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Identifiability of a Linear Relation Between Variables Which Are Subject To Error.

–by Olav Reiersøl

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1. Introduction.

Let $x_{1t}, x_{2t}, \ldots, x_{kt}$ ($t = 1, 2, \ldots, T$) be observed variables and let y_{it} be the "true" value of x_{it}. The error v_{it} will be defined by

$$(1.1) \qquad v_{it} = x_{it} - y_{it} \quad (i = 1, 2, \ldots k;\ t = 1, 2, \ldots, T).$$

We shall denote the vector $[x_{1t}, x_{2t}, \ldots, x_{kt}]$ by $\underline{x}_t$ with similar interpretations of $\underline{y}_t$ and $\underline{v}_t$. The subscript t may denote a point of time but it may also denote any other ordering of the observations.

We shall assume that there exists a linear relation between the variables y_{it}

$$(1.2) \qquad \sum_{i=1}^{k} \alpha_i y_{it} + \alpha_0 = 0$$

which can be written in vector notation as

$$(1.2') \qquad \underline{\alpha}\, \underline{y}_t' + \alpha_0 = 0$$

We shall consider different models, each of which contains Specifications (1.1) and (1.2) and additional specifications. When all parameters and all distributions in the model are specified we shall talk about a structure.[1] A structure is thus a particular realization of the model, and the model is the set of all structures compatible with the given specifications.

The variables x_{it} are observed variables. The variables y_{it} and v_{it} are not observed and are called latent variables.

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1) The terms defined here and in the rest of this introductory section are those used in publications from the Cowles Commission for Research in Economics. (For instance, Monograph No. 10).

A structure generates one and only one distribution $P(\underline{x})$ of the observed variables. On the other hand there may be several structures generating the same distribution $P(x)$. If two or more structures generate the same joint probability distribution of the observed variables, the structures are said to be equivalent. If a parameter has the same value in all equivalent structures it is said to be identifiable. In other words, a parameter is identifiable if it can be uniquely determined from a knowledge of the joint probability distribution of the observed variables. If a parameter is not identifiable, no consistent estimate of the parameter will exist. Conversely, if an estimate of a parameter has been proved to be consistent, the parameter must be identifiable.

Equation (1.2) remains the same if $\alpha_1, \ldots, \alpha_k$ and α_0 are all multiplied by the same constant. In order to make the coefficients determinate, we must have a normalization rule for these coefficients. In the following we shall tacitly assume that such a rule has been imposed.

2. Survey of previous results.

In this section I shall give a survey of some previous studies, formulated in modern terminology.

As far as I know, the first studies having some bearing on questions of identifiability in models of the type considered in the introduction, were published by Gini (1921) and Frisch (1934).

Koopmans (1937, p. 70) considered a model given by Specifications (1.1), (1.2) and the following specifications (2.1) - (2.5).

(2.1) The vector variable $[\underline{y}_t \;\; \underline{v}_t]$ is independent of the vector variable $[\underline{y}_{t'} \;\; \underline{v}_{t'}]$ when $t \neq t'$.

(2.2) The distribution of $[\underline{y}_t \;\; \underline{v}_t]$ is independent of t.

Let $[\underline{y} \;\; \underline{v}]$ be a vector variable which has the same distribution as each of the variables $[\underline{y}_t \;\; \underline{v}_t]$.

(2.3) $\underline{y}$ is independent of $\underline{v}$.

(2.4) The variables $v_1, v_2, \ldots, v_k$ are independent.

(2.5) The variables of y and v are jointly normally distributed.

Koopmans points out that $\underline{\alpha}$ is not identifiable in this model.

On the other hand he indicates that $\underline{\alpha}$ is identifiable in the case $k = 2$ if the vector $\underline{y}$ instead of being normally distributed, can take only two values.

