On a Theorem of Scarf

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In [3], Herbert Scarf has given a remarkable solution for a classical problem of economics. In this note, I wish to suggest a simplification of his proof, and a slight weakening of his assumptions.

Let $\Omega$ denote the non-negative orthant of the commodity space $\mathbb{R}^e$. The economy is made up of $N$ infinite sequences of consumers. For each $j = 1, \ldots, N$, all the consumers of the $j^{th}$ sequence have the same resources $I_j$ in the interior of $\Omega$, and the same preference preordering $\preceq_j$ on $\Omega$ satisfying

(1) \[ \{ x \in \Omega \mid x \preceq_j x' \} \quad \text{and} \quad \{ x \in \Omega \mid x \preceq_j x'' \} \] are closed for every $x'$ in $\Omega$,

(2) for every $x$ in $\Omega$, there is $x'$ in $\Omega$ such $x' \preceq_j x$,

(3) $x' \preceq_j x$ implies $tx' + (1 - t)x \preceq_j x$ for every $t$

such that $0 < t < 1$.

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1 Research undertaken by the Cowles Commission for Research in Economics under Task NR 047-006 with the Office of Naval Research. I thank Herbert Scarf for the privilege of seeing his ideas develop that he gave me last spring. To these conversations I owe my interest in the subject of his article.
(4) \( x \succ_j x' \) for some \( x' \) implies that \( x \) is interior to \( \Omega \).

An allocation is an \( N \)-tuple of infinite sequences \( (x_1^i, \ldots, x_N^i) \) of points of \( \Omega \), where \( x_j^i \) is the consumption of the \( i \)th consumer in the \( j \)th sequence, such that

\[
\lim_{n \to \infty} \left( \sum_{i=1}^{n} \sum_{j=1}^{N} x_j^i - n \sum_{j=1}^{N} I_j \right) = 0.
\]

A finite coalition \( S \) of consumers blocks an allocation \( (x_1^i, \ldots, x_N^i) \) if, for every consumer \( (i,j) \) in \( S \), there is a consumption \( y_j^i \) in \( \Omega \) such that \( \sum_{(i,j) \in S} y_j^i = \sum_{(i,j) \in S} I_j^i \), and \( y_j^i \succeq_j x_j^i \) for every \( (i,j) \) in \( S \), while \( y_j^i \succeq_j x_j^i \) for at least one \( (i,j) \) in \( S )^2 \).

The core of the economy is the set of allocations that no finite coalition blocks.

An allocation \( (x_1^i, \ldots, x_N^i) \) and a price system \( p \) form an equilibrium of the economy if, for every \( (i,j) \), the consumption \( x_j^i \) is a greatest element of the set \( \{ x \in \Omega \mid p \cdot x \preceq p \cdot I_j \} \) for \( j \).

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2 It is convenient, here, to identify the resources of consumer \( (i,j) \) by \( I_j^i \), although \( I_j^i \) is a constant with respect to \( i \). Given the assumptions made on preferences, our definition of a blocking coalition is easily seen to be equivalent to H. Scarf's.
Theorem: Given an allocation \( (x^i_1, \ldots, x^i_N) \) in the core, there is a price system \( p \) with which it forms an equilibrium.

Proof: By (1), there is a continuous utility function \( u_j \) on \( \Omega \) for every \( j \) ([1], p. 56). We denote \( u_j(x^i_j) \) by \( v^i_j \). Two cases have to be distinguished:

(a) for every \( j \), \( \inf_i v^i_j = \lim_i v^i_j \).

We introduce the notation

\[
C^i_j = \{ x \in \Omega \mid u_j(x) > v^i_j \} , \quad \Gamma^i_j = C^i_j - \{ I_j \};
\]

\[
C_j = \{ x \in \Omega \mid u_j(x) > \inf_i v^i_j \} , \quad \Gamma_j = C_j - \{ I_j \}.
\]

All these sets are non-empty, by (2), and convex, by (3) and (1) ([1], p. 60). They also have non-empty interiors, for every \( C^i_j \) does. Indeed, let \( x \) be a point in \( C^i_j \), i.e., such that \( x \succ^i_j x^i_j \). By (1), \( x \) has a neighborhood in \( \Omega \) all of whose elements \( \succ^i_j x^i_j \). But, in that neighborhood, there are points interior to \( \Omega \). Any one of them is interior to \( C^i_j \).

The basic property of the sets \( \Gamma_j \) is
(6) \( 0 \) is not interior to the convex hull of \( \bigcup_{j=1}^{N} \Gamma_j \).

To establish this, we denote the interior of a set \( S \) by \( \text{Int} \ S \), its convex hull by \( \text{H}(S) \), and its closure by \( \overline{S} \), and we first prove that

(7) \( \text{Int} \ \text{H}(\bigcup_{j} \Gamma_j) \subset \text{H}(\bigcup_{j} \text{Int} \Gamma_j) \).

\[
\begin{align*}
\text{Int} \ \text{H}(\bigcup_{j} \Gamma_j) & \subset \text{Int} \ \text{H}(\bigcup_{j} \text{Int} \Gamma_j) \subset \\
\text{Int} \ \text{H}(\bigcup_{j} \text{Int} \Gamma_j) & \subset \text{Int} \ \text{H}(\bigcup_{j} \text{Int} \Gamma_j) = \\
\text{Int} \ \text{H}(\bigcup_{j} \text{Int} \Gamma_j)
\end{align*}
\]

