

Remarks on a Rational Selection of a Decision Function
(Sequel to Discussion Papers, Statistics, 326, 326a)

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Here we shall consider the general problem G_n of all mixed problems corresponding to problems (u, D, S) where S is a set of n elements, and D contains a finite number of pure strategies. We shall prove:

Theorem 1.

The only solution of G_n is given by $C(Q)$ when $C(Q)$ is the set of strategies d which maximize $\sum_{s \in S} u(d, s)$.

As in the two state problem we may consider the geometrical representations of a problem Q . To each strategy d there corresponds a point in n dimensional space whose coordinates are $u(d, s_1), u(d, s_2), \dots, u(d, s_n)$ where s_1, s_2, \dots, s_n are the elements of S . A geometrical representation of a problem Q of G_n is convex because Q is mixed.

Most of the axioms and consequences in 326a may be geometrically interpreted. For example

A: If the representation of Q_1 is transformed to that of Q_2 by one of the following transformations, the set of points representing $C(Q_1)$ are similarly transformed / into the set representing $C(Q_2)$.

1. relabeling of axes $x'_i = x_{j(i)}$ j(i) takes all values from 1 to n
2. translation $x'_i = x_i + \mu_i$
3. expansion or contraction $x'_i = cx_i$

B: The set of points corresponding to $C(Q)$ form a convex set.

C: If x corresponds to an element of $C(Q)$ and $x = pu + (1 - p)v$ where $0 < p \leq 1$ and u, v are points of the geometrical representation of Q then u corresponds to an element of $C(Q)$.

D: To any convex set in n dimensional space which is generated by a finite number of points [i.e. there are m points u_1, u_2, \dots, u_m so that for all x in the convex set there is a $p_1, p_2 \dots p_m$ $0 \leq p_i$ $\sum p_i = 1$ so that $x = p_1 u_1 + p_2 u_2 + \dots + p_m u_m$] such that every component of every point of this set is an attainable utility, there is at least one problem Q of G_n for which this convex set is a geometrical representation.

E: If x and y are points of a geometrical representation of Q and x is uniformly dominated by y [i.e. $x_i \leq y_i$ for all i and $x_i < y_i$ for some i] then x does not correspond to an element of $C(Q)$.

Before proceeding with the proof it is well to note that the proof for the two dimensional case was considerably complicated by a desire to go as far as possible without axiom 12 [convexity]. Here we shall use the convexity axiom freely but not apply axiom 7 that $C(Q_1 + Q_2) \supset C(Q_1) \cap C(Q_2)$.

The Theorem is an immediate consequence of Lemmas 1, 2, 3, and 4.

Lemma 1. If Q is the problem whose geometrical representation is given by the convex set of x for which $m \leq x_i \leq M$ and $\sum x_i = c$, then $C(Q)$ consists of all strategies of Q .

Lemma 2. If Q is a problem of G_n the set of strategies d which maximize $\sum_{s \in S} u(d, s)$ are in $C(Q)$.

Lemma 3. If Q is a problem of G_n and $\sum_{s \in S} u(d^*, s) < \max_{d \in D} \sum_{s \in S} u(d, s)$. Then d^* is not in S .

Lemma 4. The solution $C(Q)$ of Theorem 1, satisfies axioms one to twelve.

Lemma 1.

The case where there is only one point of the convex set is trivial. Hence we assume that $m n < c < n M$ and thus no point has all components equal to M nor all components equal to m . Suppose that $x = (x_1, x_2 \dots x_n)$ corresponds to an element of $C(Q)$ and has r components equal to m and s equal to M where $1 \leq r + s \leq n$.

Then by a relabeling of axes one may obtain another point y corresponding to an element of $C(Q)$ so that $\frac{x+y}{2}$ which by convexity also corresponds to an element of $C(Q)$ has less than $r + s$ components equal to m or M . By induction it follows that there is an element z for which $m \leq z_1 \leq M$ which corresponds to an element of $C(Q)$. Given any point u of the convex set there is a p , $0 < p < 1$ and a point v of the convex set so that $z = pu + (1-p)v$ and hence all strategies are elements of $C(Q)$.

Lemma 2.

$$\text{Let } m = \min_{\substack{d \in D \\ s \in S}} u(d, s)$$

$$M = \max_{\substack{d \in D \\ s \in S}} u(d, s)$$

$$c = \max_{d \in D} \sum_{s \in S} u(d, s)$$

Construct a problem $Q_1 \supset Q$ so that Q_1 corresponds to the convex set [generated by a finite number of points] of points x so that $m \leq x_1 \leq M$, $\sum x_1 \leq c$. Each point of this convex set for which $\sum x_1 < c$ is uniformly dominated by a point of this set for which $\sum x_1 = c$. By axiom 6 $C(Q_1) = C(Q_2)$ where Q_2 is the problem derived from Q_1 by deleting strategies d^* for which $\sum_{s \in S} u(d^*, s) < c$. By Lemma 1, $C(Q_2) = C(Q_1)$ consists of all strategies of Q_2 and hence those strategies of Q for which $\sum_{s \in S} u(d, s) = c$. By axiom 5, these strategies are in $C(Q)$ which establishes Lemma 2.

Lemma 2.

Lemma 3.

Consider two points a, b so that $\sum a_i > \sum b_i$. There is a point c so that c is uniformly dominated by a and $\sum c_i = \sum b_i$. Let $d = \frac{a+b}{2}$ $e = \frac{c+a}{2}$. [See Figure 1]

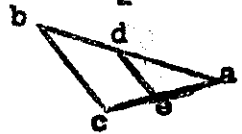


Fig. 1

Note 1. Every point in Δabc not on ab is uniformly dominated by a point on ab .

1. Let $v(p) = \frac{z - pu}{1-p}$. For p sufficiently close to 0, $v(p)$ is in the convex set.

2. Let $x = p_1 a + p_2 b + p_3 c$ where $p_1 \geq 0$, $\sum p_i = 1$, $p_3 > 0$. If we set $y = (p_1 + p_3)c + p_2 b$ then x is uniformly dominated by y .

Note 2. The quadrilateral $b c e d$ is a convex set generated by b, c, d, e . $\sum e_i \geq \sum x_i$ for x in this quadrilateral.

Consider Q_1 a problem for which the line ab is a geometrical representation.

$Q_2 \supseteq Q_1$ a problem for which the triangle abc is a geometrical representation.

$Q_3 \supseteq Q_2$ a problem for which the quadrilateral $bced$ is a geometrical representation.

To establish this lemma, it suffices by axiom 5 to show that b does not correspond to an element of $C(Q_1)$. Suppose that it did. Then by note 1 and axiom 6 b would correspond to an element of $C(Q_2)$ and hence of $C(Q_3)$. By Lemma 2 and note 2, e corresponds to an element of $C(Q_2)$. By convexity $\frac{b+e}{2}$ does also. But $\frac{b+e}{2}$ is uniformly dominated by d and axiom 3 is not satisfied. Hence b does not correspond to an element of $C(Q)$ and lemma 3 is established.

Lemma 4.

This is easily established.