Wald (1940) considers the case when $k = 2$ and when the variables x_{1t} can be divided into two groups $x_{11}, \ldots, x_{1m}$ and $x_{1,m+1}, \ldots x_{1,2m}$ such that the limit inferior of

$$\frac{1}{m}\left|(y_{11} + \ldots + y_{1m}) - (y_{1,m+1} + \ldots + y_{1,2m})\right| \quad (m = 1, 2, \ldots \text{ ad inf.}),$$

is positive. He also assumes that the distributions of v_{1t} and v_{2t} are are independent of t, that $\mathrm{cov}\,(v_{it}\, v_{it'}) = 0$ when $t \neq t'$, $i = 1$ or 2, and that $\mathrm{cov}(v_1\, v_2) = 0$. Under these assumptions, Wald shows that we can find a consistent estimate for the parameter $\frac{\alpha_2}{\alpha_1}$ and other parameters of the model. This implies that these parameters are identifiable.

The author of the present paper proved several theorems which give methods to determine the vector $\underline{\alpha}$ or limits for this vector, in terms of the parameters of the probability distribution of the observed variables. (Reiersøl 1941). Both kinds of theorems give as immediate corollaries statements about the identifiability of $\underline{\alpha}$.

In Section 4 he considers the following model. The variables $\underline{y}_t$ and $\underline{v}_t$ represent stationary sets of time series. We suppose that $\underline{y}_t$ and $\underline{v}_{t+u}$ are uncorrelated and that $\underline{v}_t$ and $\underline{v}_{t+u}$ are uncorrelated, where $u \neq 0$. Let M_u be the square k-rowed matrix $\left[\mu_{ij\,.\,u}\right]$, where $\mu_{ij\,.\,u} = \mathrm{cov}(x_{it}\, x_{j,\,t+u})$. In this model $\underline{\alpha}$ is identifiable if M_u is of rank k-1, i.e. if at least one element of the adjoint of M_u is different from zero. This statement follows as a corollary from Theorem 4 in the paper referred to. Other results concerning the identifiability of $\underline{\alpha}$ follow from Sections 2, 3 and 12 of the same paper.

In the last paragraph of Section 11, the author considers a model where we have an extra set of observed variables $x_{k+1}, x_{k+2}, \ldots, x_h$, such that x_j and v_i are uncorrelated when $i = 1, 2, \ldots, k$ and $j = k+1, k+2, \ldots, h$. Let $\mu_{ij} = \operatorname{cov}(x_i x_j)$. Then we have

$$(2.6) \quad \sum_{i=1}^{k} \mu_{ij} \alpha_i = 0 \qquad (j = k+1, \ldots, h) \quad .$$

If the rank of the matrix of this system of equations is k-1, the vector $\underline{\alpha}$ is evidently identifiable.

In a later paper (Reiersøl 1945) the author has called the set of variables $x_{k+1}, x_{k+2}, \ldots, x_h$ an instrumental set of variables. He gives several examples of sets of variables which may possibly be used as instrumental sets of variables. In the meantime the idea of using instrumental variables was introduced independently by Geary (1943). In another paper (Geary 1942) Geary considers a model given by Specifications (1.1), (1.2), and (2.1) - (2.4). Let $K(c_1, c_2, \ldots, c_k)$ denote a seminvariant of the joint distribution of $x_1, x_2, \ldots, x_k$ which is of degree c_i in the variable x_i $(i = 1, 2, \ldots, k)$. Geary shows that the parameters $\alpha_1, \alpha_2, \ldots, \alpha_k$ satisfy the following equation

$$(2.7) \quad \sum_{i=1}^{k} \alpha_i \; K(c_1, \ldots c_{i-1}, c_i + 1, c_{i+1}, \ldots, c_k) = 0$$

for each set of non-negative integers $c_1, c_2, \ldots, c_k$ where at least two of the c's are positive. If the matrix of this system of equations is of rank k-1, the vector $\underline{\alpha}$ is identifiable.

It should be noted that this survey is restricted to investigations of identifiability in the case when there is only one linear relation between the true variables. Some of the papers mentioned also consider sampling problems or problems of identifiability in the case of several equations between the true variables. Other papers giving contributions to the last mentioned problems have been published by Geary (1948 and 1949) and Tintner (1944, 1945, and 1946).

3. Conditions for the identifiability of a linear relation between two variables when the joint distribution of the errors is normal.

