At this point it seems advisable to axiomatize the concept of the rational selection of a decision function. There seem to be various methods of phrasing the actual problem in an axiomatic framework. We shall consider one such framework and a set of axioms which seem to be reasonable requirements.

Whether or not a criterion satisfies the axioms is closely related to the set of problems (domain) over which the criterion is supposed to exist. For example it is conceivable that a suitable criterion may be framed for the class of all problems with one or two states of nature (columns) but that none may exist for the class of all problems with three or less states of nature. Another consideration in this direction arises from the question as to whether it makes sense to insist that a criterion apply to the selection from any set of strategies. One may feel that it is not proper to insist on applicability to problems where not all mixed strategies are available.

In the following the solution of a problem will represent a set of the available strategies so that no one of these strategies is "worse" than any available strategies and so that there seems no indication to prefer one strategy of the solution to any others of the solution. This nonpreference is not necessarily indifference.

In what follows considerable space is devoted to framing relatively simple concepts such as the invariance of the solution of a problem if several columns are interchanged. The complications arise in part out of an attempt to give the notation enough generality to be applicable to problems where an infinity of states of nature and of strategies are present. This notation is needed because
it is impossible to avoid the infinite cases even in simple problems as the existence of mixed strategies makes evident. Another source of complications is the need to compare two problems where one is the same as the other except for the interchange of several rows.

At this point a change in notation will be made. Instead of a risk matrix, we shall use the payoff matrix, the elements of which are the payoffs in utility.¹

DEFINITION 1: A problem Q is a real valued function (with +∞ and −∞ as possible values) defined on the Cartesian product of two non-null spaces D and S. The value of the function is called the payoff \( u = u(d, s); d \in D, s \in S \). D = the set of strategies, S = the set of states of nature. We may write \( Q = (u, D, S) \).

DEFINITION 2: A general Problem G is a class of problems Q.

DEFINITION 3: In a problem Q, to an element \( d \) of D, there corresponds the function on S, \( u_d = u(d, s) \), \( s \in S \). This function is the "d-row" of the problem Q. We may write "d-row of \( Q \)" = \( d_q = (u_d, S) = (u, \{d\}, S) \).

DEFINITION 4: Similarly the "s-column of \( Q \)" = \( s_q = (u_s, D) = (u, D, \{s\}) \) where \( u_s = u(d, s) \), \( d \in D \).

It is evident the problem Q is uniquely determined by its d-rows or by its s-columns.

DEFINITION 5: \( \xi_D \) is a probability defined on D and is called a mixed strategy of D. \( D_\ast \) will represent a set of mixed strategies of D. When there is no ambiguity D may be omitted from \( \xi_D \).²

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¹. From a historical point of view risk was originally phrased as the negative of the payoff, but it was actually used and thought of as a regret. Thus the two terms risk and regret are ambiguous and will be dropped.

². We may look upon a mixed strategy on \( D_\ast \) as a mixed strategy on D since their interpretations make them identical.
DEFINITION 6: $\gamma^S$ is a probability defined on $S$ and is called a mixed state of $S$. $S^*$ will represent a set of mixed states of $S$. When there is no ambiguity $S$ may be omitted from $\gamma^S$.

DEFINITION 7: Corresponding to $Q = (u, D, S)$, $D_\alpha$ and $S_\alpha^*$ ($u, D_\alpha, S^*$) is the problem where the new $u$ is an extension of the old one.

$$u = u(\xi_D, \gamma^S) = E[u(d, s)]$$

with respect to the probability $\xi_D \otimes \gamma^S$ on $(D \otimes S)$.

This is the extension of $Q$ with respect to $S^*$, and $D_\alpha$. ($E = \text{Expectation}$)

It should be noted that $E[u(d, s)]$ may be undefined depending on the original $u$, $\xi_D$, and $\gamma^S$. In general we shall have $D_\alpha$ and $S^*$ such that $u(\xi_D, \gamma^S)$ is defined (and possibly $+\infty$ or $-\infty$).

DEFINITION 8: If $D_\alpha$ is the set of all mixed strategies $\xi_D$ of $D$ so that $u(\xi_D, S)$ is defined, $(u, D_\alpha, S)$ is the mixed problem of $(u, D, S)$.

DEFINITION 9: $Q_1 \supset Q_2$ (read $Q_1$ is strategically contained in $Q_2$) if $D_1 \subseteq D_2$, $S_1 = S_2$ and for $d \in D_1$, $u_1(d, s) = u_2(d, s)$.

DEFINITION 10: $Q_3 \supset d \otimes Q_2$ (read $Q$ is derived by deleting repetitious rows of $Q_2$) if $Q_1 \supset Q_2$ and for $d_1 \in D_1$, there is a $d_2 \in D_2$ so that $u_1(d_1, s) = u_2(d_2, s)$.

