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Stochastic Choice and Cardinal Utility*

By

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D. Davidson and J. Marschak [6] consider the following problem. A set S of actions is given; an agent is presented with a pair (a, b) of actions in S and asked to choose one; he chooses a with probability $p(a, b)$ and b with probability $p(b, a) = 1 - p(a, b)$. The relation $p(a, b) > p(c, d)$ will be read « a is preferred to b more than c is preferred to d ». One is naturally led to seek a real-valued (cardinal utility) function u on S such that $p(a, b) > p(c, d)$ be equivalent to $u(a) - u(b) > u(c) - u(d)$. Formally:

(1) <u>S is a set, p is a function from $S \times S$ to $[0, 1]$ such that $p(a, b) + p(b, a) = 1$ for every (a, b) in $S \times S$.</u>

<u>Definition</u>: <u>A utility function for (S, p) is a real-valued function u on S such that</u>

$$[\, p(a, b) \leqq p(c, d) \,] \iff [\, u(a) - u(b) \leqq u(c) - u(d) \,].$$

The problem is thus to find further assumptions on (S, p) which will insure the existence of a utility function.

Since $u(a) - u(b) \leqq u(c) - u(d)$ is equivalent to $u(a) - u(c) \leqq u(b) - u(d)$; if there is a utility function for (S, p), then ([6], section II)

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(2) $[\, p(a, b) \leqq p(c, d) \,] \iff [\, p(a, c) \leqq p(b, d) \,]$.

This necessary condition for the existence of a utility function will be taken as an axiom. It has the immediate consequence

(2') $[\, p(a, b) = p(a', b')$ and $p(b, c) = p(b', c') \,] \implies [\, p(a, c) = p(a', c') \,]$.

Proof: By (2), $p(a, a') = p(b, b') = p(c, c')$. Applying (2) again to the equality of the first term and the third term, one obtains the conclusion.

The formulation of the third axiom requires the introduction of a new function P. Consider a point (x, y) of the square $[0, 1] \times [0, 1]$. That point belongs to the domain of P if and only if there are three elements a, b, c of S such that $p(a, b) = x$ and $p(b, c) = y$. Then the value of P at (x, y) is $P(x, y) = p(a, c)$. This definition is legitimate since, by (2'), a different triple a', b', c' of points of S such that $p(a', b') = x$ and $p(b', c') = y$ would give the same value for P. It is postulated that

(3) The function P is continuous.

In other words, $p(a, c)$ depends continuously on $p(a, b)$ and $p(b, c)$. It is easy to check, using (2), that P is also increasing in each one of its two variables.

The last, and least satisfactory, axiom is

(4) If $p(b, a) \leqq q \leqq p(c, a)$, then there is d in S such that $p(d, a) = q$.

Theorem. Under assumptions (1), (2), (3), (4), there is for (S, p) a utility function determined up to an increasing linear transformation.

The trivial case where p is constant on $S \times S$ is solved by taking a constant utility function on S ; it will now be excluded.

The theorem will be proved by means of a representation of S in [0, 1]. Let k be an arbitrary element of S which will be kept fixed. The generic element a of S is represented by the number $\alpha = p(a, k)$. According to (4), the image of the set S is an <u>interval</u> Σ, contained in [0, 1]. Given two elements a, b of S, one has $p(a, b) = \underline{P}[\, p(a, k), p(k, b)\,] = \underline{P}[\alpha, 1-\beta]$. The last term will be denoted by $\pi(\alpha, \beta)$. The function π from $\Sigma \times \Sigma$ to [0, 1] defined in this way satisfies

(5) $\quad p(a, b) = \pi(\alpha, \beta)$

and thus corresponds to p in the representation. According to (3), π is <u>continuous</u>.

It is clear that finding a utility function u for (S, p) is <u>equivalent</u> to finding a utility function v for (Σ, π), the two utility functions being related by

$$u(a) = v(\alpha)$$

The second problem, however, is notably easier than the first.

Summing up the data of the new problem. Σ is a non-degenerate interval in [0, 1]; π is a function from $\Sigma \times \Sigma$ to [0, 1] such that $\pi(\alpha, \beta) + \pi(\beta, \alpha) = 1$; moreover π is continuous, increasing in α, decreasing in β, and satisfies (2).

