^* \geq p^* s$ for all $s \in Y$;

(b) (utility maximization subject to after-tax budget constraint) if $r_j^* > 0$, $y_j^*$ maximizes $u_j^i(g_j^*, y_j^*)$ on

$$\{y_j^* \in \mathbb{R}^n_+ | p^* y_j^* \leq (p^* w^* + \theta^i p^* s^*)(1 - t_j^*)\}.$$
(c) (mobility equilibrium) if \( r_{ij}^* > 0 \), \( u_j^i(g_j^*, y_{ij}^*) \geq u_j^i(g_j^*, y_{ij}^*) \) \( \forall j = 1, ..., \gamma \), where \( y_{ij}^* \) maximizes \( u_j^i(g_j^*, y_{ij}^*) \) on
\( \{ y_{ij}^* \in R_+^n | p^i y_{ij}^* \leq (p^i w^i + \theta^i s^i)(1 - t_j^*) \} \);

(d) (weak democracy) if \( \sum_{j=1}^{\gamma} r_{ij}^* > 0 \), \( (g_j^*, x_j^*) \in Y_j \) and \( u_j^i(g_j^*, y_{ij}^*) > u_j^i(g_j^*, y_{ij}^*) \) \( \forall j \) \( r_{ij}^* > 0 \), then
\[-p x_j^* + p^i \sum_{j=1}^{\gamma} r_{ij}^* y_{ij}^* > p^i \sum_{j=1}^{\gamma} r_{ij}^* w^i + p^i s^i \sum_{j=1}^{\gamma} t_j^* \theta^i \; ;
\] \[r_{ij}^* > 0 \]

(e) (balanced budgets) \( t_j^* p^i \sum_{j=1}^{\gamma} r_{ij}^*(w^i + \theta^i s^i) = -p x_j^* \) and \( t_j^* = 0 \)
if \( \sum_{j=1}^{\gamma} r_{ij}^* = 0 \).

Condition (c) requires that the equilibrium level of utility of a consumer of type \( i \) who lives in region \( j \) is not less than the level he could obtain by moving to another region and maximizing his utility, given the public good provisions and the tax rate prevailing in that region.
Condition (d) states that no nonempty region can afford an alternative public sector proposal that would be unanimously preferred by its residents.
The right-hand side of the inequality in (d) represents the total before tax income or wealth of region \( j \), and the left-hand side represents the total cost of the preferred alternative bundles. After subtracting the total expenditure on the preferred bundles of private goods from before tax wealth, there would not be enough wealth left to tax away (using any tax scheme, including ones involving subsidies to some consumers) to cover the cost \(-p x_j^*\) of the alternative bundle of public goods. \(-p x_j^*\) is the cost of the private good inputs purchased by the local government in order to produce \( g_j^* \).

Condition (e) requires that each local government set its tax rate so that the revenue raised (i.e., the tax rate times the region's before tax wealth) equals the cost of producing the public goods. If region \( j \)
is empty, then its before tax wealth is 0 and the balanced budget condition implies \( x_j^* = 0 \), and hence the region provides no public goods. However, this still leaves \( t_j^* \) indeterminate, and hence we explicitly require a zero tax rate if a region is empty.

The definition of an equilibrium allocation allows for a trivial level of non-uniqueness in the sense that if \( r_j^* = 0 \), then \( y_j^* \) can be chosen arbitrarily.

**Theorem 2.1.** If \((r^*, g^*, x^*, y^*, s^*)\) is an equilibrium allocation as described in Definition 2.5, then it is locally Pareto efficient.

**Proof.** Suppose not. Then \( \exists j \) a feasible allocation \((r^*, g, x, y, s)\) with \( u_j^i(g_j, y_j^i) > u_j^i(g_j^*, y_j^* \) \( \forall j \geq r_j^* > 0 \). Then (d) of Definition 2.5 implies

\[-p^* \sum_{j \geq r_j^*} x_j^i + p^* \sum_{i \in I_j} r_j^i y_j^i > p^* \sum_{j \geq r_j^*} r_j^i w_j^i + p^* s^* \sum_{i \in I_j} r_j^i s_j^i.\]

(We have taken the summations on the right-hand side over all \( i \) and \( j \), since terms involving \( r_j^i = 0 \) obviously do not affect the sum.) Since \( x_j \leq 0 \) \( \forall j \) and \( p^* > 0 \), we can let the first summation on the left-hand side of the inequality run over all \( j \) and maintain the inequality. The second summation can trivially be extended to run over all \( i \). Incorporating these changes, using \( \sum_{i \in I} r_j^i w_j = w \) and \( \sum_{i \in I} r_j^i s_j = 1 \) on the right-hand side, and rearranging yields \( p^* (\sum_{i \in I} r_j^i y_j^i - \sum_{j \in I} x_j - w) > p^* s^* \). Since \((r^*, g, x, y, s)\) is feasible, the term in parentheses equals \( s \), and hence we obtain \( p^* s > p^* s^* \), contradicting condition (a) of equilibrium.

Q.E.D.
Since our equilibrium allocations are efficient in the restricted sense of Theorem 2.1, perhaps they can be found by maximizing some sort of social welfare function. This is the motivation for the following definition. Let \( S_{\mathcal{M}Y} \equiv \{(k^i_j) \in \mathbb{R}_+^{m \times y_i} | \sum_{i,j} k^i_j = 1\} \).

**Definition 2.6.** The social welfare function \( U : S_{\mathcal{M}Y} \times \mathbb{R}_+^{m \times y_i} \times \mathbb{R}_+^y \times \mathbb{R} \)

is defined by \( U(k, a) \equiv \sum_{i,j} k^i_j \cdot u^i_j(g_j, y^i_j) \), where \( a \equiv (g, x, y, s) \).

Even though \( U \) is independent of \( x \) and \( s \), we use the larger domain for notational convenience.

3. **Existence Proof—Heuristic Discussion**

In the next section we rigorously demonstrate the existence of a proportional wealth tax local public competitive equilibrium. In this section an overview of the underlying strategy of the existence proof is provided.

The proof is a generalization of an idea of Negishi's [10]. In proving the existence of an equilibrium in a competitive economy with only private goods, Negishi confines his attention to the set of feasible allocations which maximize nonnegative linear combinations of the individual utilities, and hence are Pareto efficient. For any given vector of utility weights for this social welfare function a maximizing allocation, and a vector of shadow prices having many of the properties of an equilibrium price vector, are generated. In particular, profits will be maximized at these prices, and any bundle of goods preferred by a consumer to the one he receives in the maximizing allocation will cost more at these prices. However, for an arbitrary vector of weights, there is no guarantee that
at these prices, the value of the consumer's assigned bundle will be less than or equal to his endowment plus profit income.

In order to find a vector of weights such that all consumers will satisfy their budget constraints when the dual prices are used, Negishi constructs a "penalty" mapping. Given a vector of weights and a corresponding efficient allocation, the basic idea is to lower the weights of consumers whose budget constraints are violated when evaluated at the efficient allocation and corresponding dual prices, and raise the weights of those whose budget constraints are not binding. At a fixed point of such a mapping, the weights will not be adjusted, implying everyone's budget constraint is just satisfied.

Foley's [5] proof of existence in the context of a single region public goods model embodied such a welfare maximization approach, and Matsuda [8] suggested a similar approach for demonstrating existence of equilibrium in the local public goods model studied by Richter [11] and Greenberg [9], in which consumer mobility was explicitly prohibited.

In the next section we present a proof which is very much in the spirit of Negishi's original work. By Theorem 2.1, we can confine our search for an equilibrium to the set of locally efficient allocations corresponding to all the various partitions. For given vectors $k$ (the social welfare function weights) and $r$ (a partition), a feasible allocation which maximizes the social welfare function of Definition 2.5 will (in most cases) be locally Pareto efficient. (Unlike Negishi, our social welfare maximization process is parameterized by two vectors rather than just the vector of utility weights, because $r$ enters the characterization of feasible allocations.) Furthermore, we can find a vector of dual prices which, together with the efficient allocation, will satisfy all the require-
ments of the equilibrium of Definition 2.5 except possibly two—the budget constraint in condition (b) may be violated (where the tax rates are defined by (e)), and the mobility requirement (condition (c)) may also not hold. Then we construct two penalty mappings. One of these mappings is very similar to Negishi's penalty mapping, with the welfare weights being adjusted with reference to the after-tax budget constraints evaluated at the dual prices. The other penalty mapping adjusts the partition in order to fix up the mobility condition. For each type of consumer, it penalizes a region which doesn't "measure up" in utility terms to the "best" regions for that type by assigning the region zero consumers of that type. We then show that at a fixed point of a correspondence involving these penalty mappings, both the budget constraints and the mobility condition will be satisfied. Having built all the other requirements of Definition 2.5 into the maximization process, an equilibrium will have been determined.

4. The Existence Proof

As noted above, assumption (B.2) implies the set

\[(g, x, s) \in Y_G \times Y | s + \sum_{i} \gamma_{i} x_{i} + w \geq 0 \] is bounded. Hence there exist positive vectors \( \bar{b}_{g}, \bar{b}_{x}, \bar{b}_{s} \) such that for all \((g, x, s)\) in this set, \( g < \bar{b}_{g}, x > -\bar{b}_{x} \) and \(-\bar{b}_{s} < s < \bar{b}_{s}\). We define \( \hat{Y}_{G} = \{(g, x) \in Y_G | g \leq \bar{b}_{g}, x \geq -\bar{b}_{x}\} \) and \( \hat{Y} = \{s \in Y | -\bar{b}_{s} \leq s \leq \bar{b}_{s}\} \).