In the rest of this paper we shall suppose that Specifications (2.1) and (2.2) are satisfied, in other words that each set of observations $\left[x_t \; v_t \right]$ can be considered as a random drawing from a probability distribution of the variables y and v. We shall furthermore restrict ourselves to the case when k=2. Equations (1.1) and (1.2) now may be written in the form

(3.1) $\quad x_i = y_i + v_i, \quad (i = 1,2)$

(3.2) $\quad y_2 = \beta y_1 + \alpha$

In order not to exclude the case $\alpha_2 = 0$, we shall consider $\beta = \infty$ and $\alpha = \infty$ as admissible values of β and α.

We shall introduce two further specifications

(3.3) The set of variables v is independent of the set of variables y.

(3.4) The joint distribution of the errors v_1 and v_2 is normal.

(3.5) $E(u_1) = E(u_2) = 0$

The model defined by Specifications (3.1) - (3.5) will be called Model A. We shall discuss the identifiability of the parameter β in this model.

We shall use the letter φ for the characteristic function of a distribution. Characteristic functions for different distributions will be distinguished by putting the variables of the distribution as subscripts of the letter φ, for instance

$$\varphi_{x_1 x_2}(t_1\, t_2) = E(e^{ix_1t_1 + ix_2t_2})$$

The logarithm of a characteristic function φ will be denoted by ψ.

From Specifications (3.1) and (3.3) we obtain

(3.6) $$\psi_{x_1x_2}(t_1, t_2) = \psi_{y_1y_2}(t_1, t_2) + \psi_{v_1v_2}(t_1, t_2)$$

and from (3.2) we obtain

$$\begin{aligned}\phi_{y_1y_2}(t_1, t_2) &= E(e^{y_1 it_1 + y_2 it_2}) \\ &= E(e^{y_1 it_1 + (\alpha + \beta y_1)it_2}) \\ &= e^{\alpha it_2}(e^{y_1 i(t_1 + \beta t_2}) = e^{\alpha it_2}\phi_{y_1}(t_1 + \beta t_2)\end{aligned}$$

or

(3.7) $$\psi_{y_1y_2}(t_1, t_2) = \alpha it_2 + \psi_{y_1}(t_1 + \beta t_2)$$

Because of (3.4) and (3.5) we have

(3.8) $$\psi_{v_1v_2}(t_1, t_2) = -\frac{1}{2}(\lambda_{20}t_1^2 + 2\lambda_{11}t_1t_2 + \lambda_{02}t_2^2)$$

where $\lambda_{20} = \text{var}(v_1)$, $\lambda_{02} = \text{var}\ v_2$ $\lambda_{11} = \text{cov}(v_1, v_2)$

Inserting (3.7) and (3.8) in (3.6) we have

(3.9) $$\psi_{x_1x_2}(t_1, t_2) = \psi_{y_1}(t_1 + \beta t_2) + \alpha\ it_2 - \frac{1}{2}(\lambda_{20}t_1^2 + 2\lambda_{11}t_1t_2 + \lambda_{02}t_2^2)$$

We shall introduce the following difference operator

$$\underset{h_1}{\Delta}1\ \psi(t_1, t_2) = \psi(t_1 + h_1, t_2) - \psi(t_1, t_2)$$

Applying the operator $\underset{h_1}{\Delta}{}_1^2$ to both sides of (3.9) we get

(3.10) $$\underset{h_1}{\Delta}{}_1^2\ \psi_{x_1x_2}(t_1, t_2) = \underset{h_1}{\Delta}^2\psi_{y_1}(t_1 + \beta t_2) - \lambda_{20}h_1^2$$

The left-hand side of (3.10) depends on the distribution of the observable variables only. Let us now consider two different structures

$$S = \left\{\beta, \alpha, \lambda_{20}, \lambda_{11}, \lambda_{02}, \psi_{y_1}(t)\right\}$$

and

$$S^* = \left\{\beta^*, \alpha^*, \lambda^*_{20}, \lambda^*_{11}, \lambda^*_{02}, \psi^*_{y_1}(t)\right\}$$

which generate the same distribution $P(x)$ of the observed variables.

