COWLES COMMISSION DISCUSSION PAPER: STATISTICS: NO. 331

THE EXISTENCE OF MEASURABLE UTILITY and PSYCHOLOGICAL PROBABILITY

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1. Notation. We shall use the following language in this paper:

(1.1) $A\ B$ The statements A and B are both true.

(1.2) $A \vee B$ Either the statement A is true or the statement B is true or both are true.

(1.3) $A \supset B$ A implies B (A is false or B is true).

(1.4) $A \equiv B$ A is true if and only if B is true.

(1.5) $\exists x \ni A(x)$ There is an x such that A(x) is true.

(1.6) $A(x)$ For every x, A(x) is true.

(1.7) $A \equiv_{df} B$ The statement A is defined to be the statement B.

(1.8) $-A$ A is false.

Dots will be used for punctuation, e.g., in the tautology.

(1.9) $A \vee B : \equiv : A \supset B . \supset . B$

If we used parentheses instead of dots this would be

(1.10) $(A \vee B) \equiv [(A \supset B) \supset B]$.

(1.11) $X \varepsilon A \equiv_{df}$ X is an element of the set A.

(1.12) $A \subset B . \equiv_{df} . X \varepsilon A \supset X \in B$.

(1.13) $\mathcal{S} \equiv_{df}$ The space of all states of nature.

(1.14) $\mathcal{F}$ is a convex space of distributions of future histories of states of nature.

(1.15) $\{x \mid A(x)\} \equiv_{df}$ The set of all x such that A(x) is true.

(1.16) $\mathcal{F}^{\mathcal{S}} =_{df} \{f \mid f \text{ is a function. domain } f = \mathcal{S} . x \varepsilon \mathcal{S} \supset f(x) \varepsilon \mathcal{F}\}$

(1.17) $\mathcal{C} =_{df} \{f \mid f \varepsilon \mathcal{F}^{\mathcal{S}} : \exists \xi \varepsilon \mathcal{F} \ni . y \varepsilon \mathcal{S} \supset f(y) = \xi\}$

(1.18) $\mathcal{K}$ is a subspace of $\mathcal{F}^{\mathcal{S}}$ such that $\mathcal{C} \subset \mathcal{K}$.

(1.19) a, b, c, ... are real numbers.

(1.20) $\alpha, \beta, \gamma, \ldots$ are real numbers between 0 and 1 inclusive.

(1.21) $x, y, z, \ldots$ are elements of $\mathcal{S}$.

(1.22) $X, Y, Z, \ldots$ are elements of $\mathcal{X}$.

(1.23) $\xi, \eta, \zeta, \ldots$ are elements of $\mathcal{F}$.

(1.24) R is a fixed relation on $\mathcal{X}$.

(1.25) $i \equiv_{df}$ That function mapping $\mathcal{F}$ into $\mathcal{C}$ such that $i\xi(x) = \xi$ for all x in $\mathcal{S}$.

(1.26) $XIY :\equiv_{df}: XRY . YRX.$

(1.27) $XPY :\equiv_{df}: XRY . -YRX.$

(1.28) $\xi \left\{\begin{matrix} Ri \\ Ii \\ Pi \end{matrix}\right\} \eta :\equiv_{df}: i\xi \left\{\begin{matrix} R \\ P \\ I \end{matrix}\right\} i\eta.$

2. Axioms.

I. $\mathcal{X}$ is convex.

II. $XRY . YRZ . \supset . XRZ.$

III. $X . Y \in \mathcal{X} .:\supset:. \exists \alpha \ni : \beta . \supset . \alpha X + (1-\alpha) Y R \beta X + (1-\beta) Y.$

IV. $X . Y . Z \in \mathcal{X} . 0 < \alpha :\supset: XRY . \equiv . \alpha X + (1-\alpha) Z \; R \; \alpha Y + (1-\alpha) Z$

V. $X . Y \in \mathcal{X} : x . \supset . X(x) \; Ri \; Y(x) .:\supset:. XRY.$

VI. $XRY . YRZ . \supset . \exists \alpha \ni \alpha X + (1-\alpha) Z I Y.$

VII. $\exists X . \exists Y \ni : -XIY.$

VIII. $X . Y \in \mathcal{X} : x . \supset . X(x) \; Ri \; Y(x) \vee Y(x) Ri X(x) ::\supset:: \exists Z \in \mathcal{X} \ni :. x :\supset: Z(x) Ri \; X(x)$
$. Z(x) Ri Y(x) . Z(x) Ii X(x) \vee Z(x) Ii Y(x).$

3. Comment on the axioms.

Let us call elements of $\mathcal{X}$ prospects, elements of $\mathcal{C}$ certain prospects (a certain prospect may be random).

Axiom III states that for any two prospects, there is a random combination which is as good as any other random combination.