DEFINITION 11: $Q_3 \supset d \otimes Q_2$ (read $Q$ is derived by deleting repetitious columns of $Q_2$) if $S_1 \subseteq S_2$, $D_1 = D_2$ and for $s \in S_1$ and $s_2$, $s_1q_1 = s_2q_2$ and for $s_2 \in S_2$ but not in $S_1$, there is an $s_1 \in S_1$ so that $s_1q_1 = s_2q_2$.

DEFINITION 12: $Q_1 \supset Q_2$ (read $Q_1$ is $D$ equivalent to $Q_2$) if there is a one to one transformation $f$ of $D_1$ onto $D_2$ so that $u_1(d_1, s) = u_2(f(d_1), s)$.

DEFINITION 13: $Q_1 \supset Q_2$ (read $Q_1$ is $S$ equivalent to $Q_2$) if there is a one to one transformation $g$ of $S_1$ onto $S_2$ so that $u_1(d_1, s_1) = u_2(d_2, g(s_1))$.

DEFINITION 14: $Q_1$ is isomorphic to $Q_2$ if there is a one to one transformation $f$ of $D_1$ onto $D_2$ and a one to one transformation $g$ of $S_1$ onto $S_2$ so that $u_1(d_1, s_1) = u_2(f(d_1), g(s_1))$. 
DEFINITION 15: If \( Q_1 = (u_1, D_1, S) \), \( Q_2 = (u_2, D_2, S) \), and \( u_1(d, s) = u_2(d, s) \), \( \forall d \in D_1 \cap D_2 \) then \( Q_1 \circ Q_2 = (u, D_1 \cap D_2, S) \), note that \( Q_1 \circ Q_2 \supseteq Q_1 \).

DEFINITION 16: If \( Q_1 = (u_1, D_1, S) \), \( Q_2 = (u_2, D_2, S) \), and \( u_1(d, s) = u_2(d, s) \), \( \forall d \in D_1 \cap D_2 \) then \( Q_1 + Q_2 = (u, D_1 \cup D_2, S) \) where \( u(d, s) = u_1(d, s) \) if \( d \in D_1 \) and \( u_2(d, s) \) if \( d \in D_2 \). Note that \( Q_1 \circ Q_1 + Q_2 \). Further if \( Q_1 \supseteq Q_2 \), there is a \( Q_3 \) so that \( Q_1 + Q_3 = Q_2 \).

DEFINITION 17: \( Q_1 \otimes Q_2 \) is a problem \( Q = (x, D, S) \)
where \( D = \{(d_1, d_2) \mid d_1 \in D_1 \, d_2 \in D_2\} \), \( S = \{(s_1, s_2) \mid s_1 \in S_1 \, s_2 \in S_2\} \),
and \( u((d_1, d_2), (s_1, s_2)) = u_1(d_1, s_1) + u_2(d_2, s_2) \).

The above notation mean that \( D \) is the set of ordered pairs such that the first element is in \( D_1 \) and the second in \( D_2 \). Essentially, \( Q_1 \otimes Q_2 \) is the problem of playing \( Q_1 \) and then \( Q_2 \).

DEFINITION 18: A solution \( C \) of a problem \( Q \) is a subset of \( D \).

DEFINITION 19: A solution of a general problem \( G \) is a function \( C(G) \)
defined on \( G \) so that for each \( Q \) in \( G \), \( C(Q) \) is a solution of \( Q \).

DEFINITION 20: In a problem \( Q \), the \( d_1 \)-row \( d_{1q} \) is uniformly better than the \( d_2 \)-row \( d_{2q} \) if \( u(d_1, s) \geq u(d_2, s) \) for all \( s \in S \) and \( u(d_1, s) > u(d_2, s) \) for some \( s \in S \). \( d_{1q} \) is uniformly equal to \( d_{2q} \) \( u(d_1, s) = u(d_2, s) \) for all \( s \in S \).

Using these relations we may define a partial ordering on the set of \( d \)-rows of a problem \( Q \), we write \( d_{1q} \geq d_{2q} \) (preference) if \( d_1 \) is uniformly better than \( d_2 \)
\( d_{1q} \approx d_{2q} \) (indifference) if \( d_1 \) is uniformly equal to \( d_2 \)
\( d_{1q} \geq d_{2q} \) if \( d_{1q} \geq d_{2q} \) or \( d_{1q} \approx d_{2q} \)
\( d_{1q} \not\geq d_{2q} \) (non comparability) if neither \( d_{1q} \geq d_{2q} \) nor \( d_{2q} \geq d_{1q} \).

Thus \( d_{1q} \not\leq d_{2q} \) if there are \( s_1 \) and \( s_2 \in S \) so that \( u(d_1, s_1) > u(d_2, s_1) \) and \( u(d_1, s_2) < u(d_2, s_2) \). It should be noted that the partial ordering may be
undefined if some risks are $\pm \infty$ or $\mp \infty$ or are undefined. In general we shall avoid this case although it may sometimes be treated. In place of the above notation we shall use $d_1p_2$, etc., whenever there is no ambiguity, always remembering that the partial ordering is always with reference to a particular problem.