Then we must have

$$(3.11) \qquad \underset{h_1}{\Delta}^2 \psi_{y_1}(t_1 + \beta t_2) - \lambda_{20} h_1^2 = \underset{h_1}{\Delta}^2 \psi^*_{y_1}(t_1 + \beta^* t_2) - \lambda^*_{20} h_1^2$$

Putting t_2 equal to zero in this equation and replacing t_1 by $t_1 + \beta^* t_2$,

we obtain

$$(3.12) \qquad \underset{h_1}{\Delta}^2 \psi_{y_1}(t_1 + \beta^* t_2) - \lambda_{20} h_1^2 = \underset{h_1}{\Delta}^2 \psi^*_{y_1}(t_1 + \beta^* t_2) - \lambda^*_{20} h_1^2$$

Combining (3.11) and (3.12) we have

$$(3.13) \qquad \underset{h_1}{\Delta}^2 \psi_{y_1}(t_1 + \beta t_2) = \underset{h_1}{\Delta}^2 \psi_{y_1}(t_1 + \beta^* t_2)$$

Let c and c^* be two arbitrary constants. Suppose now that $\beta \neq \beta^*$.

Then the equations

$$t_1 + \beta t_2 = c$$

$$t_1 + \beta^* t_2 = c^*$$

are always satisfied by finite values of t_1 and t_2, so that the function

$$\underset{h_1}{\Delta}^2 \psi_{y_1}(t)$$

has the same value for any two values of the variable t, i.e. it is a constant. But when the second difference of $\psi_{y_1}(t)$ is a constant the function itself must be a second degree polynomial. This shows that y_1 is normally distributed. We have thus got the result that if β is not identifiable, then y_1 is normally distributed, hence also the joint distribution of x_1 and x_2 is normal.

When (x_1, x_2) is normally distributed, we may choose an arbitrary value for and put

$$y_1 = E(x_1), \qquad \alpha = E(x_2) - \beta E(x_1),$$

$$y_2 = \alpha + \beta y_1, \quad u_1 = x_1 - y_1, \quad u_2 = x_2 - y_2,$$

and Specifications (3.1) - (3.5) are fulfilled. Hence β is not identifiable in this case and we have

A necessary and sufficient condition for β to be identifiable in Model A, is that the joint distribution of x_1 and x_2 is not normal.[1]

4. Conditions for the identifiability of a linear relation between two variables when the errors are independent.[2]

In this section we shall consider a model, called Model B, which is defined by Specifications (3.1) - (3.3), (3.5) and the following specification

(4.1) u_1 and u_2 are independent.

If $\mathcal{E}(u_1)$ and $\mathcal{E}(u_2)$ do not exist, Specification (3.5) may be replaced by the specification

(3.5") median of u_1 = median of u_2 = 0

If x_1 and x_2 are independent, then any value of β is possible, together with the assumption that y_1 is a constant. In this case β is therefore not identifiable.

From now on we shall consider the case where y_1 and y_2 are not independent. Then $\beta \neq 0$.

Instead of equation (9) we now obtain

$$(4.2) \qquad \psi_{x_1x_2}(t_1, t_2) = \psi_{y_1}(t_1 + \beta t_2) + \alpha i t_2 + \psi_{v_1}(t_1) + \psi_{v_2}(t_2)$$

To this equation we shall apply the difference operator $\underset{h_1}{\Delta_1} \; \underset{h_2}{\Delta_2}$ where the subscript 2 means a difference with respect to the variable t_2.

We then obtain

$$(4.3) \qquad \underset{h_1}{\Delta_1}\underset{h_2}{\Delta_2}\psi_{x_1x_2}(t_1, t_2) = \underset{h_1}{\Delta}\;\underset{\beta h_2}{\Delta}\psi_{y_1}(t_1 + \beta t_2)$$

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1) In the case when certain seminvariants of the joint distribution of x_1 and x_2 exist, β may be expressed in terms of such seminvariants. (Koopmans and Reiersøl 1950, Section 1.2).

2) Part of the results of this section were published previously (Biometrics, Vol. 5, p. 88) as an abstract of a paper presented at meetings in Cleveland, December 1948.

