Problems in the theory of statistics seem to reduce to the selection of a strategy from several available strategies. The problem of the selection of such a strategy is analogous to the selection of a strategy in the two person zero sum game where nature is the statistician's opponent. One difference between the two problems lies in the fact that it does not seem proper to regard the unknown state of nature as a strategy of an opponent who wishes to maximize your loss.

In the case where there are a finite number of pure strategies \( d_1, d_2, \ldots, d_m \) available to the statistician and a finite number of conceivable states of nature \( S_1, S_2, S_n \), there exists a finite risk matrix \( R = \begin{bmatrix} r_{ij} \end{bmatrix} \)

\[ r_{ij} = \text{expectation of the loss incurred by statistician if he selects strategy } d_i \text{ when } S_j \text{ is the state of nature.} \]

The development and use of the Risk Matrix depend on the existence of a utility which has the property that if a statistician has probability \( p \) of reaching at a certain time a state in which he has utility \( u_1 \) and \( q = 1 - p \) of reaching a state at that time in which he has utility \( u_2 \), then his utility for that time is \( p u_1 + q u_2 \).

Clearly, if there exists a \( d_i \) so that \( r_{ij} \leq r_{kj} \) for all \( k \) and \( j \), then also \( d_i \) is at least as good as all the other strategies and if for every \( k \) there is some \( j \) so that \( r_{ij} < r_{kj} \) then \( d_i \) is uniformly better than the other strategies. Unfortunately it is seldom possible to find such a strategy. The problem remains of phrasing a reasonable criterion for selecting a strategy in the case where uniformly best strategy fails to exist.

Wald has mentioned that a conceivable criterion would be to minimize the maximum risk. That is to choose a strategy \( d_i \) if \( \max_j r_{ij} \leq \max_j r_{kj} \) for all \( k \).
This is the criterion which is used in the Theory of Games. This criterion is equivalent to ranking strategies in preference according to \(-\max r_{ij}\)

\[ \max_j r_{ij} \leq \max_j r_{kj} \quad d_i \text{ is better than } d_k. \]

As in the Theory of Games we may introduce mixed strategies i.e.

strategies which consist of selecting a pure strategy \(d_i\) at random where the probability of selecting \(d_i\) is \(\xi_i\) and \(\sum_i \xi_i = 1\) \(\xi_i \geq 0\). A result in the Theory of Games and Economic Behavior by Von Neumann and Morgenstern is that \(v = \min \max \sum_j \xi_j \eta_j r_{ij} = \max \min \sum_i \eta_i \xi_i r_{ij}\). This result seems to imply that in the zero sum two person game the mixed strategies corresponding to \(v\) are rational strategies or the part of the opponents.

Some criticism was leveled at the criterion of minimizing the maximum risk for the following reason. It is a criterion which is useful if "nature" is the statisticians opponent, but one can hardly assume this.

If the statistician applies this criterion in the case where one of the states is a state of disaster then one finds oneself forced to decide by just considering the risks in the case of the state of disaster. Consider the following example of a risk matrix:

<table>
<thead>
<tr>
<th></th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(\max r_{ij})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_1)</td>
<td>1</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>(d_2)</td>
<td>99</td>
<td>99</td>
<td>99</td>
</tr>
</tbody>
</table>

The criterion of minimizing the maximum risk prefers \(d_2\) to \(d_1\).