Now that a number of formal definitions have been made to facilitate future discussion we are faced with defining a rational solution of a general problem. Such a definition will involve satisfying axioms of rationality. To frame such axioms we must ask what is intuitively meant by a solution and what role does $G$ play.

First of all a solution in the following should be considered a set of strategies which remain after throwing away "bad" strategies and eliminating some which while not necessarily bad, still fail to make the grade. A rational statistician must then make a choice of one of strategies in the solution. If we were to consider any one of these to be as "good" as any other, then it would make no difference which he chose. However it is the author's intention to assume that they are not necessarily as good as one another, that one element of the solution may be replaced by a uniformly better strategy without evicting the other element from the solution. It seems reasonable that if two strategies are in the solution, any mixed strategy of the two should also be in the solution. This will be incorporated into the axioms but it is felt that the consequences of omitting this axiom will prove useful in later work. These consequences will be considered.

Since a solution is phrased with respect to a class $G$, the question arises of what to do if $Q \in G_1$, $Q \in G_2$, and $C(Q)$ with respect to $G_1$ has no strategies in common with $C(Q)$ with respect to $G_2$. That the use of general problems is not altogether meaningless follows from the fact that for every $Q \in G$, $D$ should contain its mixed strategies. It must be admitted that another reason for the
use of $G$ stems from the author's feeling that if $G$ included all problems where $D$ contains all of its mixed strategies no solution would be found. Then it would be nice if a justification could be made for the existence of a meaningful set of $G$'s which are disjunct. The only such justification that the author can attempt (though with some misgivings) is the following. In any problem that is to be faced there is a certain specific set of states of nature. Thus the set of all $Q$'s may be put into $G$'s where $Q_1$ and $Q_2$ are in a $G$ if and only if there is a one to one correspondence between $S_1$ and $S_2$. (Each $G$ corresponds to a distinct cardinal number.)

Another justification for the use of $G$'s which however fails to guard against a $Q$ being in two $G$'s, is a claim that is often made that in certain classes of problems, each row is unimodal. That is, there is a complete order, of $S$, so that for each row there is an $s_d$, so that $u(d, s)$ is monotone increasing for $s < s_d$ and monotone decreasing for $s > s_d$.

**DEFINITION 21:** A general problem $G$ is rationally solvable by $C(Q)$ if the following axioms are satisfied.

This definition is quite pretentious in that it is conceivable that $C(Q)$ may satisfy the following axioms but fail to satisfy a relevant axiom which has been forgotten but which should be satisfied for one to claim the solution to be rational.

**AXIOM 1:** $C(Q)$ is non-null for every $Q$ in $G$.

**AXIOM 2:** If $d_1, d_2 \in C(Q)$, $Q \in G$, then $d_2 \in C(Q)$.

**AXIOM 3:** If $d_1 \in C(Q)$, $Q \in G$, then there is no $d_2 \in D$ so that $d_2 P d_1$.

**AXIOM 4:** If $r_0 \notin Q_1, Q_2 \in G$, then $C(Q_1) = D_1 \cap C(Q_2)$. This means that all strategies of the solution which are available after deletion are in the solution and none others.
AXIOM 5: If $Q_1 \subseteq Q_2$, $Q_1, Q_2 \in G$, $C(Q_2) \subseteq C(Q_1) \cup \{D_2 \setminus D_1\}$. This axiom says that if the available strategies of $Q_1$ are increased, then each element of the solution of the new problem is an element of the solution of the old problem or one of the new strategies. This axiom is equivalent to $C(Q_2) \cap D_1 \subseteq C(Q_1)$.

AXIOM 6: If $Q = Q_1 + Q_2$, $Q, Q_1 \subseteq G$, where for every $d_2 \in D_2$ there is a $d_1 \in D$, so that $d_1 d_2$ then $C(Q) = C(Q_1)$.

It is not too unreasonable to consider extending Axiom 6 to Axiom 6a. However it would be wise to avoid making use of the extension.

AXIOM 6a: If $Q = Q_1 + Q_2$, $Q, Q_1, Q_2 \subseteq G$, where $C(Q)$ contains no elements of $D_2$ then $C(Q_1) = C(Q)$.

This may be interpreted to mean that if from a problem $Q$ strategies not in the solution are deleted, the solution remains unchanged.

AXIOM 7: If $Q = Q_1 + Q_2$, $Q, Q_1, Q_2 \subseteq G$, then $C(Q_1) \cap C(Q_2) \subseteq C(Q_1 + Q_2)$.

AXIOM 8: If $Q_1 \cap Q_2 = \emptyset$, $Q_2, Q_1, Q_2 \subseteq G$, then $C(Q_1) = C(Q_2)$. In other words if two columns representing states $s_1$ and $s_2$ are the same, the solution is is the same as that of the problem where the states $s_1, s_2$ are replaced by the state $s_1$ or $s_2$.

AXIOM 9: If $Q_1, Q_2 \subseteq G$, $Q_1 \supseteq Q_2$, then $C(Q_2) = f[C(Q_1)]$. This says that the solution does not depend on the names of the strategies but on the payoff's.