In the same way as before we find that if β is not identifiable, then, $\underset{h_1}{\Delta}\ \underset{\beta h_2}{\Delta}\ \psi_{y_1}(t)$ is constant, so that $\psi_{y_1}(t)$ must be a second degree polynomial and y_1 must be normally distributed. As before we therefore have the result that <u>β is identifiable if y_1 is not normally distributed.</u> But the condition will be different from the condition in Model A if we express it in terms of the observed variables. Let κ_{rs} be the seminvariants of the joint distribution of x_1 and x_2. The condition may now be formulated thus:

If there exists a non-zero (finite or infinite) κ_{rs} with $r \geq 1$, $s \geq 1$ and r and s not both equal to 1, then β is identifiable in Model B.

When certain seminvariants exist and are different from zero, β may be expressed in terms of such seminvariants by one of the equations

$$\kappa_{r+s} = \beta\,\kappa_{r+1,\,s-1} \quad (r \geq 1,\ s \geq 2) \tag{4.4}$$

which are obtained when we put k=2 in equation (2.7).

We shall next examine the identifiability of β in the case when the distribution of y_1 is normal. Let κ_1 and κ_2 be the first two seminvariants of the distribution of y_1. Equation (4.2) now takes the form

$$\psi_{x_1x_2}(t_1, t_2) = i\,\kappa_1(t_1 + \beta t_2) - \frac{\kappa_2}{2}(t_1 + \beta t_2)^2 + \alpha\, it_2 + \psi_{v_1}(t_1) + \psi_{v_2}(t_2) \tag{4.5}$$

Let us again consider two equivalent structures S and S^*, and let as before the parameters and functions in S^* be denoted by starred letters and the corresponding parameters and functions in S be denoted by the same letters without asterisks. Writing equation (4.5) for the structure S^* and subtracting the two equations we obtain

$$(4.6)\qquad \psi^*_{v_1}(t_1) - \psi_{v_1}(t_1) + \psi^*_{v_2}(t_2) - \psi_{v_2}(t_2)$$
$$-\frac{1}{2}(K^*_2 - K_2)t_1^2$$
$$-\frac{1}{2}(\beta^{*2} K^*_2 - \beta^2 K_2)t_2^2$$
$$-(\beta^* K^*_2 - \beta K_2)t_1t_2$$
$$+i(K^*_1 - K_1)t_1 + i(K^*_1\beta^* - K_1\beta + \alpha^* - \alpha)t_2 = 0$$

Since this is an identity in t_1 and t_2 we immediately obtain

$$(4.7)\qquad \beta^* K^*_2 = \beta K_2$$

Applying the difference operators $\Delta^2_{h_1 1}$ and $\Delta^2_{h_2 2}$ to (4.6) we obtain

$$(4.8)\qquad \Delta^2_{h_1}\left[\psi^*_{v_1}(t) - \psi_{v_1}(t)\right] = (K^*_2 - K_2)h_1^2$$

and

$$(4.9)\qquad \Delta^2_{h_2}\left[\psi^*_{v_2}(t) - \psi_{v_2}(t)\right] = (\beta^{*2} K^*_2 - \beta^2 K_2)h_2^2$$

Suppose now that $\beta \neq \beta^*$. We shall first consider the case when

$$(4.10)\qquad |\beta^*| > |\beta|$$

From (4.7) and (4.10) follows that

$$(4.11)\qquad K^*_2 < K_2$$

and

$$(4.12)\qquad K^*_2\beta^{*2} > K_2\beta^2$$

From (4.8) and (4.11) we conclude that $\psi_{v_1}(t) - \psi^*_{v_1}(t)$ is a second-degree polynomial where the coefficient of t^2 is positive. Hence

$$(4.13)\qquad \varphi_{v_1}(t) = \varphi_{v_1^*}(t)\varphi_z(t)$$

where $\varphi_z(t)$ is the characteristic function of a normally distributed variable z which is independent of v_1^*. The variable v_1 can therefore be written as a sum of v_1 and z. In the terminology of Levy (1938) the distribution $P(v_1)$ of v_1 is said to be divisible by the distribution $P(z)$ of z.

Let us next consider the case $|\beta^*| < |\beta|$. In this case we conclude in the same way that the distribution of v_2 must be divisible by a normal distribution. Hence, when β is not identifiable, the distribution of either v_1 or v_2 is divisible by a normal distribution.

