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Some Results on Identification in Lagged Shock-error Models

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Note: The following summary presents results which form a part of the paper by Anderson and Hurwicz. Only the latter, however, is responsible for the form and contents of what follows. (Earlier results will be found in a Cowles Commission staff paper on lagged shock-error models presented by Hurwicz in 1946.)

0.0 The results given below state conditions for identification in models with both shock and errors present. The theorems are sometimes stated in a form less general than that obtained. For simplicity's sake, it is assumed that the disturbances, as well as the exogenous variables, are drawn from a normal universe with zero mean. (The covariance matrices are independent of time.)

This reduces the identification problem to one of finding whether the second order movements of the distribution of the observed variables provide a unique determination of the structural parameters.

0.1 The models treated below satisfy the relations

(0.1 - 1.1) $\sum_{\tau=0}^{t^*} \beta_\tau \eta_{t-\tau} + \gamma \zeta_t = u_t$ $(t = t_0, t_0+1, \ldots)$

(0.1 - 1.2) $|\beta_0| \neq 0$

(0.1 - 2.1) $\eta = y_t - v_t^y$

(0.1 - 2.2) $\zeta_t = z_t - v_t^z$

$\eta_1, \ldots, \eta_n$

η and ζ are the "true" variable (column) vector of G and K dimensions respectively; y and z are the corresponding observed variable vectors. The covariance matrices of the u's ("shocks") and v's ("errors")

NOTE: In proofreading, Herman Rubin suggests that the y p z superscripts be subscripts in equations (0.1 - 2.1) and (0.1 - 2.2) in order that the notation be more consistent with the rest of the manuscript.

are denoted by Σ^u and Σ^v respectively; this notation indicates the invariance with regard to time mentioned in 0.0. Both the u's and the v's are assumed to be non-auto-correlated. Furthermore, it is assumed that there is no correlation between the v_y and the v_z; it would appear, however, that this restriction could easily be removed.

The condition (0.1 - 1.2) above will be strengthened to

(0.1 -3) $$\beta_0 = I.$$

where I is the identity matrix. It should be noted, however, that the results obtained on the basis of (0.1 - 3) can be applied to cases where this relation does not hold provided that the restriction on

(0.1 -4) $$\alpha = (\beta, \gamma) = (\beta_0, \beta_1, \ldots, \beta_{\tau^*}, \gamma)$$

are such that the model would be "just identified" for v = 0 (i.e., in the pure "shock" case).

Finally, it is assumed that

(0.1 - 5) $$|\Sigma^u| > 0$$

and

(0.1 - 6) $$\beta = (\beta_0, \beta_1, \cdots, \beta_{\tau^*})$$

is such as to make the system a <u>stable</u> one.

0.2 We speak of <u>complete</u> identification when it is possible to identify <u>all</u> parameters of the model. We speak of <u>structural determinacy</u> whenever it is possible to identify $\alpha = (\beta, \gamma)$. u and v are said to be <u>non-separable</u> when it is possible to identify $\Sigma^u + \Sigma^v$ but not Σ^u and Σ^v separately.

0.3 The remainder of this note is divided into two parts: I. $\gamma = 0$ and this fact is known a priori; II. $\gamma \neq 0$ or $\gamma = 0$ but this fact is not known a priori.

Part I. Exogenous variables known to be absent ($\gamma = 0$ a priori).

1.0 This case is given by eq. (0.1) with

(1.0-1) $\gamma = 0.$

We shall write

(1.0-2) $$\mathcal{E}\eta_{t-p}\eta_t' = \mu_p'$$

and

(1.0-3) $$\mathcal{E}y_{t-p}y_t' = m_p'$$

where $\mathcal{E}x$ denotes the mathematical expectation of x and A' is the transpose of the matrix A.

1.1 Given the conditions and notation stated in 1.0 we have

(1.1-1) $$\mu_0 + \sum_{i=1}^{\tau^*} \beta_i \mu_i' = \Sigma^u$$

(1.1-2) $$(\mu_1, \cdots, \mu_{\tau^*}) + \beta \mathcal{N}_0 = 0$$

(1.1-3) $$(\mu_{\tau^*+1}, \cdots, \mu_{2\tau^*}) + \beta \mathcal{N}_{\tau^*} = 0$$

where

(1.1-4) $$\mathcal{N}_0 = \begin{pmatrix} \mu_0 & \mu_{\tau^*-1} \\ \mu'_{\tau^*-1} \cdots & \mu_0 \end{pmatrix}$$

and

(1.1-5) $$\mathcal{N}_{\tau^*} = \begin{pmatrix} \mu_{\tau^*} & \cdots & \mu_{2\tau^*-1} \\ \mu_1 & \cdots & \mu_{\tau^*} \end{pmatrix}$$

Furthermore

(1.1-6) $$m_p = \mu_p + \delta_{0p} \Sigma^{vy}$$

where δ_{0p} is the Kronecker symbol.