AXIOM 10: If $Q_1, Q_2 \subseteq G$, $Q_1 \subseteq Q_2$, then $C(Q_2) = C(Q_1)$.

Consequence 1. If $Q_1, Q_2 \subseteq G$, $Q_1$ isomorphic to $Q_2$, then $C(Q_2) = f[C(Q_1)]$.

This is immediate from axioms 9 and 10.

AXIOM 11: If $Q_1 = (u_1, D, S)$, $Q_2 = (u_2, D, S)$, $Q_1, Q_2 \subseteq G$,

$u_2(d, s) = pu_1(d, s) + (1 - p) u_0(s) \exp \leq 1$, and $u_0(s)$ is attainable\(^1\) then

$C(Q_2) = C(Q_1)$.

\(^1\) It is conceivable that there is no future for which the utility exceeds a certain number $K$. Then to insert $u_0(s) = K + 1$ would be meaningless in view of the justification of Axiom 13. The set of attainable utilities is evidently an interval.
This axiom which was just conveyed to me by Herman Rubin, is justified as follows. If one were told that he would be given a problem $Q_1$ if a coin fell heads and a strategy would be forced on him if the coin fell tails, then his strategy if the coin fell heads would be $C(Q_1)$. The game including the coin has payoff $u_2(d, s) = p u_1(d, s) + (1 - p) u_0(s)$ where $p$ is the probability of falling heads (and is not zero) and $u_0(s)$ is the payoff under the strategy forced on him if the coin fell tails. The rationale of this axiom can be considered as a consequence of the unstated axiom of rational behaviors that a rational person will react in the same fashion to the same situation facing him irrespective of what came before. This axiom is rather powerful and has some important consequences.

Before stating these consequences let us define certain mixed strategies.

**Definition 22:** $(d_1, d_2, \ldots, d_n; p_1, p_2, \ldots, p_n)$ is the mixed strategy consisting of selecting $d_i$ with probability $p_i$, $i = 1, 2, \ldots, n$ where $p_1 + p_2 + \ldots + p_n = 1$.

**Consequence 2.** If $Q \in G$, and $Q_1 \sqsupseteq Q_2 \sqsubseteq Q$ implies $Q_1, Q_2 \in G$, then $\xi = (d_1, d_2, p, 1 - p) \in C(Q)$ for $0 < p < 1$ implies $d_1$ and $d_2 \in C(Q)$. (This is a converse of convexity)

**Proof:** Let $Q_p = (u_p, D, S)$ where $u_p(d, s) = pu(d, s) + (1 - p) u(d_2, s)$

$$Q_{2p} = (u, D_{2p}, S) \text{ where } D_{2p} = \left\{ (d, d_2; p, (1 - p)) \mid d \in D \right\}$$

Then it is evident that for $p > 0$, $Q_p \sqsupseteq Q_{2p} \sqsubseteq Q$

Thus $Q_p, Q_{2p} \in G$. By axiom 5, $\xi \in C(Q_{2p})$ by axiom 9, $d_1 \in C(Q_p)$

Then by axiom 11 applied to $Q_p$ and $Q$, where $u_0(s) = u(d_2, s)$ we have $d_1 \in C(Q)$. Similarly $d_2 \in C(Q)$.

This consequence is not only a converse of convexity but it just about states that mixed strategies have no special claim to fame. This is not too
surprising in view of the fact that the game theory interpretation of the role of mixed strategies was that they served to prevent the opponent from guessing at one's own strategy. Nature could hardly be expected to be attempting to outguess the statistician.

For future discussions it will be convenient to assume that \( u = 0 \) is an attainable value and that the interval of attainable \( u \)'s contains \( u = 0 \) as interior point. That we may make these assumptions is evident from the fact that if \( u \) is a utility indicator derived from the axioms of von Neumann and Morgenstern so is \( au + b, a > 0 \). Furthermore if the set of attainable states were indifferent (only one utility) then our problems would disappear.

Consequence 3. If \( Q_1, Q_2 \in G, Q_1 = (u_1, D, S), Q_2 = (u_2, D, S) \) where \( u_2(d, s) = au_1(d, s), a > 0 \), then \( C(Q_2) = C(Q_1) \).

We may assume \( a \leq 1 \) for otherwise we can reverse \( Q_1 \) and \( Q_2 \). If \( a = 1 \), the consequence is trivial. If \( 0 < a < 1 \), we let \( u_0(s) = 0 \) (which is attainable) and apply axiom 11.

Consequence 4. If \( Q_1, Q_2, Q_3 \in G, Q_1 = (u_1, D, S), Q_2 = (u_2, D, S), Q_3 = (u_3, D, S), 0 < p < 1 \), where \( u_2(d, s) = u_1(d, s) + u_0(s) \), \( u_3(d, s) = pu_2(d, s) \), then \( C(Q_2) = C(Q_1) \).