Axiom IV states that it is immaterial in which order choice or a random event occur, provided that a decision can be made before the random event occurs which corresponds to an arbitrary decision made afterward.

Axiom V states that if, regardless of the state of nature, if it were known, X is as good as Y, then X is as good as Y.

Axiom VII states that there is a possibility of choice.

Axiom VIII states that the "maximum" of two prospects is a prospect.

4. Construction of measurable utility.

$(4.1)\ \alpha > 0 . \alpha X + (1-\alpha) YRY . \supset . XRY .$ (IV).

I.e., if $\alpha > 0$, and $\alpha X + (1-\alpha)YRY$, then XRY. This is proved by Axiom IV.

(4.2) $XRY \vee YRX$ (III) (4.1)

(4.3) $XPY \equiv -YRX$ (4.2) (1.27)

(4.4) $X.Y.Z \varepsilon \mathcal{X} . 0 < \alpha : \supset : XIY . \equiv . \alpha X + (1-\alpha) Z I \alpha Y + (1-\alpha) Z$ (IV)

(4.5) $\alpha \neq \beta . \alpha X + (1-\alpha) Y P \beta X + (1-\beta) Y . \supset . -XIY$ (4.4)(1.26)(1.27)

(4.6) $XPY : \equiv : \alpha > \beta . \supset . \alpha X + (1-\alpha) Y P \beta X + (1-\beta) Y$ (4.5)(IV)

(4.7) $\exists X_1 . X_2 . X_3 . X_4 \varepsilon \mathcal{X} \ni : i \neq j . 1 \leq i \leq 4 . 1 \leq j \leq 4 . \supset . -X_i I X_j$ (4.6)(VII)

(4.8) $\exists f \ni : X \varepsilon \mathcal{X} . \supset . f(X)$ real $: f(X) \geq f(Y) \equiv XRY : f(\alpha X + (1-\alpha) Y)$ (I)(II)(VI)

$= \alpha f(X) + (1-\alpha) f(Y)$ (4.2)(4.4)(4.7)

(4.9) $X \varepsilon \mathcal{X} . \supset . g(X)$ real $: g(X) \geq g(Y) \equiv XRY : g(\alpha X + (1-\alpha) Y) = \alpha g(X) + (1-\alpha) g(Y)$

$. : \equiv : . \exists 1a . \exists 1b \ni : a > 0 . X \varepsilon \mathcal{X} \supset g(X) = af(X) + b$ (I)(II)(VI) (4.2)(4.4)(4.7)

(4.10) g is a <u>measurable utility scale</u>

$. : =_{df} : . X \varepsilon \mathcal{X} . \supset . g(X)$ real $: g(X) \geq g(Y) \equiv XRY : g(\alpha X + (1-\alpha) Y) = \alpha g(X) + (1-\alpha) g(Y) .$

5. Construction of psychological probability.

(5.1) $\xi . \eta \varepsilon \mathcal{F} . \supset . \xi Ii \eta : \supset : X.Y \varepsilon \mathcal{X} . \supset . XIY$ (1.26)(1.28)(V)

(5.2) $\exists \xi . \eta \varepsilon \mathcal{F} \ni . - \xi Ii \eta$ (5.1)(VII)

(5.3) g a function on $\mathcal{X} . \supset . gi(\xi) =_{df} g(i\xi)$

(5.4) g a measurable utility scale $. : \supset : . \xi \varepsilon \mathcal{F} . \supset . gi(\xi)$ real $: gi(\xi) \geq gi(\eta)$

$\equiv \xi Ri \eta : gi(\alpha \xi + (1-\alpha) \eta) = \alpha gi(\xi) + (1-\alpha) gi(\eta)$ (4.10)(5.3)

(5.5) $\exists g_0 . \xi_0 \ni : g_0$ a measurable utility scale $. \xi_0 \varepsilon \mathcal{F} . g_0^*(\xi_0) = 0.$ (4.9)

(5.6) $X \varepsilon \mathcal{X} .\supset. \overline{X}(x) =_{df} g_0 i X(x)$

(5.7) $G(g_0 i X) =_{df} g_0(X)$

(5.8) $\mathcal{E} =_{df} \left\{ \varphi / \exists n \ni . n \text{ positive integer} :. 1 \leq i \leq n :\supset: \exists a_i . \exists \overline{X}_i \ni . \varphi = \sum_{i=1}^{n} a_i \overline{X}_i \right\}$

(5.9) $\sum_{i=1}^{n} a_i X_i = \sum_{j=1}^{m} b_j Y_j . \sum_{i=1}^{n} a_i = 1 . \sum_{j=1}^{m} b_j = 1 : 1 \leq i \leq n .\supset. a_i \geq 0 : 1 \leq j \leq n .\supset. b_j \geq 0$