If "nature" is malevolent it will select strategy \(S_2\) and our statistician is wise to select strategy \(d_2\). If one does not assume that nature is malevolent it seems unreasonable to allow the slight preference of \(d_2\) to \(d_1\) is the case of disaster to dictate the choice of \(d_2\). Indeed if our statistician were to follow the policy of minimizing the maximum risk, he might very well commit suicide rather than risk being unpleasantly killed in an accident. Savage
felt that a better criterion would be to minimize the maximum regret \(^{(1)}\)

Consider \(\min_i r_{ij}\). This represents the minimum risk that can be incurred by
the statistician if state \(j\) is the actual state of nature. If the statistician
select \(d_1\) and state \(j\) turns out to be the actual state one may say that his
regret for choosing strategy \(i\) would be \(r_{ij} - \min_i r_{ij} = \rho_{ij}\). (Here there is
an assumption that the "regret" of attaining a state of utility \(a\) instead of
one of utility \(b\) is measured by \(b-a\).) The Savage criterion of minimizing
the maximum regret would be to rank the available strategies in preference
according to \(\max_j \rho_{ij}\). i.e. if \(\max_j \rho_{ij} < \max_j \rho_{kj}\), \(d_1\) is as good as \(d_k\).

In the previous example:

\[
\begin{array}{ccc}
R & S_1 & S_2 \\
S_1 & & \rho \\
S_2 & & \\
\end{array}
\]

The criterion of minimizing the maximum regret prefers \(d_1\) to \(d_2\).

While the method of minimizing the maximum regret seems to overcome
the difficulties facing the method of minimizing the maximum regret, it
too seems to have some disadvantageous properties. Consider the risk matrix:

\[
R
\begin{array}{ccc}
S_1 & S_2 & S_3 \\
S_1 & 10 & 0 & 14 \\
S_2 & 0 & 8 & 15 \\
S_3 & 80 & 60 & 0 \\
\end{array}
\]

If \(d_3\) were not an available decision the matrix would be:

\[
R
\begin{array}{ccc}
S_1 & S_2 & \\
S_1 & 10 & 0 & 14 \\
S_2 & 0 & 8 & 15 \\
\end{array}
\]

\(^{(1)}\) The criterion of minimizing the maximum regret was transmitted to me
via K. J. Arrow.
In the first case $d_1$ is the strategy that would be selected by the principle of min max regret. (Let us ignore mixed strategies for the time being.) In the second case $d_2$ is the one which would be selected. Thus it happens that when a new strategy is made available, the preferred strategy is neither the one that was the preferred nor the new one, but the one that was formerly not preferred.

Arrow conjectured that this was probably due to some inherent insensitivity on the ranking and with his help the following case was constructed.

$$
\begin{array}{ccc}
S_1 & S_2 & S_3 \\
\hline
d_1 & 10 & 0 & 14 \\
d_2 & 0 & 8 & 15 \\
d_3 & 5 & 7 & 8 \\
\end{array}
$$

If comparisons where respectively $d_1$ and $d_2$, $d_2$ and $d_3$, $d_3$ and $d_4$ are considered, one sees that $d_1$ is preferred to $d_2$, $d_2$ to $d_3$, and $d_3$ to $d_4$ if the min max regret is applied.

Some question arose as to whether the first criticism of min max regret was fair inasmuch as mixed strategies were ignored. The following example was constructed to show that the use of mixed strategies does not avoid the pitfall.

$$
\begin{array}{ccc}
S_1 & S_2 & S_3 \\
\hline
d_1 & 10 & 0 & 14 \\
d_2 & 0 & 8 & 15 \\
d_3 & 80 & 80 & 0 \\
d_4 & 12.922 & 12.922 & 12.922 \\
\end{array}
$$
If a best (by the min max regret criterion) strategy were to be constructed when \(d_1, d_2, d_3, d_4\) were available it would be equivalent to \(d_4\) in that the risks would be the same. If, however, one considers only \(d_1, d_2, d_4\) available then obviously \(d_2\) would be better than \(d_4\). Indeed the "best" mixed strategy involving \(d_1, d_2, d_4\) (\(d_3\) unavailable) would be equivalent to some \(d_5\) which is a mixed strategy of \(d_1, d_2, d_3, d_4\) but is not best as long as \(d_3\) is available. From another point of view while \(d_4\) was always available it did not become the "best" until \(d_3\) was thrown into the picture.