(1.1-6) implies that $\mathcal{N}_{\tau^*}$ may be regarded as known from observation.

Hence β is identified if $\mathcal{N}_{\tau^*}$ is non-singular.

Now it can be verified that

(1.1-7) $$\mathcal{N}_{\tau} = \mathcal{J}^{\tau^*} \mathcal{N}_0$$

where

(1.1-8) $$\mathcal{J} = \begin{pmatrix} -\beta_1, \dots, -\beta_{\tau-1}, -\beta_{\tau^*} \\ I, \quad 0, \dots, 0, \quad 0, \\ \dots \\ 0, \dots, \quad I, \quad 0 \end{pmatrix}$$

all the submatrices in $\mathcal{J}$ being G rowed and square.

1.2 On the basis of the above facts it is possible to prove

<u>Proposition I</u>

$|\beta_{\tau^*}| \neq 0$ is a sufficient condition for the complete identification of the model given by 1.1. When $|\beta_{\tau^*}| \neq 0$, the structural parameters can be obtained from the following relations

(1.2-1) $$\beta = -(\mu_{\tau^*+1}, \dots, \mu_{2\tau^*}) \mathcal{N}_{\tau^*}^{-1}$$

(1.2-2) $$\Sigma^{\Gamma} = \beta_{\tau^*}^{-1}(\mu_{\tau^*} + \sum_{i=1}^{\tau^*} \beta_i \quad m_{\tau^*-i} \quad)$$

(1.2-3) $$\Sigma^{u} = m_0 + \sum_{i=1}^{\tau^*} \beta_i \mu_i' - \Sigma^{vy}$$

where all μ's equal the corresponding m's and can therefore be regarded as given by observation.

1.3 <u>Proposition II</u>

Let G = 1 (i.e., y a scalar) while $\tau^* \geqq 1$.

Then the necessary and sufficient condition for complete identification of the system in 1.1 is that β be of rank 1 (one).

1.4. <u>Proposition III</u>

Let β be of rank zero while G $\geqq$ 1 and $\tau^* \geqq 1$. (In this case $\tau^* > 1$ is trivially equivalent to $\tau^* = 1$.) Then there is structural determinacy.

but u and v are non-separable, with

(1.4-1) $$m_0 = \Sigma^u + \Sigma^v$$

1.5 Proposition IV

Let $\tau^* = 1$, $G \geq 1$ and let $\beta = \beta_1 = \beta_{\tau}^{\#}$ be of the form

$$\begin{pmatrix} \beta^{(11)} & \beta^{(12)} \\ \beta^{(21)} & \beta^{(22)} \end{pmatrix}$$

where $\beta^{(11)}$ is H by H $(0 < H < G)$, $(\beta^{(11)}, \beta^{(12)})$ is of rank H and $(\beta^{(21)}, \beta^{(22)})$ is of rank zero.

Then $\beta^{(11)}, \beta^{(21)}, \beta^{(22)}$ are identifiable; $\beta^{(12)}$ is not.

1.6 Proposition V

Let $\tau^* = 1$, $G \geq 1$. Then the necessary and sufficient condition for complete identification of the system is that β be of rank G.

1.7 It will be noted that both Proposition II and Proposition V give β of rank G as a necessary and sufficient condition for the complete identification of the system.

Part II. Lack of knowledge concerning the presence or absence of the exogenous variables.

2.1 Write

(2.1-1) $$\beta_p = \left\{ \gamma_{t-p} \right\},$$

Then:

Proposition VI

A necessary condition for structural determinacy (hence, a fortiori for complete identification) is that

(2.1-2) $\rho_i \neq 0$ for some $i \neq 0$,

i.e., that $\mathcal{J}$ be auto-correlated.

2.2 Proposition VII.

If either of the two matrices β and γ is of rank zero, there is no complete identification.

2.3 Proposition VIII.

Let $G = K = \tau^* = 1$ and lèt $1 > \rho_i \neq 0$ for some $i \neq 0$.

Then there is complete identification if and only if $\beta \neq 0$, $\gamma \neq 0$; if either β or γ vanish, there is structural determinacy.