Let \( \zeta = (\zeta_1, \ldots, \zeta_T) \), where \( \zeta \) is the saturation parameter of assumption (C.4). (The dimensionality of \( \zeta \) will be implied by its context.) Then we define \( \hat{W} = Y_G \times \{y \in R^{\text{mm}}_{+} | y \leq \zeta \} \times Y \) and \( \hat{\hat{W}} = \hat{Y}_{G} \times \{y \in R^{\text{mm}}_{+} | y \leq \zeta \} \times \hat{Y} \).
If \( a \in \hat{W} \) and \( z(r, a) \leq 0 \) for some \( r \in P \), then \( a \in \hat{\hat{W}} \).

The set \( P \times \hat{\hat{W}} \) is a convex, compact subset of the set of allocations.
A (see Definition 2.1), and includes all allocations \((r,g,x,y,s)\) in which \(y \leq \xi\) and \(z(r,g,x,y,s) \leq 0\). We will confine our search for an equilibrium allocation to the set \(P \times \hat{W}\).\(^5\)

Given \((k,r) \in S_{m\gamma} \times P\), consider the problem

\[
\max_{a \in \hat{W}} U(k,a) \text{ subject to } z(r,a) \leq 0
\]

where \(U\) is defined in Definition 2.6. Since \(\hat{W}\) is compact and \(U\) is continuous in \(a\) (by (C.1)), (1) has a non-empty solution set for each \((k,r) \in S_{m\gamma} \times P\).

**Definition 4.1.** \(N : S_{m\gamma} \times P \rightarrow \hat{W}\) is defined by \(N(k,r) = \{a \in \hat{W}| a\text{ is an optimal solution to (1) for the given } (k,r)\}\).

Given \((k,r) \in S_{m\gamma} \times P\), let \(\bar{a} \in N(k,r)\). Let

\[
C_{kr}(\bar{a}) \equiv \{z \in \mathbb{R}^n | z = z(r,a) \text{ for some } a \in \hat{W} \text{ with } U(k,a) \geq U(k,\bar{a})\} \text{ and } D \equiv \{z \in \mathbb{R}^n | z < 0\}.
\]

Since \(\bar{a} \in N(k,r)\), \(C_{kr}(\bar{a}) \cap D = \emptyset\). For suppose not. Then \(\exists a \in \hat{W} \exists z(r,a) < 0 \text{ (hence } a \in \hat{W}) \text{ and } U(k,a) > U(k,\bar{a})\).

Hence from (C.3) and (B.3) \(\exists \hat{a} \in \hat{W} \text{ with } z(r,\hat{a}) \leq 0 \text{ and } U(k,\hat{a}) > U(k,\bar{a})\), contradicting \(\bar{a} \in N(k,r)\).

---

\(^5\)Even though all allocations \((r,g,x,y,s)\) with \(z(r,g,x,y,s) \leq 0\) have the property that \(\sum_{j} r^i_j y^i_j \leq s + \sum_{j} x_j + w \leq b + w\), this inequality is not sufficient to bound the \(y^i_j\) because the \(r^i_j\) do not have a positive lower bound. Hence we have invoked (C.4) in order to confine our search to a compact subset of the consumption set.

\(^6\)Since \(k \in S_{m\gamma}\), \(\exists \sum_{j} k^i_j > 0\). Let \(a = (g,x,y,s)\). Form \(\hat{W}\) by changing only the components \(g^i_j\) and \(x^i_j\) of \(a\), in the following manner.

Since \(z(r,a) < 0\), we can choose \(\hat{x}^i_j < x^i_j\) so that \(z(r,\hat{a}) \leq 0\) and obtain (by (B.3)) \(\hat{g}^i_j > g^i_j\). Then \(\hat{a} \in \hat{W}\), and by (C.3), \(u^i_j(\hat{g}^i_j, y^i_j) > u^i_j(g^i_j, y^i_j)\) and hence \(U(k,\hat{a}) > U(k,a) \geq U(k,\bar{a})\).
$W$ is convex, and $U$ is concave in $a$ because the $u^i_j$ are concave. Hence $C_{kr}(\bar{a})$ is convex, and by the separating hyperplane theorem $\parallel p \neq 0$, with $\sum p^i_k = 1 \geq p^i_j \geq 0 \forall z \in C_{kr}(\bar{a})$. Since $D$ contains points with arbitrarily large (in absolute value) negative coordinates, it follows that $p > 0$. Furthermore, (C.3) and (B.3) imply $p >> 0$ with $\sum p^i_k = 1$ such that $p^i(r, a) \geq 0 \forall a \in W$ with $U(k, a) \geq U(k, \bar{a})$. Hence the following correspondence is well-defined, where $S^o_n$ denotes the

---

7 Since $z(r, \bar{a}) \in C_{kr}(\bar{a})$, and $z(r, a) \leq 0$ and $p > 0$, it follows that $p^i(r, \bar{a}) = 0$. Thus if $p^i_k > 0$ it follows that the $k^{th}$ coordinate of $z(r, a)$ equals 0, i.e., the $k^{th}$ coordinate of $\sum_{j} r^i_j y^i_j - \sum_{j} x^i_j - \bar{s} - w$ is 0. Since $w >> 0$, it follows that if $p^i_k > 0$, then either (a) $\exists j \ni r^i_j > 0$ and $(y^i_k)$ the $k^{th}$ coordinate of $y^i_j$ is positive or (b) $\exists j \ni (x^i_j) < 0$ or (c) $\bar{s} > 0$. (Of course these cases are not mutually exclusive.) Suppose $p^i_k = 0$ for some $i$ and (a) holds for some $k$ with $p^i_k > 0$. Then by decreasing $(y^i_k)$ slightly and increasing some component of $x^i_j$ by (algebraically) decreasing $(x^i_j)$ (we appeal here to (B.3)), we can obtain an allocation $\hat{a} \in W \ni U(k, \hat{a}) > U(k, \bar{a})$ by (C.1) and (C.3) and $p^i(r, \hat{a}) < 0$. This contradiction rules out case (a) holding for any $k$ for which $p^i_k > 0$ under the assumption that $p^i_k = 0$. Thus suppose $p^i_k = 0$ and (b) holds for some $k$ with $p^i_k > 0$. Then by (algebraically) increasing $(x^i_j)$ slightly and decreasing (algebraically) $(\bar{x}^i_j)$ sufficiently (and hence by (B.3) increasing some component of $\bar{x}^i_j$) we can (by (C.1) and (C.3)) obtain an allocation $\hat{a} \in W$ with $U(k, \hat{a}) > U(k, \bar{a})$ and $p^i(r, \hat{a}) < 0$. This contradiction rules out (b) holding for any $k$ with $p^i_k > 0$. Thus neither (a) nor (b) can hold for any $k$ with $p^i_k > 0$ under the assumption that $p^i_k = 0$. Thus (c) must hold for all $k$ with $p^i_k > 0$. But then $p^i < 0$, and by changing $\bar{s}$ to 0 and leaving the other components of $\bar{a}$ the same, we would obtain an allocation $\hat{a} \in W$ with $U(k, \hat{a}) = U(k, \bar{a})$ and $p^i(r, \hat{a}) < 0$. Thus the supposition that $p^i_k = 0$ has led to a contradiction in all possible cases. Hence $p >> 0$. 

interior of the \((n-1)\)-dimensional simplex \( S_n \equiv \{ p \in \mathbb{R}_+^n \mid \sum p_i = 1 \} \).

**Definition 4.2.** \( M : S_{\mathcal{M} \times \mathcal{P} \times \hat{S}_n} \) is defined by
\[
M(k,r) = \{ (\alpha, p) \in \hat{S}_n \times S_n \mid \alpha \in N(k,r) \text{ and } p_z(r, \alpha) \geq 0 \ \forall a \in \mathcal{W} \Rightarrow U(k, a) \geq U(k, \alpha) \}.
\]
(Note that \( \alpha \in \hat{S}_n \), but that the \( a \) need only belong to \( \mathcal{W} \).)

As we vary \((k,r)\) over \( S_{\mathcal{M} \times \mathcal{P}} \), most allocations \((r,a)\), where \( a \in N(k,r) \), will be locally Pareto efficient, and hence \( M \) essentially represents the locus of locally Pareto efficient allocations together with the associated dual price vectors. We formalize these remarks in the following lemma.

**Lemma 4.1.** If \((a, p) \in M(k,r)\) and \( r^i_j > 0 \) whenever \( k^i_j > 0 \), the allocation \((r,a)\) is locally Pareto efficient.

**Remark.** The hypothesis of Lemma 4.1 rules out the placing of any positive weight in the social welfare function on nobody. The correspondence \( M \) will be part of the mapping whose fixed points will correspond to equilibria.

We will show that the vectors \( k \) and \( r \) determined by such a fixed point will satisfy the property that \( r^i_j > 0 \) whenever \( k^i_j > 0 \).

**Proof.** For \((r,a)\) to be locally Pareto efficient, we must first show it is feasible in the sense of Definition 2.3. Since \( a \in N(k,r) \), \( z(r,a) \leq 0 \). But \( p >> 0 \) and \( p_z(r, a) \geq 0 \), and hence \( z(r, a) = 0 \).

Suppose \((r,a)\) is not locally Pareto efficient. Let \( a = (g,x,y,s) \).