Proof: \( u_3(d, s) = pu_1(d, s) + (1 - p)\left(\frac{p}{1 - p} u_0(s)\right) \)

\( \frac{p}{1 - p} u_0(s) \) is an attainable utility since

\( |u_0(s)| \leq |u_2(d, s)| + |u_1(d, s)|. \) Then \( C(Q_3) = C(Q_1) \) by axiom 11. But by consequence 3, \( C(Q_3) = C(Q_2) \).

Finally we have the convexity axiom.

AXIOM 12: If \( Q \in G \), and \( \xi \) is a mixed strategy of the elements of \( C(Q) \) [i.e. \( \xi \) is a probability on \( D \) such that the probability of \( C(Q) \) is one] then \( \xi \in D \) implies \( \xi \in C(Q) \).
As was mentioned before we shall avoid using the implications of twelve until we have exhausted the other axioms. It may be noted that for any \( G \), the solution \( C(Q) \) equal the set of all admissible strategies of \( Q \) [those for which there are no uniformly better ones] constitutes a solution satisfying axioms two to eleven but not twelve and not one. By restricting \( G \) so that every \( Q \in G \) contains admissible strategies, axiom one will be satisfied. That this axiom system is consistent follows from considering \( G \) the set of \( Q \)’s with one column and attainable payoffs and the least upper bound payoff of each problem attained in that problem. Of course axiom 8 is trivially satisfied in this case.

Let us now consider the class \( G \) of all problems \( Q \) containing two columns and such that \( Q \) is the mixed problem of a problem with a finite number of rows [every payoff must of course be attainable]. Thus every mixed strategy of \( D \) is in \( D \). We denote the interval of attainable utilities by \((A, B)\). This finite, interval may be open, half open, closed, semi-infinite or infinite.

A problem \( Q \) in \( G \) is \( S \) equivalent to a problem in which one of the states of nature is \( I \) and the other \( II \). [These are the labels put on the states of nature.] Thus it may be geometrically represented by a set of points in the \((x_1, x_2)\) plane where \( x_1 = u(d, I) \), \( x_2 = M(d, II) \).

Since \( D \) contains all mixed strategies we have for each \( x, y, p \), \( x, y \) in the set, \( 0 \leq p \leq 1 \), the point \( px + (1 - p) y \) in the set. The original pure strategies correspond to a finite number of points and hence the mixed problem consists of the smallest \( \text{closed convex set} \) containing those points. This set is of course a polygon and every vertex corresponds to a pure strategy.

Two different strategies may correspond to the same point (if they are indifferent). This convex set associated with the problem \( Q \) is called a geometrical representation of \( Q \). The questions arise as to how many representations may a given problem
have and of what problems is a given set the geometrical representation.

First of all a given problem may have (at the most) two representations, for once the states are labeled, the problem determines the representation. There are two possible labelings and thus at most two representations. To answer the other question we first construct a problem which is uniquely determined by a geometrical representation. \( \tilde{\mathcal{Q}} = (\tilde{\mathcal{U}}, \tilde{\mathcal{D}}, \tilde{\mathcal{S}}) \) where \( \tilde{\mathcal{S}} = \{I, II\} \), \( \tilde{\mathcal{D}} = \{(x_1, x_2)\mid \}

\( (x_1, x_2) \) is in the convex set of the representation \( \mathcal{U}(x_1, x_2), I = x_1 \), \( \tilde{u}(x_1, x_2), II = x_2 \). There is no ambiguity in identifying this problem \( \tilde{\mathcal{Q}} \) with the set of points described above and calling both a geometrical representation of \( \mathcal{Q} \) for there is a one to one correspondence between the sets and the \( \tilde{\mathcal{Q}} \).

[It may be noted that while \( \tilde{\mathcal{Q}} \) is not a mixed problem \( \tilde{\mathcal{Q}} \) is not a mixed problem for this \( \mathcal{Q} \), it is evident that there is a two column problem \( \mathcal{Q}_1 \) with attainable payoffs so that \( \mathcal{Q}_1 d \triangleright d rr \mathcal{Q} \) and \( \tilde{\mathcal{Q}} \) is isomorphic to \( \mathcal{Q}_1 \). Also if there is a two column problem \( \mathcal{Q}_1 \) with attainable payoffs so that \( \mathcal{Q}_1 d b d rr \mathcal{Q} \) and \( \tilde{\mathcal{Q}} \) is isomorphic to \( \mathcal{Q}_1 \) then \( \tilde{\mathcal{Q}} \) is a geometrical representation of \( \mathcal{Q} \).