On Fig. 1, the square $\Sigma \times \Sigma$ has been drawn and, in it, isoprobability lines corresponding to a few values of π. (The diagonal corresponds to the value 1/2). The problem is to find an increasing transformation

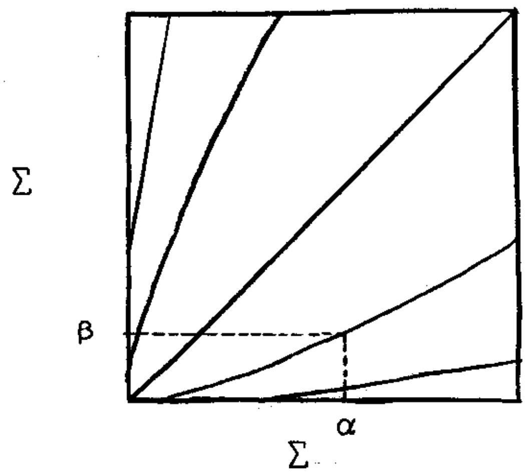

Fig. 1

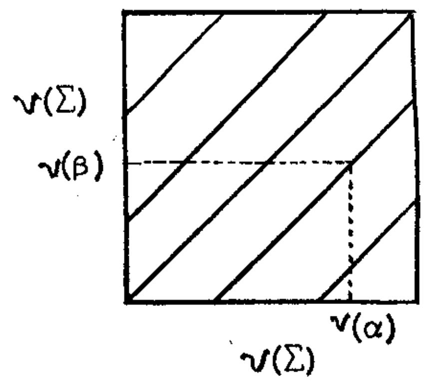

Fig. 2

ν of Σ into the reals such that these isoprobability lines become straight lines parallel to the diagonal of $\nu(\Sigma) \times \nu(\Sigma)$ (see Fig. 2). This, however, is but a particular case of a problem of plane topology* solved in 1927 by G. Thomsen [15] and W. Blaschke [4] (See also W. Blaschke and G. Bol [5] pp. 1-42). Instead of proceeding to a painstaking identification of assumptions and conclusions it seems preferable to sketch a proof adapted to the present situation.

Select two arbitrary points α_0 and α_1 of Σ such that $\alpha_0 < \alpha_1$, and take $\nu(\alpha_0) = 0$ and $\nu(\alpha_1) = 1$. Consider the function f defined by $f(\alpha) = \pi(\alpha, \alpha_0) - \pi(\alpha_1, \alpha)$; it is continuous and increasing, $f(\alpha_0)$ is

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* The problem can be roughly described as follows. Given three families of curves in a plane, when does there exist a topological transformation carrying them into three families of parallel straight lines? On Fig. 1 the three families are the isoprobability lines, the verticals, the horizontals. After the transformation, on Fig. 2, they are the parallels to the diagonal, the verticals, the horizontals.

negative and $f(\alpha_1)$ is positive; hence there is a unique $\alpha_{1/2}$ where $f(\alpha_{1/2}) = 0$, i.e.

$$\pi (\alpha_1, \alpha_{1/2}) = \pi (\alpha_{1/2}, \alpha_0).$$

The value of v at $\alpha_{1/2}$ is necessarily 1/2. That dichotomy of the interval $[\alpha_0, \alpha_1]$ will be repeated ad infinitum. At the n^{th} stage, one has points of the form $\alpha_{\frac{i}{2^n}}$ and the value of v at that point is necessarily $\frac{i}{2^n}$. In this way the function v is defined on the set Δ of points obtained by the preceding process. Checking that v satisfies the definition of utility on Δ amounts to checking that

$$(6) \quad \pi(\alpha_{\frac{i+1}{2^n}}, \alpha_{\frac{i}{2^n}}) = \pi(\alpha_{\frac{j+1}{2^n}}, \alpha_{\frac{j}{2^n}}) \text{ for every } i, j \text{ from } 0 \text{ to } 2^n - 1.$$

This, however, is a direct consequence of assumption (2) for π. The common value of the probabilities in (6) depends only on n, it will be denoted by $\Theta (n)$. It is not difficult to show, using the continuity of π, that $\lim_{n \to +\infty} \Theta (n) = 0$. It is then easy to derive from this fact that the set Δ is dense in $[\alpha_0, \alpha_1]$. The utility function v constructed on Δ is a one-to-one correspondence between Δ and the set of dyadic numbers of [0, 1], which is dense in [0, 1]. The extension of v from Δ to $[\alpha_0, \alpha_1]$ is therefore immediate and the resulting function is clearly continuous. It remains only to extend v from $[\alpha_0, \alpha_1]$ to Σ. For this, a procedure similar to that of Herstein-Milnor ([9], pp. 296-297) can be used.

The only arbitrariness in the construction comes from the choice of the values 0 and 1 for α_0 and α_1 respectively. Given two utility functions, one is derived from the other by an increasing linear transformation.

The discussion on cardinal utility in the thirties is well known (O. Lange [10], E. H. Phelps Brown, H. Bernardelli, O. Lange [12], R. G. D. Allen [1], F. Zeuthen [16], P. A. Samuelson [13], W. E. Armstrong [3]). However the important paper by F. Alt [2] has generally been overlooked. Noteworthy in the recent revival of interest in that topic is the article by P. Suppes and M. Winet [14]. In connection with the problem of stochastic choice the work of N. Georgescu-Roegen ([7] section VI, [8]) and of A. G. Papandreou, in collaboration with O. H. Sauerlender, O. H. Brownlee, L. Hurwicz, and W. Franklin [11] must be mentioned.

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