Then \( \exists \) an allocation \((r,\tilde{g}, \tilde{x}, \tilde{y}, \tilde{s})\) with \( z(r, \tilde{g}, \tilde{x}, \tilde{y}, \tilde{s}) = 0 \) such that
\[
\forall i \exists j \quad u^i_j(\tilde{g}_j, \tilde{y}_j) > u^i_j(g_j, y_j) \quad \forall j \ \exists r^i_j > 0.
\]
If \( \vec{y} \neq \zeta \), form \( \hat{y} \) by lowering all coordinates of \( \vec{y} \) which exceed \( \zeta \) to the level \( \zeta \). Then \( z(\hat{\vec{y}}, \hat{\vec{z}}, \hat{\vec{x}}, \hat{\vec{s}}) \leq 0 \) and hence \( \hat{a} = (\hat{\vec{g}}, \hat{\vec{x}}, \hat{\vec{y}}, \hat{\vec{s}}) \in \hat{W} \).

By (C.4), \( u_j^i(\hat{g}_j, \hat{\vec{y}}_j) \geq u_j^i(\vec{g}_j, \vec{y}_j) \forall j \) and hence (2) holds with \( \vec{y}_j^i \) replaced by \( \hat{y}_j^i \). Then \( U(k, \hat{a}) - U(k, a) = \sum_{j} k_j^i(u_j^i(g_j, y_j) - u_j^i(g_j, y_j)) \), \( \forall j \)

Since \( r_j^i > 0 \) if \( k_j^i > 0 \), each term of this summation is positive.

Hence \( U(k, \hat{a}) > U(k, a) \). Since \( \hat{a} \in W \) with \( z(\hat{\vec{a}}, \hat{\vec{z}}) \leq 0 \), this contradicts \( a \in N(k, r) \).

Q.E.D.

Lemma 4.2. \( M \) is upper semicontinuous on \( S_{m \gamma} \times P \), and for each \( (k, r) \in S_{m \gamma} \times P \), the image set \( M(k, r) \) is convex.

Proof. See the Appendix.

The next task is to construct a mapping which penalizes consumers who violate their budget constraints. Let \( \delta \) be an arbitrary positive number, and let

\[
\begin{equation}
I \equiv \{ t = (t_1, \ldots, t_\gamma) \mid t_j \in [0, 1+\delta] \forall j \}
\end{equation}
\]

Definition 4.3. The budgetary deficits function \( \eta : \hat{W} \times S_m \times I \rightarrow R^{m \gamma} \) where \( \eta(a, p, t) = (\eta_j^i(a, p, t)) \) and \( a = (g, x, y, s) \), is defined by

\[
\eta_j^i(a, p, t) = (pw^i + ps^i)(1 - t_j) - py_j^i.
\]

Interpreting \( t_j \) as a tax rate (even though for technical reasons we allow it to exceed 1) applied to wealth (which equals endowment income plus profit income), \( \eta_j^i(a, p, t) \) represents after-tax income minus private goods expenditure by a type \( i \) consumer residing in region \( j \).
\( \eta \) is a continuous function defined on a compact set. Hence
\[
\exists \varepsilon > 0 \ \exists \sum_{i,j}^{i} \langle a, p, t \rangle > -\frac{1}{\varepsilon} \forall (a, p, t) \in \hat{W} \times \mathbb{S}_{n} \times I. \text{ We use } \varepsilon \text{ in the}
\]
following definition.

Definition 4.4. The budget penalty mapping \( K : \mathbb{S}_{m} \times \hat{W} \times \mathbb{S}_{n} \times I \to \mathbb{S}_{m} \) is defined by \( K(k, a, p, t) = \hat{k} \) where
\[
\hat{k}_{j}^{i} = [\max(0, k_{j}^{i} + \varepsilon \eta_{j}^{i}(a, p, t))]/[\sum_{i,j} \max(0, k_{j}^{i} + \varepsilon \eta_{j}^{i}(a, p, t))].
\]

Let \( D \geq \sum_{i,j} \max(0, k_{j}^{i} + \varepsilon \eta_{j}^{i}(a, p, t)) \) for arbitrary \( (a, p, t) \in \hat{W} \times \mathbb{S}_{n} \times I. \) Then
\[
D \geq \sum_{i,j} k_{j}^{i} + \varepsilon \sum_{i,j} \eta_{j}^{i}(a, p, t) = 1 + \varepsilon \sum_{i,j} \eta_{j}^{i}(a, p, t). \text{ Since } \varepsilon \sum_{i,j} \eta_{j}^{i}(a, p, t) > -1
\]
from the definition of \( \varepsilon, \) we have \( D > 0. \) Hence the mapping \( K \) is well-defined, and obviously maps into \( \mathbb{S}_{m} \). Since \( \eta \) is continuous, so is \( K. \)

The numerator of this mapping raises a consumer's weight in the social welfare function if his after-tax budget constraint is not binding for the given vector \( (a, p, t), \) and lowers his weight if his budget constraint is violated. (Note that violation of the budget constraint can result even if private expenditure is zero, either because the tax rate exceeds 1 or because \( ps \) is sufficiently negative.)

The next task is to define a regional penalty mapping, which distributes consumers among the regions so that they will have no incentive to move. The following two definitions will be used in constructing this regional penalty mapping. Recall the definitions of \( \rho^{i} \) and \( K_{j}^{i} \) from (C.6).

Definition 4.5. \( \lambda : \mathbb{S}_{n} \to R_{+} \) is defined by \( \lambda(p) = \overline{\lambda} \min_{i} (r^{i}p^{i}) \) where \( \overline{\lambda} > 0 \) is any constant satisfying \( \overline{\lambda} \leq \min(1/[\max_{i,j} K_{j}^{i}], \min_{i} [(1 - \rho^{i})r^{i}]/\gamma). \)

Since \( w^{i} > 0 \forall i, \lambda(p) > 0 \forall p \in \mathbb{S}_{n}. \)
Definition 4.6. $v : P \times \hat{W} \times \hat{S}_n \rightarrow \mathbb{R}^n$, where $v(r, a, p) = (v^i_j(r, a, p))$ and $a = (g, x, y, s)$, is defined by $v^i_j(r, a, p) = u^i_j(g_j, y^i_j)$ where $y^i_j$ is an optimal solution to the problem:

\[
\max_{y^i_j \geq 0} u^i_j(g_j, y^i_j)
\]

subject to \(py^i_j \leq \max \left\{ 0, (pw^i + s^i ps) \frac{1 + \frac{px^i_j}{\lambda(p)}}{\lambda(p)} \right\} \).

Suppose that $\frac{px^i_j}{\lambda(p)} \geq \lambda(p)$ and $-px^i_j/\left[\lambda(p) \frac{px^i_j}{\lambda(p)} \right] \leq 1$. Then $v^i_j(r, a, p)$ is the maximum utility level a type $i$ consumer could achieve by moving to region $j$, taking the region's public goods vector $g_j$ and tax rate $-px^i_j/\left[\lambda(p) \frac{px^i_j}{\lambda(p)} \right]$ as given. This tax rate would balance the budget in region $j$ for the given allocation $(r, a)$. The max operator in the denominator of the tax rate rules out undefined tax rates stemming, for example, from empty regions. (The specific choice of the function $\lambda$ is motivated by technical reasons.) The other max operator insures "after-tax income" is nonnegative.

For each $(r, a, p)$ in the domain of $v$, the constraint set in (4) is compact (since $p \gg 0$), and hence an optimal solution always exists. Furthermore $p \gg 0$ rules out the "exceptional case" and hence $v$ is continuous on its domain.

Definition 4.7. The regional penalty mapping $R : P \times \hat{W} \times \hat{S}_n \rightarrow P$, where $a = (g, x, y, s)$, is defined by $R(r, a, p) = \{ \hat{r} \in P \mid v^i_j = 0 \}$ if

\[\max(v^i_j(r, a, p), u^i_j(g_j, y^i_j)) < \max_{j \in \{1, \ldots, \gamma\}} \max(v^i_j(r, a, p), u^i_j(g_j, y^i_j))\]
This mapping is well-defined because for some value of $j$, the left-hand side of the inequality equals the right-hand side, thus insuring that $\hat{r}_j^i$ is not restricted to zero for all $j$. To make transparent the intuition underlying this mapping, suppose for the moment that $v^i_j \geq u^i_j \bar{v}_j$. Then the mapping penalizes a region $j$ by assigning it no type $i$ consumers if the region does not measure up, in the budget constrained utility sense underlying the definition of the $v^i_j$, to some of the other regions (i.e., $v^i_j < \max_j v^i_j$) for this consumer type. Thus it should be clear that this mapping is designed to insure satisfaction of the mobility condition of equilibrium. The presence of the additional terms $u^i_j$ will insure, as we shall see later, that some type $i$ consumers are assigned to region $j$ if the social welfare function assigns a positive weight to type $i$ consumers residing in region $j$.

**Lemma 4.3.** $R$ is upper semicontinuous on $P \times \hat{W} \times S_n$, and for each $(r, a, p) \in P \times \hat{W} \times S_n$, $R(r, a, p)$ is a convex set.

**Proof.** Convexity is obvious, and upper semicontinuity follows straightforwardly from the continuity of $v$ and the $u^i_j$.

In order to insure continuity of $v$, we restricted our attention to positive price vectors in Definition 4.6. Hence $R$, which inherits its domain from $v$, is not defined on a compact set. However, we need a correspondence defined on a compact set for our fixed point arguments. Hence we extend $R$ to an upper semicontinuous correspondence on the closure of its domain. Let $G_R = \{(r, a, p, \hat{r}) \in P \times \hat{W} \times S_n \times P | \hat{r} \in R(r, a, p)\}$, the graph of $R$. Let $\overline{G}_R$ denote the closure of $G_R$ in $P \times \hat{W} \times S_n \times P$. 
We then extend $R$ to an upper semicontinuous correspondence $\hat{R}$ on the domain $P \times \hat{W} \times S_n$ as follows.