Under what conditions do two problems \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) in \( \mathcal{C} \) have the same geometrical representation. This may occur if and only if there are two problems \( \mathcal{Q}_3 \) and \( \mathcal{Q}_4 \) in \( \mathcal{C} \) so that \( \mathcal{Q}_1 d b d rr \mathcal{Q}_3 \), \( \mathcal{Q}_2 d b d rr \mathcal{Q}_4 \). \( \mathcal{Q}_3 \) is isomorphic to \( \mathcal{Q}_4 \). Furthermore \( \mathcal{Q}_3 \) and \( \mathcal{Q}_4 \) have the same representations as \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) and a row of \( \mathcal{Q}_3 \) corresponds to the same point \([\text{i.e. row of } \tilde{\mathcal{Q}}]\) as does the corresponding row if the states are properly labelled of \( \mathcal{Q}_4 \) in the isomorphism. By axiom 2 if a row of \( \mathcal{Q}_3 \) is in \( C(\mathcal{Q}_3) \), every uniformly equal strategy in \( C(\mathcal{Q}_3) \) and thus every row of \( \mathcal{Q}_3 \) corresponding to a point in \( \tilde{\mathcal{Q}} \) which corresponds to a strategy of \( C(\mathcal{Q}_3) \) is in \( C(\mathcal{Q}_3) \). Thus \( C(\mathcal{Q}_3) \) uniquely determines a subset of \( \tilde{\mathcal{Q}} \) which in turn uniquely determines \( C(\mathcal{Q}_3) \). By consequence 1, \( C(\mathcal{Q}_4) = f(C(\mathcal{Q}_3)) \). The nature of this isomorphism is that a strategy of \( \mathcal{Q}_4 \) corresponds to a strategy of \( \mathcal{Q}_3 \) only if they both correspond to the same point.
Thus \( C(Q_3) \) and \( C(Q_4) \) determine the same subset of \( \tilde{Q} \). But by axiom 4 this subset is exactly the subset determined by \( C(Q_1) \) and \( C(Q_2) \). Thus this subset corresponds to all problems which have the representation \( \tilde{Q} \). Let us then call this subset \( C(\tilde{Q}) \) even though \( \tilde{Q} \) is technically not in \( G \). It is well to note here that consequence 1 implies that if \( \tilde{Q} \) is reflected about the line \( x_1 = x_2 \), \( C(\tilde{Q}) \) is similarly reflected. We may also note the partial ordering is maintained in going from two rows in \( Q \) to the corresponding ones in \( \tilde{Q} \) and vice versa.

Thus far we have been less specific than we may have been in the hope of extending these concepts. However it is well to note that for the particular \( G \) we are discussing \( \tilde{Q} \) is a closed convex polygon with a finite number of vertices (each representing at least one pure strategy) and this polygon lies in the rectangle where \( (x_1, x_2) \) are both in the interval \( (A, B) \). Conversely for every such \( \tilde{Q} \) there are problems of \( G \) which have this \( \tilde{Q} \) as a geometrical representation.

Now we may consider what the axioms imply with respect to \( C(\tilde{Q}) \). We have already discussed an implication of consequence 1. Consequences 3 and 4 imply that \( C(\tilde{Q}) \) is invariant with respect to a translation and an expansion or contraction about the origin which makes the transformed set a \( \tilde{Q} \). The transformed set is obviously a closed convex set. However one must be sure that \( x_1, x_2 \) still lie in the interval of attainable payoffs.

The problem where \( \tilde{Q} \) has only one point has by axiom 7 that point for a solution. Consider the problem \( \tilde{Q} \) of the line connecting \((0, 0)\) to \((a, b)\).

[We denote a line connecting two points \( x \) and \( y \) by \([x \text{ to } y]\).] If the line is in the closed first quadrant \((a, b)\) \( P \) \((a, b)\), \( 0 \leq t \leq 1 \) and thus by axioms 1 and 3, \( \{(a, b)\} = C(\tilde{Q}) \). Similarly if the line is the closed third quadrant \( \{(0, 0)\} = C(\tilde{Q}) \).
We associate with the line $x$ to $y$ an angle $\theta$ which the vector from $x$ to $y$ makes with the vector from $(0, 0)$ to $(1, -1)$. Thus we have yet to consider the lines where $-45 < \theta < 45$, and $135 < 225$. The cases where $135 < \theta < 225$ can be reduced to those where $-45 < \theta < 45$ by translating so that $(a, b)$ goes into $(0, 0)$ after contracting if necessary to keep within the bounds of attainability. The cases where $-45 < \theta < 0$ can be reduced to the case of $0 < \theta < 45$ by translating so that $(a, b)$ goes into $(0, 0)$ and reflecting with respect to the line $x_1 = x_2$ after carrying through the necessary contractions.

Suppose that an interior point of the line is in $C(\bar{Q})$. By consequence 2, both end points of the line are in $C(\bar{Q})$, the whole line is in $C(\bar{Q})$. Then by contraction $(ta, tb)$ is in $C(Q_1)$ where $Q_1$ is the line $(0, 0)$ to $(ta, tb)$. By translating $(ta, tb)$ to the origin $(ta, tb)$ is in $C(Q_2)$ where $Q_2$ is the line $(ta, tb)$ to $(a, b)$. Axiom 7 implies that $(ta, tb)$ is in $C(\bar{Q})$ since $\bar{Q} = \bar{Q}_1 + \bar{Q}_2$. Hence, $C(\bar{Q})$ consists of one end point or of the whole line. Indeed the above type of argument can be used to show that if $\theta = 0$, $C(\bar{Q})$ consists of the whole line.

