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The Approximate Distribution of Calculated Disturbances

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Suppose we have a system of equations

(1) $$\beta y_t' + \Gamma z_t' = u_t',$$

and estimates B, C, of β, Γ. Suppose that u_t is independent of B, C, and that B, C are approximately distributed with mean β, Γ and some covariance matrix.

Let us put

(2) $$B = \beta + B^*,$$

(3) $$C = \Gamma + C^*.$$

Then we wish to calculate the covariance matrix of

(4) $$u_t^{*\prime} = By_t' + Cz_t'.$$

From (1), we have

(5) $$y_t' = \beta^{-1}u_t' - \beta^{-1}\Gamma z_t'.$$

Let us substitute (5) in (4). We obtain

(6) $$u_t^{*\prime} = u_t' + B^*\beta^{-1}\mu_t' + (C^* - B^*\beta^{-1}\Gamma) z_t'.$$

Let us make the usual assumptions about μ_t, namely, $\mathcal{E}(u_t) = 0$, $\mathcal{E}(u_t'u_t) = \Sigma$.

Let us rewrite (6) in tensor form:

(7) $$u^*_{ti} = u_{ti} + b^*_{ij}\beta^{jk}u_{tk} + (c^*_{ik} - b^*_{ij}\beta^{jl}\gamma_{lk}) z^k_t.$$

Since $\mathcal{E}(u_{ti}) = 0$, $\mathcal{E}(b^*_{ij}) = \mathcal{E}(c^*_{ik}) = 0$,

we see that the three terms on the right-hand side of (7) are uncorrelated. Hence

(8) $$\mathcal{E}(u^*_{ti}u^*_{th}) = \mathcal{E}(u_{ti}u_{th}) + \mathcal{E}(b^*_{ij}\beta^{jk}u_{tk}b^*_{mg}\beta^{gb}u_{tf}) + z^k_t z^m_t\,\mathcal{E}\left[(c^*_{ik} - b^*_{ij}\beta^{jl}\gamma_{ek})(c^*_{hm} - b^*_{hg}\beta^{gb}\gamma_{fm})\right].$$

Let

(9) $$(\Lambda_{ij})_{\alpha\beta} = \mathcal{E}(b^{*\prime}_{i\alpha}\, b^{*}_{j\beta}).$$

(10) $$(M_{ij})_{\alpha\beta} = \mathcal{E}(b^{*}_{i\alpha}\, c^{*}_{j\beta}).$$

(11) $$(N_{ij})_{\alpha\beta} = \mathcal{E}(c^{*}_{i\alpha}\, c^{*}_{j\beta}).$$

Then

(12) $$\mathcal{E}(u^{*}_{ti}u^{*}_{th}) = \sigma_{ih} + \mathrm{tr}\,\Lambda_{ih}\,\beta^{-1}\,\Sigma\,\beta^{-1\prime} + z_t\,(N_{ih} - \Gamma'\beta^{-1\prime}M_{ih} - M'_{hi}\beta^{-1}\Gamma + \Gamma'\beta^{-1\prime}\Lambda_{ih}\beta^{-1}\Gamma)\,z'_t.$$

We may simplify this expression. For let

(13) $$\beta^{-1}\,\Sigma\,\beta^{-1\prime} = \Omega,$$

(14) $$-\beta^{-1}\Gamma = \Pi.$$

Then (12) becomes

(15) $$\mathcal{E}(u^{*}_{ti}u^{*}_{th}) = \sigma_{ih} + \mathrm{tr}\,\Lambda_{ih}\,\Omega + z_t\,(N_{ih} + \Pi'\,M_{ih} + M'_{hi}\,\Pi + \Pi'\,\Lambda_{ih}\,\Pi)\,z'_t.$$

A special case of interest is that in which $h = i$, and the estimates are reduced form estimates. In this case, we have

(16) $$M_{ii} = (-\Lambda_{ii}\,\Pi^{*} \quad 0)$$

(17) $$N_{ii} = \begin{pmatrix}\Pi^{*\prime} \\ 0\end{pmatrix} \Lambda_{ii}\,(\Pi^{*} \quad 0) + \frac{\sigma_{ii}}{T}\begin{pmatrix} M^{-1}_{z^*z^*} & 0 \\ 0 & 0 \end{pmatrix}$$

Substituting in (15), we obtain

(18) $$\mathcal{E}(u^{*2}_{ti}) = \sigma_{ii}\left(1 + \frac{1}{T} z^{*}_{t}\, M^{-1}_{z^*z^*}\, z^{*\prime}_{t}\right) + \mathrm{tr}\,\Lambda_{ii}\,\Omega.$$