**Definition 4.8.** $\hat{R} : P \times \hat{W} \times S_n \rightarrow P$ is defined by

$$\hat{R}(r,a,p) = \{\hat{t} \in P | (r,a,p,\hat{t}) \in \hat{C}_R \} .$$

Of course on the set $P \times \hat{W} \times S_n$, $\hat{R}$ agrees with $R$.

We have now defined 3 mappings—$M$, $K$, and $\hat{R}$—which will be components of our fixed point mapping. The final component mapping is the content of the next definition. Recall that $I = Y[0, 1+\delta]$.

**Definition 4.9.** The tax mapping $T : P \times \hat{W} \times S_n \rightarrow I$, where $T(r,a,p) = (T_j(r,a,p))$ and $a = (g,x,y,s)$, is defined by

$$T_j(r,a,p) = \min \left\{ 1+\delta, \frac{-px_j}{E_j(r,a,p)} \right\} \text{ if } E_j(r,a,p) \equiv \sum_{j=1}^{\gamma} \omega_j (px_j + \delta \max(0,ps)) > 0$$

(5) $$T_j(r,a,p) = 1+\delta \text{ if } -px_j > 0 \text{ and } E_j(r,a,p) = 0$$

(6) $$T_j(r,a,p) = 0 \text{ if } px_j = 0 \text{ and } E_j(r,a,p) = 0$$

(7) $$T_j(r,a,p) = [0, 1+\delta] \text{ if } px_j = 0 \text{ and } E_j(r,a,p) = 0$$

Suppose profits $ps$ are nonnegative. Then $E_j(r,a,p)$ is region $j$'s before tax income, and if this income is positive and region $j$'s expenditure on public goods does not exceed it, then $T_j(r,a,p)$ equals the proportional wealth tax rate which will balance the region's budget.

Technical reasons motivate the other complications in the definition.

**Lemma 4.4.** $T$ is upper semicontinuous, and for each $(r,a,p) \in P \times \hat{W} \times S_n$, $T_j(r,a,p)$ is a convex set.
Proof. Convexity is obvious. \( T_j \) is obviously upper semicontinuous at any point \((r, a, p)\) with \( E_j(r, a, p) > 0\). Thus suppose \( E_j(r, a, p) = 0 \) and \(-px_j > 0\). Let \((r^v, a^v, p^v) + (r, a, p)\) (where \(a^v = (g^v, x^v, y^v, s^v)\)), \( t_j^v \in T_j(r^v, a^v, p^v) \forall v\), and \( t_j^v \to t_j^v\). According to (6), we must show \( t_j = 1+\delta \). Clearly \( \exists v_0 \exists v \geq v_0 \) : \(-p^vx_j^v > 0\) and either \( E_j (r^v, a^v, p^v) = 0 \) or \( E_j (r^v, a^v, p^v) > 0 \) and \((-p^vx_j^v/E_j(r^v, a^v, p^v)) > 1+\delta\). If \( E_j (r^v, a^v, p^v) = 0 \), then by (6), \( t_j^v = 1+\delta \). If \( E_j (r^v, a^v, p^v) > 0 \), then by (5), \( t_j^v = \min(1+\delta, [-p^vx_j^v/E_j(r^v, a^v, p^v)]) = 1+\delta \). Thus \( v^v \geq v_0 \), \( t_j^v = 1+\delta \), and hence \( t_j = 1+\delta \). Thus \( T_j \) is upper semicontinuous at all points \((r, a, p)\) with \( E_j(r, a, p) = 0 \) and \(-px_j > 0\). Finally it is obvious that \( T_j \) is upper semicontinuous at the remaining points \((r, a, p)\) where \( px_j = 0 \) and \( E_j(r, a, p) = 0 \). Thus \( T_j \) is upper semicontinuous everywhere, and hence so is \( T \).

Q.E.D.

Definitions 4.2, 4.4, 4.8, and 4.9 define 4 upper semicontinuous mappings \( M \), \( K \), \( \tilde{K} \), and \( T \). \( M \), \( K \), and \( T \) have convex image sets at all points of their respective domains, and \( \tilde{K} \) has convex image sets (at least) at all points in the relative interior of its domain. Hence the mapping \( M \times K \times \tilde{K} \times T : S_{\gamma} \times P \times \tilde{W} \times S_n \times I \to S_{\gamma} \times P \times \tilde{W} \times S_n \times I \) is upper semicontinuous, and has convex image sets on the relative interior of its domain. Thus by a slight extension of Kakutani's Fixed Point Theorem, \(8\) there exists \( (k^*, r^*, a^*, p^*, t^*) \in S_{\gamma} \times P \times \tilde{W} \times S_n \times I \) such that

\[
(8) \quad (a^*, p^*) \in M(k^*, r^*)
\]

\(8\) The only change from the standard version that we are exploiting is that the image sets need not be convex at boundary points of the domain. See \([9]\) for a proof.
(9) \( k^* \in K(k^*, a^*, p^*, t^*) \)

(10) \( r^* \in K(r^*, a^*, p^*) \)

(11) \( t^* \in T(r^*, a^*, p^*) \)

where \( a^* = (g^*, x^*, y^*, s^*) \).

The price vector \( p^* \), the allocation \((r^*, a^*)\), and a vector of tax rates \( \tau^* \), where \( \tau^*_j = t^*_j \) if region \( j \) is nonempty and \( \tau^*_j = 0 \) if region \( j \) is empty in the partition \( r^* \), correspond to an equilibrium (see Theorem 4.1 below). Essentially (8) insures \((r^*, a^*)\) is feasible and that the profit maximization condition of Definition 2.5 is satisfied. (8) also does the lion's share of the work in guaranteeing the weak democracy condition is satisfied. (8) and (9) insure satisfaction of the utility maximization condition, (10) bears most of the burden in guaranteeing the mobility condition, and (11) aims primarily at balancing the regional budgets. However, there are some subtle interactions among the mappings. Thus before stating and proving the major theorem, we will prove a series of lemmas.

We first derive some implications which stem primarily from property (8).

**Lemma 4.5.**

(12) \( z(r^*, a^*) = 0 \)

(13) \( p^*s^* \geq p^*s \ \forall s \in Y \)

(14) If \( r_j^* > 0 \), \( u_j^i(g_j^*, y_j^i) > u_j^i(g_j^*, y_j^i) \), then \( p^*y_j^i > p^*y_j^i \).
If \( \sum_{k_j^i} r_j^i \geq 0 \), \((\tilde{g}_j, \tilde{y}_j) \in Y_j\), \( u_j^i(\tilde{g}_j, \tilde{y}_j^i) > u_j^i(g_j^*, y_j^i) \), \( \forall i \in r_j^i > 0 \), and \( \forall i \), \( r_j^i > 0 \) whenever \( k_j^i > 0 \), then

\[
-p^*_x_j + p^*_r \sum_{i \in \Theta} r_j^i y_j^i \geq -p^*_x_j + p^*_r \sum_{i \in \Theta} r_j^i y_j^i
\]

where \( r_j^i > 0 \) whenever \( r_j^i > 0 \).

\[
\sum_{i,j} r_j^i n_j^i (a^*, p^*, r^*) \geq 0
\]

If \( k_j^i = 0 \) and \( r_j^i > 0 \), then \( y_j^i = 0 \).

If \( \sum_{i,j} k_j^i = 0 \), then \( x_j^* = 0 \).

Proof. Since \((a^*, p^*) \in M(k^*, r^*)\) it follows from Definition 4.2 that

\[
p^* z(r^*, a^*) \geq 0 \quad \forall \hat{a} \in W = Y_G \times \{y \in R_+^{m \times n \times y} | y \leq \zeta \} \times Y
\]
such that

\[
U(k^*, \hat{a}) > U(k^*, a^*).
\]

Since \( z(r^*, a^*) \leq 0 \), \( p^* \gg 0 \), and \( \hat{a} = a^* \) satisfies (19), we have

\[
p^* z(r^*, a^*) = 0
\]

and hence \( z(r^*, a^*) = 0 \), verifying (12). Combining (19) and (20) yields

\[
p^* z(r^*, a^*) \geq p^* z(r^*, a^*) \quad \forall \hat{a} \in W \ni U(k^*, \hat{a}) \geq U(k^*, a^*)
\]

Let \( s \) be any element of \( Y \). Then \( \hat{a} = (g^*, x^*, y^*, s) \in W \) and it follows from (21) that \(-p^* s \geq -p^* s^*\), verifying (13).