Assume now that (6) does not hold. According to (7), there are, for each \( j \), a joint \( y_j^* \) in \( \text{Int} \Gamma_j \), and a non-negative real number \( \alpha_j \), with \( \sum_{j=1}^{N} \alpha_j = 1 \), such that

\[
\sum_{j} \alpha_j y_j^* = 0.
\]

Thus, one can find, for each \( j \), a point \( y_j \) in \( \Gamma_j \), and a non-negative rational number \( r_j \), with \( \sum_{j=1}^{N} r_j = 1 \), such that
\[ \sum_j r_j y_j = 0. \]

Multiplying by a common denominator of the \( r_j \), we obtain

\[ \sum_j k_j y_j = 0 \]

for an \( N \)-tuple \( (k_j) \) of non-negative integers, not all zero. Since
\( y_j \in T_j \), one has \( u_j (y_j + I_j) > \inf_j v^i_j \). Therefore, according to (a), we can select, in the \( j \)th sequence, \( k_j \) consumers whose \( v^i_j \) are less than \( u_j (y_j + I_j) \). This means that \( y_j \) belongs to the set \( T^i_j \) of each one of these \( k_j \) consumers. Consequently, \( 0 \) belongs to the sum of the sets \( T^i_j \) of the \( k_1 + \ldots + k_N \) consumers we have selected. And the coalition of these consumers would block the given allocation.

Having established (6), we apply Minkowski's theorem to the situation it describes, and we obtain a hyperplane through \( 0 \), with normal \( p \), bounding for \( \bigcup_{j=1}^N T_j \), hence for every \( T_j \). We write this as

\[ p \cdot T_j \geq 0, \text{ or } p \cdot C_j \geq p \cdot I_j. \]

However, \( C^i_j \) is contained in \( C_j \) for every \( (i,j) \). In addition, by (5), every \( x \) such that
\[ x \geq x^i_j \] is adherent to \( C^i_j \). Therefore
(8) for every \((i,j)\), \(x_j^i\) implies \(p \cdot x \geq p \cdot I_j\).

In particular, \(p \cdot x_j^i \geq p \cdot I_j\) for every \((i,j)\). If any of these inequalities were strict, the inner product of \(p\) and the vector in the parenthesis of (5) would not tend to zero when \(n \to \infty\). Hence

\[
p \cdot x_j^i = p \cdot I_j \quad \text{for every} \quad (i,j).
\]

Finally, since \(I_j\) is interior to \(\Omega\), it follows readily from (8) ([1], p. 69), that \(x_j^i\) is a greatest element of the set

\[
\left\{ x \in \Omega \mid p \cdot x \leq p \cdot I_j \right\} \quad \text{for} \quad j
\]

(b) for some \(j'\), \(\inf_{j'} v_{j'}^i < \lim_{j \to \infty} v_{j'}^i\).

We will show that this case cannot occur. Notice first that, according to (5),

\[
\lim_{n \to \infty} \left( \sum_{j=1}^{N} x_j^n - \sum_{j=1}^{N} I_j \right) = 0 .
\]

Therefore the sequence of \(N\)-tuples \((x_j^n)\) is bounded, and we can extract a subsequence converging to the \(N\)-tuple \((x_j^0)\). Clearly
(9) \[ \sum_{j=1}^{N} x_j^o = \sum_{j=1}^{N} r_j. \]

Moreover

\[ u_j(x_j^o) \geq \inf_j v_j^i \quad \text{for every } j, \quad \text{and } u_j(x_j^o) > \inf_j v_j^i. \]

The last inequality, which follows from (b), implies \( x_j^o \succ_j x_j^1 \) for some \( i \), hence, by (4),

\[ x_j^o \] is interior to \( \Omega \).

Let \( s(x,r) \) denote the open sphere with center \( x \) and radius \( r > 0 \). We can choose \( r \) small enough for \( s(x_j^o, r) \) to be contained in \( \Omega \), and for the utility of every consumption in \( s(x_j^o, r) \) to be greater than \( \inf_j v_j^i \). By (2) and (3), there is, for every \( j \neq j' \), a consumption \( x_j^* \) in \( s(x_j^o, r/N) \) such that

\[ u_j(x_j^*) > u_j(x_j^o) \quad (j \neq j') \]

We define \( x_j^* \) as equal to \( \sum_{j=1}^{N} x_j^o - \sum_{j \neq j'} x_j^* \).

Thus \( |x_j^* - x_j^o| < r \). Consequently \( x_j^* \) is in \( \Omega \) and
\[ u_j^i (x_j^*) > \inf_j v_j^1. \]

Also, by (9),
\[ \sum_{j=1}^{N} x_j^* = \sum_{j=1}^{N} I_j. \]

To conclude, select for each \( j \) a consumer \( (i,j) \) such that \( x_j^i \not< x_j^* \). The coalition of these \( N \) consumers blocks the given allocation.

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The theorem can be generalized without modification of the proof. For instance, the common consumption set \( X_j \) of the consumers of the \( j^{th} \) sequence may be any closed, convex set with a non-empty interior (instead of being \( \Omega \) ), provided that the asymptotic cone of \( X = \sum_{j=1}^{N} X_j \) satisfies \( AX \cap (-AX) = \{ 0 \} \) (to insure that the sequence \( (x_j^n) \), at the beginning of (b), is bounded). Assumptions (1), (2), (3), and (4) are made on the preferences \( \preceq_j \) on \( X_j \). Then, given an allocation in the core, there is a price system with which it forms a quasi-equilibrium (a definition of this concept, and a discussion of its relation to the concept of equilibrium will be found in [2]).

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References