Suppose that $0 < \theta < 45$. Consider the problem $\bar{Q}_1$ consisting of the triangle whose vertices are $(0, 0), (a, b), (a + b, 0)$. Let $\bar{Q}_2$ be the problem of the line $(a + b, 0)$ to $(a, b)$ and $\bar{Q}_3$ of the triangle except for the line $\bar{Q}_2$. $\bar{Q}_1 = \bar{Q}_2 + \bar{Q}_3$ and for every $d_3$ of $\bar{Q}_3$ there is a $d_2$ of $\bar{Q}_2$ such that $d_2$ $Pd_3$. Thus axiom 6 implies that $C(\bar{Q}_1) = C(\bar{Q}_2)$. By translation within the bounds of attainability $C(\bar{Q}_2)$ consists of the whole line $\bar{Q}_2$ and $(a, b) \in C(\bar{Q}_1)$. Hence $(a, b) \in C(\bar{Q})$ by axiom 5.

Suppose that $\bar{Q}_1$ and $\bar{Q}_2$ are two lines from $(0, 0)$ to $(a_1, b_1), (a_2, b_2)$ respectively and $0 < \theta_2 < \theta_1 < 45^\circ$ and furthermore that the entire line $\bar{Q}_1$ lies
in \( C(\overline{a_2}) \). Contract or expand \( \overline{a_2} \) so that \( b_2 \) goes into \( b_1 \). Let this new line be \( \overline{a_1} \). \([\text{This operation keeps the line within the bounds of attainability.}]\)

By axiom 6, if \( \overline{a} \) is the triangle bounded by \( \overline{a_1}, \overline{a_2}, \) and the horizontal line connecting them, \( C(\overline{a}) = C(\overline{a_1}) \) which contains all strategies of \( \overline{a_1} \). Thus \((0, 0)\) is in \( C(\overline{a}) \) and by axiom 5 in \( C(\overline{a_2}) \). Since both end points of \( \overline{a_2} \) are then in \( C(\overline{a_2}) \), the whole line is 1. Thus we have

**THEOREM 1.** Corresponding to a solution \( C(\overline{a}) \) defined on \( G \) the set of two column problems which are mixed problems of problems with a finite number of pure strategies with attainable payoffs, there is an angle \( \theta_0 \), \( 0 \leq \theta_0 \leq 45 \) and a case (inclusive or exclusive) so that if \( \overline{a} \) the line from \( x \) to \( y \) has an angle \( \theta \), then \( C(\overline{a}) \) is

1) The whole/if 
\[ -\theta_0 < \theta < \theta_0 \] exclusive case
\[ -\theta_0 < \theta < \theta_0 \] inclusive case

2) \( y \) if 
\[ \theta_0 < \theta < 180^\circ - \theta_0 \] exclusive case
\[ \theta_0 < \theta < 180^\circ - \theta_0 \] inclusive case

3) The whole/if 
\[ 180^\circ - \theta_0 < \theta < 180^\circ + \theta_0 \] exclusive case
\[ 180^\circ - \theta_0 < \theta < 180^\circ + \theta_0 \] inclusive case

4) \( x \) if 
\[ 180^\circ + \theta_0 < \theta < 360^\circ - \theta_0 \] exclusive case
\[ 180^\circ + \theta_0 < \theta < 360^\circ - \theta_0 \] inclusive case

Also, if \( \theta_0 = 0 \), the case is the inclusive case and if \( \theta_0 = 45^\circ \) it is the exclusive case.

Now consider any closed convex polygon \( \overline{a} \) with a finite number of vertices which lies in the region of attainable payoffs.
Let \( b_1 = \sup \{ x_1 | (x_1, x_2) \in Q \} \)
\( b_2 = \sup \{ x_2 | (b_1, x_2) \in Q \} \)
\( a_2 = \sup \{ x_2 | (x_1, x_2) \in Q \} \)
\( a_1 = \sup \{ x_1 | (x_1, a_2) \in Q \} \).

Traversing the boundary of \( Q \) in a clockwise direction from \( a \) to \( b \) one passes the vertices \( c_0 = a, c_1, c_2, \ldots, c_{n-1}, c_n = b \). Let \( \bar{Q} \) be the line from \( c_1 \) to \( c_i \) and \( Q_1 \). The corresponding angle of the line translated to the origin \( 45^\circ > \theta_1 > \theta_2 > \ldots > \theta_n > -45^\circ \). Furthermore, the angle of the line connecting any two points between \( c_{i-1} \) and \( c_j \) is between \( Q_1 \) and \( \theta_j \) (if the orientation is taken in the correct direction). Suppose \( Q_1 \) has an angle \( \theta_k \) so that
- \( \theta_k < \theta_1 < \theta_0 \) in the exclusive case or
- \( \theta_0 < \theta_1 < \theta_k \) in the inclusive case.