Next suppose \( \hat{a} = (g^*, x^*, \hat{y}, s^*) \) where \( \hat{y} \) is identical to \( y^* \) except possibly in the component vector \( \hat{y}_j^i \) corresponding to some fixed
pair \( ij \) for which \( r^*_j > 0 \). If \( u^i_j(g^*_j, \hat{y}^i_j) \geq u^i_j(g^*_j, y^i_j) \), then
\[ U(k^*, a^*) \geq U(k^*, a^*) \] if \( \hat{y}^i_j \leq \zeta \), then \( a^* \in W \) and (21) implies
\[ p^*y^i_j \geq p^*y^i_j \]. Suppose \( u^i_j(g^*_j, \hat{y}^i_j) > u^i_j(g^*_j, y^i_j) \) and \( p^*y^i_j = p^*y^i_j \).
Then by reducing some positive component of \( \hat{y}^i_j \) a little (since \( y^*_j \geq 0 \), clearly \( \hat{y}^i_j > 0 \)), we could obtain a vector \( \hat{y}^i_j \leq \zeta \) for which
\[ u^i_j(g^*_j, \hat{y}^i_j) \geq u^i_j(g^*_j, y^i_j) \] (by continuity of \( u^i_j \)) and \( p^*y^i_j < p^*y^i_j \) (since \( p^* \gg 0 \)), leading to a contradiction of (21). Thus \( p^*\hat{y}^i_j > p^*y^i_j \) if
\[ u^i_j(g^*_j, \hat{y}^i_j) > u^i_j(g^*_j, y^i_j) \] and \( \hat{y}^i_j \leq \zeta \). If \( \hat{y}^i_j \not\leq \zeta \), then by (C.4) a vector \( y^i_j < \hat{y}^i_j \) with \( y^i_j \leq \zeta \) and \( u^i_j(g^*_j, y^i_j) \geq u^i_j(g^*_j, \hat{y}^i_j) \). Thus
\[ u^i_j(g^*_j, y^i_j) > u^i_j(g^*_j, y^i_j) \] and since \( y^i_j \leq \zeta \), \( p^*y^i_j > p^*y^i_j \). Since
\[ y^i_j < \hat{y}^i_j, \quad p^*y^i_j > p^*y^i_j \], and hence \( p^*\hat{y}^i_j > p^*y^i_j \). Thus \( u^i_j(g^*_j, \hat{y}^i_j) > u^i_j(g^*_j, y^i_j) \)
implies \( p^*\hat{y}^i_j > p^*y^i_j \) even if \( \hat{y}^i_j \not\leq \zeta \), verifying (14).

Next suppose \( \sum_{j \in I} r^*_j > 0 \) for some fixed \( j \) and define \( \hat{a} = (\hat{g}, \hat{x}, \hat{y}, s^*) \),
where \((\hat{g}, \hat{x})\) is identical to \((g^*, x^*)\) except possibly for the component vectors \( \hat{g}_j \) and \( \hat{x}_j \) (where \((\hat{g}_j, \hat{x}_j) \in Y_j \)), and \( \hat{y} \geq 0 \) is identical to \( y^* \), except possibly in the component vectors \( \hat{y}^i_j \) for \( i \in r^*_j > 0 \).
Then \( U(k^*, \hat{a}) - U(k^*, a^*) = \sum_{j \in I} k^*_j (u^i_j(\hat{g}_j, \hat{y}^i_j) - u^i_j(g^*_j, y^i_j)) \). Suppose
\[ k^*_j > 0 \]
the hypotheses of (15) hold. Then either this summation is vacuous (i.e., \( k^*_j = 0 \) \( \forall i \)) or each of its terms is positive. (The latter assertion follows from the fact that if \( k^*_j > 0 \), then \( r^*_j > 0 \) and hence
\[ u^i_j(\hat{g}_j, \hat{y}^i_j) > u^i_j(g^*_j, y^i_j) \).) Thus \( U(k^*, \hat{a}) \geq U(k^*, a^*) \) and if \( \hat{y} \leq \zeta \),
(21) directly implies the conclusion of (15). The argument can easily be extended to allow for \( \hat{y} \not\leq \zeta \), as we did in proving (14), and hence (15) is verified.

It follows from (13) and \( 0 \in Y \) that
(22) \( p^s s^* \geq 0 \).

Hence \( E_j(r^*, a^*, p^*) \) (see Definition 4.9) equals before-tax wealth in region \( j \), and we have

(23) \[ E_j^* = E_j(r^*, a^*, p^*) = \sum_{x_j} r^{x_j}_j (p^s w^* + \theta^0 p^s s^*) \geq 0 \quad \forall j. \]

If \( E_j^* > 0 \), then since \( t^* \in T(r^*, a^*, p^*) \), (5) implies

(24) \[ \frac{t^* E_j^*}{x_j^*} < -p^s x_j^*. \]

If \( E_j^* = 0 \), then since \( x_j^* \leq 0 \), (24) also holds. Thus (24) holds \( \forall j \), implying

(25) \[ \sum_{j} \frac{t^* E_j^*}{x_j^*} < -p^s \sum_{j} x_j^*. \]

Using Definition 4.3, \[ \sum_{i_j} r^{x_i}_j n^i_j (a^*, p^*, t^*) = \sum_{i_j} r^{x_i}_j (p^s w^1 + \theta^0 p^s s^*) (1 - t^*_j) - p^s \sum_{i_j} r^{x_i}_j y^i_j = p^s w^1 + p^s s^* - \sum_{i_j} (t^* E_j^* - \sum_{i_j} r^{x_i}_j y^i_j) \geq -p^s (w^1 + p^s s^*), \]

the inequality following from (25). But the right-hand side of the inequality equals \( -p^s z(r^*, a^*) \), which equals \( 0 \) by (12). Thus (16) is verified.

With respect to (17), suppose \( k^*_{j} = 0 \), \( r^*_{j} > 0 \) and \( y^*_{j} > 0 \).

\( \exists j \) (possibly \( j = j \)), \( \sum_{k_j} k^*_{j} > 0 \). Reduce \( y^*_{j} \) to 0 (which doesn't alter the value of \( U \) since \( k^*_{j} = 0 \)) and let \( x^*_j = x^*_j - r^*_{j} y^*_{j} \). This operation leaves the excess demand vector \( z(r^*, a^*) \) unchanged. But by (B.3), as a result we obtain a vector \( \delta^*_j > \delta^*_j \), which raises the value of \( U \) (by (C.3) and the fact that \( \sum_{k_j} k^*_{j} > 0 \)), contradicting the maximality of \( a^* \). Thus (17) is verified.
Finally, with respect to (18), suppose \( \sum_{i,j} k_{j}^{*} = 0 \) and \( x_{j}^{*} < 0 \).

Then by reducing \( x_{j}^{*} \) to 0 (and consequently reducing \( g_{j}^{*} \) to 0, which does not alter the value of \( U \) since \( \sum_{i,j} k_{j}^{*} = 0 \)) and algebraically decreasing \( x_{j}^{*} \) so as to maintain a nonpositive excess demand vector (and hence by (8.3) increasing \( g_{j}^{*} \)), we can increase the value of \( U \) (by (C.3) and \( \sum_{i,j} k_{j}^{*} > 0 \)), thus contradicting the maximality of \( a^{*} \). Thus (18) is verified.

Q.E.D.

Lemma 4.6. If \( r_{j}^{*} > 0 \), \( (p^{*} \omega_{j}^{*} + s^{*} p^{*} s^{*})(1 - t_{j}^{*}) - p^{*} y_{j}^{*} = 0 \).

Proof. Since \( k^{*} \in K(k^{*}, a^{*}, p^{*}, t^{*}) \), it follows from Definition 4.4 that

\[
D^{*} k_{j}^{*} = \max(0, k_{j}^{*} + \epsilon \eta_{j}^{*})
\]

where \( D^{*} \equiv \sum_{i,j} \max(0, k_{j}^{*} + \epsilon \eta_{j}^{*}) > 0 \) and \( \eta_{j}^{*} \equiv \eta_{j}^{*}(a^{*}, p^{*}, t^{*}) \). It follows directly from (26) that:

\[
W_{ij} \in k_{j}^{*} > 0 \text{, all the corresponding } \eta_{j}^{*} \text{ have the same sign;}
\]

\[
\text{if } k_{j}^{*} = 0 \text{, then } \eta_{j}^{*} < 0 .
\]

Furthermore, we can refine (28) so that

\[
\text{if } k_{j}^{*} = 0 \text{ and } r_{j}^{*} > 0 \text{, then } \eta_{j}^{*} = 0 .
\]

To establish (29), suppose for some fixed pair \( i_{j}^{*}, k_{j}^{*} = 0 \), \( r_{j}^{*} > 0 \), and \( \eta_{j}^{*} \neq 0 \). Then by (28), \( \eta_{j}^{*} < 0 \). By (17), \( y_{j}^{*} = 0 \). Since
\( \eta_j^* < 0 \) and \( y_j^* = 0 \). Definition 4.3 implies \( t_j^* > 1 \). Hence \( t_j^* \in T_j(r^*, a^*, p^*) \) and (5) imply \( x_j^* < 0 \). (\( E_j^* > 0 \) because \( r_j^* < 0 \).)

But \( x_j^* < 0 \) implies, by (18), that \( \sum_{k_j^*} > 0 \). Hence \( \exists I \ni k_j^* > 0 \).