Suppose conversely that a structure S is given where the distribution of v_1, say, is divisible by a normal distribution with variance θ. We shall show that there exists an equivalent structure S^* with $\beta^* \neq \beta$. Let us choose K_2^* such that $K_2 - \theta \leqq K_2^* < K_2$ and let z be a normally distributed variable with variance $K_2 - K_2^{*2} \leq \theta$. Let

$$(4.14) \qquad \psi_{v_1}^*(t) = \psi_{v_1}(t) - \tfrac{1}{2}(K_2 - K_2^*)\, t^2 + i\,(K_1 - K_1^*)\, t$$

where K_1^* is chosen such that Specification (3.5) or Specification (3.5') is satisfied. Let β^* be determined by equation (4.7) and let $\psi_{v_2}^*(t)$ and α^* be determined by the equation

$$(4.15) \qquad \psi_{v_2}^*(t) = \psi_{v_2}(t) + \tfrac{1}{2}(\beta^{*2} K_2^* - \beta^2 K_2)\, t^2 - i\,(\alpha^* - \alpha)\, t$$

such that Specification (3.5) or (3.5') is satisfied. Since (4.7), (4.14) and (4.15) are satisfied also (4.6) must hold good. Hence the structure

$$S^* = \{\beta^*,\ \alpha^*,\ K_1^*,\ K_2^*,\ \psi_{u_1}^*,\ \psi_{u_2}^*\}$$

is equivalent to S, and $\beta^* \neq \beta$. Hence β is not identifiable, and we may state:

<u>When y_1 is normally distributed, a necessary and sufficient condition for the identifiability of β is that neither the distribution of u_1 nor the distribution of u_2 is divisible by a normal distribution.</u>

We shall now summarize in two tables our results concerning the identifiability of β. In Table 1 we shall give the conditions in terms of the latent variables and parameters. In Table 2 we shall give the results in terms of the probability distribution of the observed variables. The letter D in Table 2 denotes the determinant

$$D = \begin{vmatrix} K_{20} - \sigma_1 & K_{11} \\ K_{11} & K_{02} - \sigma_2 \end{vmatrix}$$

where σ_i is the maximum variance of any normal divisor of $P(x_i)$, $i = 1, 2$.

Table 1

<table>
<tr><td colspan="3">$\beta = 0$ or ∞</td><td>β not identifiable</td></tr>
<tr><td rowspan="3">$\beta \neq 0$ and finite</td><td colspan="2">y_1 not normally distributed</td><td>β identifiable</td></tr>
<tr><td rowspan="2">y_1 normally distributed</td><td>Neither $P(v_1)$ nor $P(v_2)$ divisible by a normal distribution</td><td>β identifiable</td></tr>
<tr><td>Either $P(v_1)$ or $P(v_2)$ divisible by a normal distribution</td><td>β not identifiable</td></tr>
</table>

Table 2

$\psi_{x_1x_2}(t_1, t_2) - \psi_{x_1}(t_1) - \psi_{x_2}(t_2)$		β
not of the form $c\, t_1\, t_2$		identifiable
equal to zero		not identifiable
of the form $c\, t_1\, t_2$ where $c \neq 0$	$D = 0$	identifiable
	$D < 0$	not identifiable

The cases given in Table 2 exhaust all possible cases, since D cannot be positive.

We shall next consider the identifiability of the rest of the structure in the case when β is identifiable. Writing equation (4.2) for two equivalent structures with the same β and subtracting the two equations we get

$$(4.16)\qquad \begin{aligned} &\psi^*_{y_1}(t_1 + \beta t_2) - \psi_{y_1}(t_1 + \beta t_2) \\ &+ \psi_{v_1}^*(t_1) - \psi_{v_1}(t_1) + \psi_{v_2}^*(t_2) - \psi_{v_2}(t_2) \\ &+ (\alpha^* - \alpha)\, i\, t_2 = 0 \end{aligned}$$

Applying the operator $\underset{h_1}{\Delta_1}$ to this equation we obtain

$$(4.17) \qquad \underset{h_1}{\Delta} \psi^*_{y_1}(t_1 + \beta t_2) - \underset{h_1}{\Delta} \psi_{y_1}(t_1 + \beta t_2) + \underset{h_1}{\Delta} \psi^*_{v_1}(t_1) - \underset{h_1}{\Delta} \psi_{v_1}(t_1) = 0$$

