Suppose that we have a two way analysis of variance problem where the random variable $X_{ij}$ can be written as

$$X_{ij} = a_i + b_j + \mu + c_{ij} + u_{ij} \quad i = 1, \ldots, I$$
$$j = 1, \ldots, J$$

The $a_i$'s, $b_j$'s, $\mu$, and $c_{ij}$'s are to be regarded as fixed parameters. The variable $u_{ij}$ is the random disturbance in the equation and $\text{Var} u_{ij} = \sigma^2$ and $E u_{ij} = 0$. The problem as stated so far is the usual model for a two way analysis of variance.

However in addition to these requirements suppose that it is known that $\sum_{j} u_{ij} = K$ where $K$ is some constant for all $i$. This means that the error terms are dependent. In order to proceed with the analysis it becomes necessary to make an assumption regarding the nature of the dependence of the $u_{ij}$'s for a given $i$. We shall assume that $E u_{ij} u_{ij'}$ are equal for each $i, j, j'$. This implies that the covariance of the $u_{ij}$'s have a particularly simple value.
Consider \( \text{Var} \sum_j u_{ij} \). It is clear that \( \text{Var} \sum_j u_{ij} = 0 \)

\[
\text{Var} \sum_j u_{ij} = E \left( \sum_j u_{ij} \right)^2
\]

\[
E \left( \sum_j u_{ij} \right)^2 = E \left( \sum_j \sum_k u_{ij} u_{ik} \right) = \sum_j \sum_{k=1}^J E (u_{ij} u_{ik})
\]

But when \( j = k \) we have \( E u_{ij} u_{ik} = \sigma^2 \) and when \( j \neq k \) we have

\[
E u_{ij} u_{ik} = \text{Cov} u_{ij} u_{ik} = \lambda \text{ say.}
\]

Therefore

\[
E \left( \sum_j u_{ij} \right)^2 = J \sigma^2 + 2 \binom{J}{2} \lambda = 0
\]

Hence

\[
\lambda = - \frac{J \sigma^2}{2 \binom{J}{2}} = - \frac{\sigma^2}{J-1}
\]

Since there is no replication it is impossible to estimate the interaction term of first order \( c_{ij} \). Furthermore we assume that \( \sum_j X_{ij} = 0 \) for all \( i \).

This is the motivation for the assumption that \( \sum_j u_{ij} = K \). In particular

this means that \( \sum_i \sum_j X_{ij} = X_+ = 0 \). Hence the grand mean of the observation will also be zero. Suppose we want Markoff estimates of \( a_i, b_j, \) and \( \mu \).

\[
\sum_j X_{ij} = Ja_i + b_j + J\mu + \sum_j u_{ij} = 0
\]

To obtain the Markov estimates we wish to

\[
\min_{i,j} (X_{ij} - a_i - b_j - \mu)^2 \text{ subject to } \sum_j u_{ij} = -(Ja_i + b_j + J\mu)
\]

so that we have \( J \) constraints. We introduce \( J \) Lagrangean parameters \( \rho_i \) and minimize the following expression:
\[ \sum_{i,j} (x_{ij} - a_i - b_j - \mu)^2 + \sum_i \rho_i (Ja_i + b_i + J\mu) \]
\[ \frac{\partial}{\partial a_i} : 2\sum_j (x_{ij} - a_i - b_j - \mu) + \rho_i J = 0 \]
\[ \frac{\partial}{\partial b_j} : 2\sum_i (x_{ij} - a_i - b_j - \mu) + \sum_i \rho_i = 0 \]
\[ \frac{\partial}{\partial \mu} : 2\sum_{i,j} (x_{ij} - a_i - b_j - \mu) + J\sum_i \rho_i = 0 \]

\[ Ja_i + b_i + J\mu = 0 \]

These equations simplify to:

\[ x_{1.} - Ja_1 - b_1 - J\mu + \frac{\rho_1}{2} J = 0 \]
\[ x_{.j} - \bar{a} - Ib_j - J\mu + \frac{1}{2} \sum_i \rho_i = 0 \]
\[ x_{..} - Ja_1 - Ib_j - J\mu + \frac{1}{2} \sum_i \rho_i = 0 \]

\[ Ja_1 + b_1 + J\mu = 0 \]