But since \( t_j^* > 1 \), \( \eta_j^* < 0 \). Hence by (27), \( \sum_{i,j} r_j^{*i} \eta_j^* \leq 0 \). Then \( \sum_{k_j^*} > 0 \)

by (16), \( 0 \leq \sum_{i,j} r_j^{*i} \eta_j^* = \sum_{k_j^*} r_j^{*i} \eta_j^* + \sum_{k_j^*} r_j^{*i} \eta_j^* \). The first summation

\( \sum_{k_j^*} > 0 \)

\( \sum_{k_j^*} = 0 \)

\( \sum_{r_j^{*i}} > 0 \)

is nonpositive, and hence the second summation is nonnegative. But using (28), every term in this second summation is nonpositive, and by assumption \( r_j^{*i} \eta_j^* < 0 \). This contradiction establishes (29).\(^9\)

Given (29), if \( k^* r^* = 0 \), the proof is complete, since if \( r_j^{*i} > 0 \), then \( k_j^{*i} = 0 \) and hence \( \eta_j^* = 0 \). Thus suppose \( k^* r^* > 0 \). We first need to show that (16) holds with equality.

\[ 0 \leq \sum_{i,j} r_j^{*i} \eta_j^* = \sum_{k_j^{*i}} r_j^{*i} \eta_j^* + \sum_{k_j^{*i}} r_j^{*i} \eta_j^* \]

\( \sum_{k_j^{*i}} > 0 \)

\( \sum_{k_j^{*i}} = 0 \)

\( \sum_{r_j^{*i}} > 0 \)

\( \sum_{r_j^{*i}} = 0 \)

---

\(^9\)Since the logic underlying the verification of (29) is somewhat intricate, it may be useful to review what has been done. The key is to show there exists some consumer type in region \( j \) which receives positive weight in the social welfare function (SWF) and violates its budget constraint. (For then (27) and the weak form of the Walras law (16) can be used straightforwardly to establish a contradiction.) Essentially we assumed there were consumers in region \( j \) who receive no weight in the SWF who violate their budget constraints. Since they receive no weight, they are assigned 0 private goods, and hence violation of their budget constraints must stem from a tax rate larger than 1. But a tax rate larger than 1 implies public goods are being provided in their region, and hence some consumer type in the region must be receiving positive weight in the SWF (even though none of this type may be living in the region). But then this type would be violating its budget constraint, since the tax rate exceeds 1.
Either the last summation is vacuous, or (29) implies it is 0, and hence

\[
\sum_{i,j \in \mathbb{N}} r_{ij}^* \eta_{ij}^* > 0.
\]

This summation is not vacuous, since we are assuming \(k^* r^* > 0\). (27) and (30) imply

\[
\eta_{ij}^* > 0 \quad \forall i,j \ni k_{ij}^* > 0.
\]

If \(\eta_{ij}^* > 0\), then Definition 4.3 implies \(t_{ij}^* < 1\). Thus (31) implies

\[
t_{ij}^* < 1 \text{ if } \sum_{\ell} k_{ij}^\ell > 0.
\]

We also claim

\[
t_{ij}^* < 1 \text{ if } \sum_{\ell} r_{ij}^\ell > 0.
\]

There are 2 subcases. If \(\sum_{\ell} k_{ij}^\ell > 0\), (33) follows from (32). If \(\sum_{\ell} k_{ij}^\ell = 0\), then \(\exists i \ni k_{ij}^* = 0\) and \(r_{ij}^* > 0\), and hence by (29), \(\eta_{ij}^* = 0\), which in turn implies \(t_{ij}^* < 1\). Thus (33) is verified.

It follows from (33), \(t^* \in T(r^*, a^*, p^*)\) and (5) that

\[
t_{ij}^* = \frac{-p_{ij}^*}{E_{ij}^*} \quad \forall i,j \ni \sum_{\ell} r_{ij}^\ell > 0.
\]

(Note that \(\sum_{\ell} r_{ij}^\ell > 0 \iff E_{ij}^* > 0\).) Then using Definition 4.3,
\[ \sum_{i,j} r_{ij}^i \eta_{ij}^i = p^w + p^s - \sum_{j} t_j^* E_j - p^* \sum_{i,j} r_{ij}^i y_{ij}^i, \]
which equals, since (34) holds
\[ \forall j \] with \( E_j > 0 \) (or equivalently \( \sum_{i,j} r_{ij}^i > 0 \)),

\[ \text{(35)} \quad p^w + p^s - \sum_{j} (-p^x_j) - p^* \sum_{i,j} r_{ij}^i y_{ij}^i. \]
\[ \sum_{i,j} r_{ij}^i > 0 \]

Furthermore,

\[ \text{(36)} \quad x_j^* = 0 \text{ if } \sum_{i,j} r_{ij}^i = 0. \]

If \( \sum_{i,j} r_{ij}^i = 0 \), (36) follows directly from (18). Thus suppose \( \sum_{i,j} r_{ij}^i = 0 \),
\[ \sum_{i,j} k_{ij}^i > 0 \] and \( x_j^* < 0 \). (32) implies \( t_j^* \leq 1 \). But \( t^* \in T(r^*, a^*, p^*) \),
\[ x_j^* > 0, \sum_{i,j} r_{ij}^i = 0 \] and (6) imply \( t_j^* = 1 + \delta \). This contradiction establishes
\[ \text{(36).} \] Thus we can add \( \sum_{j} p^x_j \) to (35) without altering its value.
\[ \sum_{i,j} r_{ij}^i = 0 \]

But then (35) becomes \(-p^x(r^*, a^*)\), which equals 0 since \( z(r^*, a^*) = 0 \).

Thus we have established that (16) holds with equality and hence

\[ \text{(37)} \quad 0 = \sum_{i,j} r_{ij}^i \eta_{ij}^i = \sum_{i,j} r_{ij}^i \eta_{ij}^i + \sum_{i,j} r_{ij}^i \eta_{ij}^i. \]
\[ k_{ij}^i > 0 \quad k_{ij}^i = 0 \quad r_{ij}^i > 0 \quad r_{ij}^i > 0 \]

(29) implies the second summation is 0, and then (31) implies \( \eta_j^i = 0 \)
if \( r_{ij}^i > 0 \) and \( k_{ij}^i > 0 \). This result, coupled with (29), proves the
lemma for the case where \( k^* r^* > 0 \). Since (29) proves the lemma by itself
if \( k^* r^* = 0 \), the proof is complete. 10

Q.E.D.
Next we state and prove the final lemma before getting to the main theorem of the paper. Recall the definition of $v$ (Definition 4.6).

**Lemma 4.7.**

(38) If $r_j^t > 0$, $y_j^t$ maximizes $u_j^i(g_j^t, y_j^t)$ on

$$\{y_j^t \in R^*_+, p*y_j^t \leq (p*w^i + e^i p*s*)(1 - t_j^*) \};$$

(39) $r_j^t > 0$ whenever $k_j^t > 0$;

(40) If $r_j^t > 0$, then $u_j^i(g_j^t, y_j^t) \geq v_j^i(r^*, a^*, p^*) v_j^t$.

**Remark.** Since $p^* >> 0$, $v_j^t$ in (40) is defined at the point $(r^*, a^*, p^*)$. In the proof we shall use the notation $v_j^t = v_j^i(r^*, a^*, p^*)$.

**Proof.** Suppose $r_j^t > 0$. Then (14), Lemma 4.6, and $p^* >> 0$ directly imply (38). Since $p*y_j^t = (p*w^i + e^i p*s*)(1 - t_j^*)$, $t_j^* \leq 1$. Then $t_j^* \in T_j(r^*, a^*, p^*)$ and (5) imply $t_j^* = -p*x_j^*/E_j^*$. ( $E_j^* > 0$ since $r_j^t > 0$.) Thus $y_j^t$ maximizes $u_j^i(g_j^t, y_j^t)$ subject to the budget constraint $p*y_j^t \leq (p*w^i + e^i p*s*)(1 + [p*x_j^*/E_j^*])$. If we now add the assumption that $E_j^* \geq \lambda(p^*)$ (where $\lambda$ is defined in Definition 4.5), inspection reveals that this budget constraint is identical to the one underlying the definition of $v_j^t$ (see (4)). Hence

---

10. It may be useful to review the overall logic of the proof. Starting from (29), the penalty mapping $K$ together with the weak form of the Walras law (16), tell us that if $r_j^t > 0$, then $\eta_j^t > 0$. (This is the content of (29) and (31).) The fact that these budget constraints are not violated tells us that we are on the part of the tax mapping (see (34)) where the strong Walras law holds. Then the strong Walras law allows us to strengthen the claim that $r_j^t > 0 \implies \eta_j^t > 0$ to the claim that $\eta_j^t = 0$. 
\[(41) \quad u^i_j(g_j^*, y_j^*) = v_j^* \quad \text{if} \quad r_j^* > 0 \quad \text{and} \quad E_j^* \geq \lambda(p^*) .\]

Since \( r^* \in \hat{R}(r^*, a^*, p^*) \), \( p^* \gg 0 \), and \( \hat{R} \) agrees with \( R \) on \( P \times \hat{w} \times \hat{s}_n \), \( r^* \in R(r^*, a^*, p^*) \). Hence by (41) and Definition 4.7,

\[(42) \quad u^i_j(g_j^*, y_j^*) = \max \{ \max (v_j^*, u^i_j(g_j^*, y_j^*)) \} \quad \text{if} \quad r_j^* > 0 \quad \text{and} \quad E_j^* \geq \lambda(p^*) .\]

To verify (39), suppose \( k_j^* > 0 \) but \( r_j^* = 0 \). Then since \( a^* \in M(k^*, r^*) \), \( u^i_j(g_j^*, y_j^*) = \max_{y_j^* \in R^+_n} u^i_j(g_j^*, y_j^*) \) (i.e., since \( r_j^* y_j^* = 0 \)).

\( y_j^* \in R^+_n \) with \( y_j^* \leq \zeta \), \( y_j^* \) must correspond to a maximum of \( u^i_j(g_j^*, y_j^*) \) on \( y_j^* \in R^+_n \cap y_j^* \leq \zeta \), and hence by (C.4) to a maximum on \( R^+_n \) as well.

Since \( \gamma \leq (1-\rho i) \lambda \) (see Definition 4.5) and there are only \( \gamma \) regions, at least \( \rho \lambda \) consumers of type \( i \) live in regions \( j \) in which \( \sum_{k} r_j^k \geq \lambda \). Since \( \sum_{k} r_j^k \geq \lambda \implies E_j^* \geq \lambda(p^*) \), these consumers must obtain the utility level, call it \( \underline{u}^{-i} \), equal to the right-hand side of (42).