$:\supset: . \sum_{i=1}^{n} a_i g_0(X_i) = \sum_{j=1}^{m} b_j g_0(Y_j)$ (5.5)(4.10)

(5.10) $\sum_{i=1}^{n} a_i \overline{X}_i = \sum_{j=1}^{m} b_j \overline{Y}_j . \sum_{i=1}^{n} a_i = 1 . \sum_{j=1}^{m} b_j = 1 : 1 \leq i \leq n .\supset. a_i \geq 0$

$: 1 \leq j \leq n .\supset. b_j \geq 0 . :\supset: . \sum_{i=1}^{n} a_i G(\overline{X}_i) = \sum_{j=1}^{m} b_j G(\overline{Y}_j)$ (5.7)(5.9)

(5.11) $\sum_{i=1}^{n} a_i \overline{X}_i = \sum_{j=1}^{m} b_j \overline{Y}_j . \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j :\supset: \sum_{i=1}^{n} a_i G(\overline{X}_i) = \sum_{j=1}^{m} b_j G(\overline{Y}_j)$ (5.10)

(5.12) $\overline{i \xi_0} = 0.$ (5.6)(1.25)

(5.13) $\sum_{i=1}^{n} a_i \overline{X}_i = \sum_{j=1}^{m} b_j \overline{Y}_j :\supset: \sum_{i=1}^{n} a_i G(\overline{X}_i) = \sum_{j=1}^{m} b_j G(\overline{Y}_j)$ (5.11)(5.12)

(5.14) $\varphi \varepsilon \mathcal{E} . \varphi = \sum_{i=1}^{n} a_i \overline{X}_i .\supset. E\varphi =_{df} \sum_{i=1}^{n} a_i G(\overline{X}_i)$ (5.8)(5.13)

(5.15) $X . Y \varepsilon \mathcal{X} :\supset: \exists \varphi \varepsilon \mathcal{E} \ni . \varphi = \max(\overline{X}, \overline{Y}).$ (5.8)(VIII)

(5.16) $\varphi . \psi \varepsilon \mathcal{E} :\supset: \exists \eta \varepsilon \mathcal{E} \ni . \eta = \max(\varphi, \psi).$ (5.8)(5.15)

(5.17) $\varphi \varepsilon \mathcal{E} :\supset |\varphi| \varepsilon \mathcal{E}$ (5.16)

(5.18) $\overline{X} \geq \overline{Y} .\supset. G(\overline{X}) \geq G(\overline{Y})$ (V)(4.10)(5.7)

(5.19) $\varphi \varepsilon \mathcal{E} . \varphi \geq 0 .\supset. E(\varphi) \geq 0.$ (5.8)(5.18)

(5.20) E is an elementary integral on $\mathcal{E}$ (5.14)(5.17)(5.19)

(5.21) $\exists \varphi \varepsilon \mathcal{E} \ni \varphi$ is constant (5.2)(5.8)

(5.22) $\exists \varphi \varepsilon \mathcal{E} \ni \varphi = 1$ (5.21)

(5.23) $\exists \mathcal{R} . \mu \ni : \mathcal{R}$ is a field of subsets of $\mathcal{S}$

$. \mu$ is a finitely additive measure on $\mathcal{R} :. \varphi \varepsilon \mathcal{E} :\supset: \varphi$ measurable $(\mathcal{R})$

$. |\varphi| < a \supset E(\varphi) = \int_{\mathcal{S}} \varphi(x) d\mu(x)$ (5.20)(5.22)

(5.24) $|\bar{X}| < a \supset g_0(X) = \int g_0 i X(x)\, d\mu(x)$ (5.7)(5.14)(5.23)

(5.25) g a measurable utitlity scale

$: \supset : \exists a . \exists b \ni . a > 0 . X \varepsilon \mathcal{X} . \supset . g(X) = a g_0(X) + b$ (4.9)(4.10)(5.6)

(5.26) $|\bar{X}| < a \supset g(X) = \int_{\mathcal{S}} g_1 X(x)\, d\mu(x)$ (5.24)(5.25)

(5.27) $\exists a \ni |\bar{X}| < a . : \equiv : . \exists \xi_1 . \exists \xi_2 . \exists \xi_3 . \exists \xi_4 . \exists \alpha > 0 . \exists \beta > 0 \ni :$

$x \varepsilon \mathcal{S} . \supset . \alpha X(x) + (1-\alpha) \xi_1 \text{ Ri } \xi_2 . \xi_3 \text{ Ri } \beta X(x) + (1-\beta) \xi_4$ (5.5)

Therefore we can consider μ as a psychological probability with the property that for any "bounded" prospect X, the utility of X is the expected value of the utilities of the certain prospects of which X is composed.