Then extend \( \bar{Q} \) until its \( x_1 \) coordinate becomes \( b_1 \) and its \( x_2 \) coordinate becomes \( a_1 \) at \( a^* \). Replace the boundary \( \bar{Q} \) by the line \( \bar{Q}^* = (a, a^*) \) (it may be only a point). The line \( \bar{Q}^* = (a^*, b^*) \) and the line \( (b^*, b) \). Thus \( \bar{Q} \) has been increased to a new problem \( \bar{Q}^* \); \( \bar{Q}^* \) is the representation of a problem in \( C \).

\( \bar{Q}^* = \bar{Q}^* + \bar{Q}^{**} \) where \( \bar{Q}^{**} \) is the new problem except for \( \bar{Q}^* \). To every point \( d^{**} \) of \( \bar{Q}^{**} \) there is a \( \bar{d} \) of \( \bar{Q}^* \) so that \( \bar{d}^* \bar{d}^{**} \). Thus \( C(\bar{Q}^*) = C(\bar{Q}) \) by axiom \( C \). By Theorem 1, \( C(\bar{Q}^*) \) contains \( \bar{Q}^* \), which contains \( Q_1 \). Thus by axiom \( 5 \) \( \bar{Q} \) is in \( C(\bar{Q}) \).

If there is no \( \bar{Q} \) satisfying the above requirements, a similar proof can be used to show that \( C(\bar{Q}) \) contains the point \((x_1, x_2)\) for which \( x_1 + x_2 \) attains its maximum.
Conversely it can be shown that in this case \( C(\widetilde{Q}) = \{(x_1, x_2)\} \) and in the case where there are \( \widetilde{Q}_1 \)'s satisfying the above restrictions only the points on the \( \widetilde{Q}_1 \) are in the solution. Thus we have a complete characterization of the solution. To state it in an alternative fashion, we first consider the following. For every \( \theta \) in \((-\Theta_0, \Theta_0)\) (this interval is closed in the inclusive case and open in the exclusive case) there is a \( p \) so that \( 0 \leq p \leq 1 \) and \( \frac{p}{1 - p} = \tan (\theta + 45) \). Corresponding to the interval \((-\Theta_0, \Theta_0)\) there is an interval \((p_0, 1 - p_0)\).

**Theorem 2.** If \( G \) is the set of representations of \( G_3 \) and \( G \) is the class of all problems with attainable payoffs which are the mixed problems of problems a point a is in \( C(Q) \) with a finite number of rows, then if \( G \) is rationally solvable by \( C(Q) \) if and only if a is in \( G \) and for some \( p \) in \((p_0, 1 - p_0)\), \( pa_1 + (1 - p)a_2 \geq px_1 + (1 - p)x_2 \) for all \( x \) in \( G \).

This theorem is equivalent to stating that every problem of \( Q \) of \( G \) may be replaced by \( Q^* = (u, D, S^*) \) where \( S^* \) is the set of mixed strategies of \( S \) where one state has probability \( p \) and the other \( 1 - p \) in \((p_0, 1 - p_0)\). Then \( C(Q) \) is the set of admissible strategies of \( Q^* \).

**Theorem 3.** If \( G \) of Theorem 2 is rationally solvable by \( C(Q) \), \( Q = 0 \), (and the case is inclusive). This follows immediately from axiom 12.

The solution of the two column problems can be interpreted in a slightly different fashion. \( C(Q) \) is the set of strategies which minimize the maximum regret in comparisons at a time. If \( a_1 + a_2 \geq x_1 + x_2 \), then \( a_1 - x_1 \geq x_2 - a_2 \) or \( a_2 - x_2 \geq a_1 - x_1 \), and one of these statements is equivalent to the fact that in comparing \((a_1, a_2)\) with \((x_1, x_2)\), \( a_1, a_2 \) minimizes the maximum regret.

Theorem 2 can be extended to the case where \( G \) is the class of all problems where the representations are closed bounded convex sets within the region of attainable payoffs. This in turn is extendable to the case where the representations are convex in the region of attainable payoffs and the solution of the
closure of the convex set is in the original set.

Theorems 1, 2, 3 and consequences 1, 2, 3, 4 made use of all the axioms except 8 which evidently doesn't apply unless we included the one column case in G. It is easy to prove that the solution for any \((p_o, 1 - p_o)\) would satisfy all axioms except 12, and the solution for \(p_o = \frac{1}{2}\) (inclusive case) satisfies axiom 12. Thus we have

**Theorem 4:** G of Theorems 2 and 3 is rationally solvable by C(Q) where C(Q) is the set of strategies for which the sum of the payoffs attain the maximum. There is no other "rational" solution of G.

The arguments used here seem to be easily extendable to the "general" n column case. However if axiom 12 were abolished, they would still indicate a method which would apply to the class of problems with no more than n columns.

At this point a criticism of at least the use of certain G's should be made. A constructive criticism would be especially appreciated.