From \( z(r^*, a^*) = 0 \) and the definition of \( \Phi^i_j(\rho^i) \) (see the paragraph following (C.5)), it follows that \( \underline{u}^{-i} = \max_{j} \Phi^i_j(\rho^i) \). Thus

\[\max_{j} \Phi^i_j(\rho^i) \geq \underline{u}^{-i} \geq u^i_j(g_j^*, y_j^*) = \max_{y_j^* \in R^+_n} u^i_j(g_j^*, y_j^*) \geq \max_{y_j^* \in R^+_n} u^i_j(0, y_j^*) \] .

The second inequality follows trivially from the definition of \( \underline{u}^{-i} \) (\( j \) is now fixed), and the last inequality from (C.3). But this result contradicts (C.6a).

This contradiction establishes that \( k_j^* > 0 \implies r_j^* > 0 \), and thus (39) is verified.

\[\text{If} \quad \sum_{k} r_j^k \geq \lambda \quad \text{then} \quad E_j^* = \sum_{k} r_j^k(p^{*w^k} + \theta^{*} p^{*s^*}) \geq \sum_{k} r_j^k p^{*w^k} \geq (\sum_{k} r_j^k)(\min_{k} p^{*w^k}) \geq \lambda \min_{k} p^{*w^k} \geq \lambda \min_{k} r_j^k p^{*w^k} = \lambda(p^*) .\]
Next we wish to show that the restriction that \( E^*_j \geq \lambda(p^*) \) can be dropped from (42). If \( \sum_{\lambda} r^*_j \geq \overline{\lambda} \), then \( E^*_j \geq \lambda(p^*) \) (see footnote 11), and hence if \( r^*_j > 0 \), (42) would be satisfied. Using (C.6b) it can be shown that

\[
(43) \quad x^*_j = 0 \text{ if } 0 < \sum_{\lambda} r^*_j \leq \overline{\lambda}. \tag{12}
\]

Hence if \( 0 < \sum_{\lambda} r^*_j \leq \overline{\lambda} \) and \( r^*_j > 0 \), \( t^*_j = -p^*x^*_j/E^*_j = 0 \) and the budget

\[
(\overline{\lambda}) \text{ since } z(r^*, a^*) = 0, \text{ it follows from the definition of } \phi^i_j(\rho^i) \text{ that } \overline{u}^i < \phi^i_j(\rho^i) \text{ if } r^*_j > 0. \text{ Suppose } 0 < \sum_{\lambda} r^*_j < \overline{\lambda}. \text{ Hence } \exists i \in n \text{ such that } r^*_i > 0.
\]

Since the budget constraint underlying \( v^*_j \) is no more restrictive than the budget constraint in (38), \( u^i_j(g^*_j, y^*_j) \leq v^*_j \). Hence from Definition 4.7, \( u^i_j(g^*_j, y^*_j) \leq \overline{u}^i \leq \phi^i_j(\rho^i) \). Suppose \( x^*_j < 0 \). Then the hypotheses of (C.6b) are satisfied. By reducing \( g^*_j \) to 0 we can redistribute the vector of private goods \( -x^*_j \) evenly among the \( \sum_{\lambda} r^*_j \) consumers of the region without altering the value of \( z(r^*, a^*) \). Let \( \overline{a} = (\overline{g}, \overline{x}, \overline{y}, \overline{s}) \), where \( (\overline{g}_j, \overline{x}_j) = (g^*_j, x^*_j) \) \( \forall j \neq i \), \( (g_j, x_j) = 0 \), \( \overline{y}_j = y^*_j \forall i \) and \( \forall j \neq i \), and for \( j \) and \( i \neq j \), suppose

(1) \( \overline{y}^i_j \leq \overline{y}^i_j - (x^*_j/\sum_{\lambda} r^*_j) \). Then \( \overline{a} \in \hat{W} \) and \( z(r^*, a^*) \leq z(r^*, a^*) = 0 \).

Since \( 0 < \sum_{\lambda} r^*_j \leq \overline{\lambda} \leq 1/(\max K^i_j) \) (the last inequality following from Definition 4.5), if we choose (II) \( \overline{y}^i_j \leq y^*_j - K^i_j x^*_j \) if \( r^*_j > 0 \), then (I) will also be satisfied. Furthermore, (C.6b) tells us that we can choose the \( \overline{y}^i_j \) in (II) so that \( u^i_j(0, \overline{y}^i_j) > u^i_j(g^*_j, y^*_j) \). Summarizing, we have shown that there exists \( \overline{a} \in \hat{W} \) with \( z(r^*, \overline{a}) \leq z(r^*, \overline{a}) = 0 \) \( \forall i \neq j \), and (III) \( u^i_j(g^*_j, \overline{y}^i_j) > u^i_j(g^*_j, y^*_j) \) if \( r^*_j > 0 \). Since we are assuming \( x^*_j < 0 \), it follows from (18) that \( \sum_{\lambda} r^*_j > 0 \). Hence by (39) \( \exists i \neq j : k^i_j > 0 \) and \( r^*_j > 0 \), with \( k^i_j = 0 \) if \( r^*_j = 0 \). Then (III) implies \( U(k^*, a^*) > U(k^*, a^*) \), contradicting the maximality of \( a^* \). This contradiction establishes that \( x^*_j = 0 \).
constraint underlying \( v_j^* \) also involves a zero tax rate, and thus by (38),
\[ u_j^1(g_j^*, y_j^*) = v_j^*. \]
Thus (41) holds without the restriction that
\[ \lambda_j^* > \lambda(p^*) , \]
and hence (42) also holds without this restriction. Since
the right-hand side of (42) is trivially greater than or equal to \( v_j^* \),
(40) is verified.

Q.E.D.

We are now ready to prove the main theorem. (12) implies that
\( (r^*, g^*, x^*, y^*, s^*) \) is a feasible allocation. Lemma 4.6 implies \( t_j^* \leq 1 \)
if \( r_j^* > 0 \) and hence

\[ t_j^* \in [0,1] \text{ if } \sum_{k} r_j^* > 0 . \tag{44} \]

Let

\[ \tau^* = (\tau_1^*, \ldots, \tau_\gamma^*) , \text{ where } \tau_j^* = t_j^* \text{ if } \sum_{k} r_j^* > 0 \text{ and } \tau_j^* = 0 \text{ if } \sum_{k} r_j^* = 0 . \tag{45} \]

Theorem 4.1. The price vector \( p^* \gg 0 \), the feasible allocation
\( (r^*, g^*, x^*, y^*, s^*) \), and the vector of tax rates \( (\tau_1^*, \ldots, \tau_\gamma^*) \in \times (0,1) \)

constitute an equilibrium as described by Definition 2.5.

Proof. Conditions (a) and (b) have already been verified (see (13) and
(38)). (In (38), \( t_j^* = \tau_j^* \) since \( \sum_{k} r_j^* > 0 \).)

Since \( \sum_{k} r_j^* > 0 \) implies \( t_j^* < 1 \) (by (44)), (5) implies

\[ 13 \text{ If } \sum_{k} r_j^* = 0 , \text{ it is not necessarily true that } t_j^* = 0 . \]
\[ r_j^* = t_j^* = \frac{-p^* x_j^*}{E_j^*} \text{ if } \sum_k r_j^k > 0 . \]

Next note that
\[ x_j^* = 0 \text{ if } 0 \leq \sum_k r_j^k < \bar{\lambda} . \]

(If \( \sum_k r_j^k = 0 \), then (39) implies \( \sum_k x_j^k = 0 \), which in turn implies \( x_j^* = 0 \) by (18). The rest of (47) has already been established in (43).) If \( \sum_k r_j^k \geq \bar{\lambda} \), then \( E_j^* \geq \lambda(p^*) \) (see footnote 11). Hence using (46) and the fact that \( t_j^* = 0 \) if \( \sum_k r_j^k = 0 \), inspection reveals that for all \( ij \), the budget constraints underlying \( v_j^i \) (Definition 4.6) and the \( \gamma_j^i \) of condition (c) are identical. Thus
\[ v_j^i = u_j^i(g_j^i, \gamma_j^i) \text{ Wij} . \]

Then (40) and (48) directly imply condition (c).

Under the hypotheses of condition (d), (15) and (39) imply
\[ -p^* x_j + p^* \sum_{i\in \Theta} x_j^i y_j^i \geq -p^* x_j^* + p^* \sum_{i\in \Theta} x_j^i y_j^i . \]

Using (46) and Lemma 4.6, the right-hand side of (49) can be seen to equal
\[ p^* \sum_{i\in \Theta} r_j^i w_i + p^* s^* \sum_{i\in \Theta} r_j^i e_i . \]

Thus under the hypotheses of (d),
\[ v_j^i = \frac{-p^* x_j + p^* \sum_{i\in \Theta} x_j^i y_j^i \geq -p^* x_j^* + p^* \sum_{i\in \Theta} x_j^i y_j^i}{r_j^* > 0} + p^* s^* \sum_{i\in \Theta} r_j^i e_i \]

\[ r_j^* > 0 \]

Using (46) and Lemma 4.6, the right-hand side of (49) can be seen to equal
\[ v_j^i = \frac{-p^* x_j + p^* \sum_{i\in \Theta} x_j^i y_j^i \geq -p^* x_j^* + p^* \sum_{i\in \Theta} x_j^i y_j^i}{r_j^* > 0} + p^* s^* \sum_{i\in \Theta} r_j^i e_i \]

\[ r_j^* > 0 \]
where we have trivially extended the summation indices on the right-hand side to include all $i$. The right-hand side of (50) is positive, since $p^* > 0$, $w^i > 0 \forall i$, $p^*s^* \geq 0$, and by assumption, $\sum r_{j}^{*i} > 0$. Hence some term on the left-hand side must be positive. Suppose equality held in (50). Then by a straightforward argument, appealing primarily to $p^* > 0$ and the continuity of preferences, we can obtain a contradiction. Thus the strict inequality holds in (50), and hence condition (d) is verified.