We put $t_2 = 0$ in the last equation, thereafter we replace t_1 by $t_1 + \beta t_2$, subtract the resulting equation from (4.17), and obtain

$$(4.18) \qquad \underset{hi}{\Delta} \underset{\beta t_2}{\Delta} (\psi^*_{v_1}(t_1) - \psi_{v_1}(t_1)) = 0$$

In the derivation of this equation we have supposed that the functions ψ are finite. This assumption must at least hold good in the vicinity of the origin since any characteristic function is continuous and equal to 1 when the argument is 0.[1] From (4.18) we conclude that

$$\psi^*_{v_1}(t) - \psi_{v_1}(t) = c\,t$$

at least in the vicinity of the origin. Because of Specification (3.5) or (3.5') c must be equal to zero, and we have

$$(4.19) \qquad \psi^*_{v_1}(t) = \psi_{v_1}(t)$$

Combining (4.19) and (4.17) we obtain

$$(4.20) \qquad \psi^*_{y_1}(t) = \psi_{y_1}(t)$$

Finally we obtain from (4.16), (4.19) and (4.20), together with Specification (3.5) or (3.5'),

$$(4.21) \qquad \psi^*_{v_2}(t) = \psi_{v_2}(t)$$

and

$$(4.22) \qquad \alpha^* = \alpha$$

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1) The same remarks apply to preceding proofs in this section and the preceding section. For instance, the conclusion that $\psi_{y_1}(t)$ is a second degree polynomial when β is not identifiable, holds good at least in the vicinity of the origin, and this is sufficient to conclude that y_1 is normally distributed.

Equations (4.19) - (4.21) are not necessarily valid over the whole range of the variable t, but hold good at least in the vicinity of the origin. Hence we conclude that when β is identifiable in Model B, then α is identifiable and the characteristic functions $\varphi_{v_1}(t)$, $\varphi_{v_2}(t)$, and $\varphi_{y_1}(t)$, are identifiable at least in the vicinity of the origin.

If the characteristic functions have discrete zerios only, the characteristic functions will be identifiable over the whole range of the variable t, since they are continuous. In this case the whole structure will be identifiable when β is identifiable.

We shall finally give an example where β is identifiable, but where $\varphi_{y_1}(t)$ is not identifiable. Let $\varphi_1(t)$ and $\varphi_2(t)$ be the two characteristic functions given as examples in Cramér's textbook, Section 10.3 (Cramér 1945, p. 94). Let $\varphi_{u_1}(t) = \varphi_{u_2}(t) = \varphi_1(t)$ and let

$$\varphi_{y_1}(t) = \varphi_1\left(\frac{t}{1+|\beta|}\right)$$

In the structure

$$S = \{\beta, \alpha, \varphi_{y_1}(t), \varphi_{u_1}(t), \varphi_{u_2}(t)\}$$

β is identifiable since y_1 is not normally distributed. But there exists another equivalent structure

$$S^* = \{\beta, \alpha, \varphi^*_{y_1}(t), \varphi_{u_1}(t), \varphi_{u_2}(t)\}$$

where $\varphi^*_{y_1}(t) = \varphi_2\left(\frac{t}{1+|\beta|}\right)$.

From the equation

$$\varphi_{x_1x_2}(t_1, t_2) = \varphi_{y_1}(t_1 + \beta t_2)\, \varphi_{u_1}(t_1)\, \varphi_{u_2}(t_2)\, e^{i\alpha t_2}$$

follows that $\varphi_{x_1 x_2}(t_1, t_2) = 0$

in both structures when

$|t_1| > 1$ or $|t_2| > 1$. Since $\left| \frac{t_1 + \beta t_2}{1 + |\beta|} \right| < 1$

$\varphi_{y_1}(t_1 + \beta t_2)$ and $\varphi^*_{y_1}(t_1 + \beta t_2)$

are identical when $t_1 \leq 1$ and $t_2 \leq 1$. This proves that S and S^* are equivalent.

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