Hence we see that \( \rho_i = 0 \) for every \( i \) and that \( x_{.j} - \bar{a} - Ib_j - J\mu = 0 \) is the only set of independent equations remaining. If we make the usual assumption that \( a_1 = b_1 = 0 \) we see that \( \mu = 0 \) so that

\[ \hat{b}_j = \frac{x_{1.}}{1} = \bar{x}_{.j} \]

None of the \( a_i \) parameters are estimable because our restriction on the \( X_i \).
causes all of these equations to vanish identically. Ordinarily we would estimate \( \mu \) by \( \bar{X}_n \) and we would take \( \hat{b}_j = \bar{X}_{.j} - \bar{X}_n \), and \( \hat{\alpha}_1 = \bar{X}_{1.} - \bar{X}_n \). We see that the estimate for \( b_j \) is what it would be without the restrictions when we recall that \( \bar{X}_n = 0 \).

In order to make tests on the significance of the \( \hat{a}_j \)'s we have to specify a distribution for the error terms, \( \epsilon_{ij} \)'s. We shall assume that the \( \epsilon_{ij} \)'s are normal with means, variance and covariance as we have specified.

First let us learn the expected value of \( \hat{b}_j \) and its variance. Obviously

\[
\hat{b}_j = b_j
\]

\[
\text{Var} \hat{b}_j = E [\bar{X}_{.j} - b_j]^2 = \frac{1}{\bar{X}_n^2} E (X_{.j})^2 - b_j^2
\]

After some calculation we find that

\[
\text{Var} \hat{b}_j = \frac{\sigma^2}{\bar{X}_n}
\]

\[
\sum_{i,j} (X_{ij} - \mu - a_i - b_j)^2 = \sum_{i,j} (X_{ij} - \mu - \hat{\mu} - \hat{a}_1 - \hat{b}_j)^2
\]

We may as well take \( a_i = \hat{a}_i = 0 \) and \( \mu = \hat{\mu} = 0 \) since it turns out that their variance is zero and our estimates of them are identically zero. Hence

\[
\sum_{i,j} (X_{ij} - \mu - a_i - b_j)^2 = \sum_{i,j} (X_{ij} - \hat{\epsilon}_j)^2 + \sum_{i,j} (\hat{\alpha}_1 - \hat{b}_j)^2
\]

Taking expected values on both sides we have

\[
IJ \sigma^2 - IJ \cdot \frac{\sigma^2}{\bar{X}_n} = E \sum_{i,j} (X_{ij} - \hat{\epsilon}_j)^2
\]

\[
\sigma^2 J (I - 1) = E \sum_{i,j} (X_{ij} - \hat{b}_j)^2
\]
We claim that the estimate of the residual sums of squares has \((I - 1) (J - 1)\) degrees of freedom. Since there are \(I\) restrictions on the observations the total number of degrees of freedom is

\[ I J - I = I (J - 1) \]

The degrees of freedom due to the estimate of \(b_j\) is \(J - 1\). Therefore the degrees of freedom of the residual is

\[ I (J - 1) - J - 1 = (I - 1) (J - 1). \]

\[ \frac{1}{(I - 1) (J - 1)} \cdot E \sum_{ij} (X_{ij} - \hat{b}_j)^2 = \frac{\sigma^2_{J(I - 1)}}{(I - 1)(J - 1)} = \frac{\sigma^2_J}{J - 1} \]

Similarly the

\[ \frac{E \sum_{ij} (\hat{b}_j - b_j)^2}{J - 1} = \frac{\sigma^2_J}{(J - 1)} \]

\[ \cdots \frac{1}{(J - 1)} \sum_j (\bar{X}_j)^2 \]

has the \(F\) distribution with

\[ \frac{1}{(I - 1)(J - 1)} \sum_{ij} (X_{ij} - \bar{X}_j)^2 \]

\((J - 1)\) and \((I - 1) (J - 1)\) degrees of freedom respectively.

A model similar to this was used to analyze the gains and losses of traders in future markets.