If $\sum r_{j}^{*i} > 0$, condition (e) follows immediately from (46). If $\sum r_{j}^{*i} = 0$, then (47) implies $p^*x_{j}^* = 0$ and $r_{j}^* = 0$ by definition, and hence (e) is also satisfied.

Q.E.D.

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14 If $-p^*x_{j} > 0$, then by convexity of $Y_{j}$ and continuity of the $u_{j}^{i}$, $\exists (\tilde{y}_{j}, \tilde{x}_{j}) \in Y_{j}$ with $\tilde{x}_{j} > x_{j}$ and $u_{j}^{i}(\tilde{y}_{j}, \tilde{x}_{j}) > u_{j}^{i}(y_{j}, x_{j}) \forall i \exists r_{j}^{*i} > 0$. But since $p^* > 0$, $-p^*x_{j} + p^* \sum r_{j}^{*i}y_{j}^{i}$ is less than the right-hand side of (50)—contradiction. Thus suppose $\exists i \exists r_{j}^{*i}p^*y_{j}^{i} > 0$. Then by reducing some positive component of $y_{j}^{i}$ slightly we can obtain a vector $y_{j}^{i} \triangleright u_{j}^{i}(s_{j}, y_{j}^{i}) > u_{j}^{i}(s_{j}, y_{j}^{i})$. Then the left-hand side of (50), with $y_{j}$ replaced by $y_{j}^{i}$, will be less than the right-hand side—contradiction.
APPENDIX

In this appendix we prove Lemma 4.2 of the text. We first prove a preliminary lemma. Recall the definition of \( N \) (Definition 4.1).

\textbf{Lemma A.1.} \( N \) is upper semicontinuous on \( S_{\mathcal{W}^V} \times P \).

\textbf{Proof.} Let \( (k^V, r^V) \rightarrow (k, r) \), \( a^V \rightarrow a \), and \( a^V \in N(k^V, r^V) \) \( \forall v \). Since \( z(r^V, a^V) \leq 0 \) and \( a^V \in \hat{W} \) \( \forall v \), it follows that \( z(r, a) \leq 0 \) and \( a \in \hat{W} \) (since \( \hat{W} \) is closed). Suppose \( a \notin N(k, r) \). Then for \( \hat{a} \in N(k, r) \),

(i) \( U(k, \hat{a}) > U(k, a) \). Given \( \hat{a} \), suppose we can exhibit a sequence \( \hat{a}^V \rightarrow \hat{a} \rightarrow \hat{a}^V \in \hat{W} \) \( \forall v \) and \( z(r^V, \hat{a}^V) \leq 0 \) \( \forall v \). Then \( U(k^V, a^V) > U(k^V, \hat{a}^V) \) (since \( a^V \in N(k^V, r^V) \)), and it would follow from the continuity of \( U \) that \( U(k, a) > U(k, \hat{a}) \), contradicting (i). Thus the supposition that \( a \notin N(k, r) \) would be ruled out, and hence \( N \) would be upper semicontinuous at the arbitrary point \( (k, r) \).

Thus it remains to define a sequence \( \hat{a}^V \) with the desired properties.

If \( z(r, \hat{a}) < 0 \), then \( \exists v_0 \exists \psi v \geq v_0 , \ z(r^V, \hat{a}) \leq 0 \). Hence define \( \hat{a}^V = 0 \) if \( v < v_0 \) and \( \hat{a}^V = \hat{a} \psi v \geq v_0 \). This sequence clearly has the desired qualities.

Thus suppose \( z(r, \hat{a}) \leq 0 \) but the equality holds for at least one component. For any \( v \) one of the following two cases must hold:

(a) \( z(r^V, \hat{a}) \leq 0 \); (b) \( z(r^V, \hat{a}) \) has at least one positive coordinate.

If (a) holds, define \( \alpha^V = 1 \). If (b) holds, choose \( \alpha^V \in (0, 1) \exists \ z(r^V, \alpha^V \hat{a}) \leq 0 \), with equality holding in at least one coordinate. (Clearly this can be done, since \( z(r^V, 0) < 0 \) because \( w \gg 0 \), and \( z \) is continuous.)

Then define \( \hat{a}^V = \alpha^V \psi \psi v \). Since \( 0 \) and \( \hat{a} \in \hat{W} \) and \( \hat{W} \) is convex, \( \hat{a}^V \in \hat{W} \).
By definition of $\alpha^v$, $z(r^v, \hat{a}^v) \leq 0$. Thus it remains to show $\hat{a}^v \to \hat{a}$. Since the sequence $\alpha^v$ is bounded, it has a limit point. Let $\alpha^*$ be any such limit point. Then to show $\hat{a}^v \to \hat{a}$ it is sufficient to show $\alpha^* = 1$.

If case (a) holds for an infinite number of $v$, trivially such a subsequence of the $\alpha^v$ has a limit point of 1. Thus suppose (b) holds for an infinite number of $v$ and $\alpha^*$ is a limit point of this subsequence. From the continuity of $z$ it follows that $z(r, \alpha^* \hat{a}) \leq 0$, with equality holding in at least one coordinate. By assumption, $z(r, \hat{a}) \leq 0$ with equality holding in at least one coordinate. If $\alpha^* < 1$, then $z(r, \alpha^* \hat{a}) << 0$ (since $w \gg 0$). This contradiction establishes that $\alpha^* = 1$, and hence $\hat{a}^v \to \hat{a}$.

Q.E.D.

Lemma 4.2. $M$ is upper semicontinuous on $S_{M^*} \times P$, and for each $(k, r) \in S_{M^*} \times P$, the image set $M(k, r)$ is convex.

Proof. Let $(k^v, r^v) \to (k, r)$, $(a^v, p^v) \to (\overline{a}, \overline{p})$, and $(a^v, p^v) \in M(k^v, r^v)$ for all $v$. To show $(\overline{a}, \overline{p}) \in M(k, r)$ we must show (i) $\overline{a} \in N(k, r)$; (ii) $\overline{p}z(r, \overline{a}) \geq 0$ for all $\overline{a} \in \mathcal{W}$ with $U(k, \overline{a}) \geq U(k, \overline{a})$. (i) follows immediately from the upper semicontinuity of $N$. Given $\overline{a} \in \mathcal{W}$ with $U(k, \overline{a}) \geq U(k, \overline{a})$, for each $v$ either (I) $U(k^v, \overline{a}) \geq U(k^v, a^v)$ or (II) $U(k^v, a^v) > U(k^v, \overline{a})$. For each $v$ satisfying (I), $p^v z(r^v, \overline{a}) \geq 0$ since $(a^v, p^v) \in M(k^v, r^v)$. If there are an infinite number of $v$ satisfying (I), then in the limit of this subsequence we obtain $\overline{p}z(r, \overline{a}) \geq 0$, verifying (ii) for this case.

By (C.3) and (B.3) $\exists \overline{a} \in \mathcal{W}$ with $U(k, \overline{a}) > U(k, \overline{a})$, and hence $U(k, \overline{a}) > U(k, \overline{a})$. Since $(k^v, a^v) \to (k, \overline{a})$, it follows from the continuity of $U$ that $\exists v_0 \in \mathcal{W}$ with $U(k^v, \overline{a}) > U(k^v, a^v)$. Suppose (II) holds for an infinite number of $v$. Then $\forall v \geq v_0$ satisfying (II), $U(k^v, \overline{a}) > U(k^v, a^v) > U(k^v, \overline{a})$. For these $v$ define...
\( \hat{a}^v \equiv \alpha_v \hat{a} + (1 - \alpha_v)\hat{a} \) where \( \alpha_v \in (0, 1) \) is chosen so that \( U(k^v, \hat{a}^v) \equiv U(k^v, \hat{a})^v \). Such a choice is possible because \( U \) is continuous. Then \( p^v z(r^v, \hat{a}^v) \geq 0 \) since \( \hat{a}^v \in \mathcal{W} \) and \( (a^v, p^v) \in M(k^v, r^v) \). The bounded sequence \( \alpha_v \) has a convergent subsequence. Suppose \( \alpha_v \to \alpha_* \) along such a subsequence of \( v \) satisfying (II). If we can show \( \alpha_* = 0 \), then it will follow that \( \hat{a}^v \to \hat{a} \) and hence \( \overline{p}z(r, \hat{a}) \geq 0 \), verifying (ii) for this case. Suppose \( \alpha_* > 0 \). Then \( \hat{a}^v \to \alpha_* \hat{a} + (1 - \alpha_*)\hat{a} \) and \( U(k, \alpha_* \hat{a} + (1 - \alpha_*)\hat{a}) \geq \alpha_* U(k, \hat{a}) + (1 - \alpha_*)U(k, \hat{a}) \geq U(k, \hat{a}) \), the first inequality following from the concavity of \( U \) and the second from the definition of \( \hat{a}^v \). But \( U(k, \alpha_* \hat{a} + (1 - \alpha_*)\hat{a}) \equiv \lim U(k^v, \hat{a}^v) = \lim U(k^v, \hat{a}^v) = U(k, \hat{a}) \leq U(k, \hat{a}) \), the second equality following from the definition of \( \hat{a}^v \), the third from continuity of \( U \) and \( (k^v, a^v) \to (k, \hat{a}) \), and the inequality from the definition of \( \hat{a} \). This contradiction establishes that \( \alpha_* = 0 \).

The proof of convexity is straightforward.

Q.E.D.
REFERENCES


