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On the Identification Problem in Factor Analysis

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1. Introduction.

I shall adopt most of the notations of Thurstone[1] with the following changes: I shall consider a hypothetical universe and shall use greek letters for the parameters. A matrix shall be denoted by the capital letter corresponding to the small letter which denotes the elements of this matrix. Our notation will be:

y_i = score in test i ,
ζ_k = score in reference ability k , corresponds to z_k in econometric model.
π_{ik} = factor loading of common factor k in test i .
η_i = the common-factor part of y_i .
ν_i = the specific part of y_i (i.e. the specific factor + the error)
G = number of tests
ϱ = number of common factors, corresponds to K in econometric model.

We shall introduce the following row vectors.

$$y = [y_1 \quad \cdots \quad y_G]$$
$$\zeta = [\zeta_1 \quad \cdots \quad \zeta_\varrho]$$
$$\eta = [\eta_1 \quad \cdots \quad \eta_G]$$
$$\nu = [\nu_1 \quad \cdots \quad \nu_G]$$

and the matrix

$$\Pi = \Pi_{y\zeta} = \begin{bmatrix} \pi_{11} & \cdots & \pi_{1\varrho} \\ \cdot & \cdot & \cdot \\ \pi_{G1} & \cdots & \pi_{G\varrho} \end{bmatrix} = \Pi_{\eta\zeta}$$

1. This applies to a typed version of this note. In the present mimeographed version the notation has been chosen so as to facilitate comparison with the other models under discussion.

The following assumptions do not restrict the generality of the approach:

(1) $$\mathcal{E}y = \mathcal{E}\eta = \mathcal{E}v = 0$$

(2) $$\mathcal{E}\zeta = 0$$

and

(3) $$var(\zeta_k) = 1, \qquad k = 1, 2, \ldots, \rho.$$

We shall introduce the following covariance matrices:

$$\mu_{\eta\eta} = \mathcal{E}\eta'\eta$$

$$\mu_{yy} = \mathcal{E}y'y$$

$$\mu_{\zeta\zeta} = \mathcal{E}\zeta'\zeta$$

and

$$\Phi = \Phi_{vv} = \mathcal{E}v'v$$

In the following we shall always make the following assumptions:

(4) Π has rank ρ

(5) $\mu_{\zeta\zeta}$ has rank ρ

(6) There exists no y_i whose correlations with all other y_j are zero.

The assumptions (4) and (5) do not restrict the generality, and assumption (6) only excludes a case of no interest.

Thurstone's model may now be described by the following 5 assumptions.

(7) $\eta' = \Pi\zeta', \qquad y = \eta + v$

(8) $\mathcal{E}v'\eta = 0$

(9) Φ is diagonal, to be expressed by the notation $\Phi = \Delta$

(10) Certain assumptions about zeros and non-zeros in Π.

(11) An assumption about ρ (to be given later on).

Thurstone alternatively considers models where $\mu_{\zeta\zeta}$ is the unit matrix and (or) Π is non-negative. None of these models will be considered here.

From equation (7) we obtain

(12) $$\mu_{\eta\eta} = \Pi \mu_{\zeta\zeta} \Pi'$$

which together with equation (8) and the identity $y = \eta + v$ gives

(13) $$\mu_{yy} = \Pi \mu_{\zeta\zeta} \Pi' + \Delta$$

For sake of definiteness we shall assume in the following that the joint distribution of η and v is normal. However, any sufficient condition for identifiability and probably also any necessary condition will still hold good if we drop the assumption about normality.

Under the assumption of normality, knowing the probability distribution of y is equivalent to knowing the covariance matrix μ_{yy}. The identification problem can therefore be stated thus: Can the matrices Π, $\mu_{\zeta\zeta}$ and Δ be uniquely determined from (13) and the conditions (1)-(11) when μ_{yy} is known or does there exist other matrices Π^*, $\mu^*_{\zeta\zeta}$ and Δ^*, not all equal to Π, $\mu_{\zeta\zeta}$ and Δ such that

(14) $$\mu_{yy} = \Pi \mu_{\zeta\zeta} \Pi' + \Delta = \Pi^* \mu^*_{\zeta\zeta} \Pi^{*\prime} + \Delta^*$$

We can now state the assumption about ρ which corresponds to Thurstone's analysis

(11*) ρ is the least possible rank of the matrix Π which is consistent with conditions (1)-(9).

Thurstone has remarked that equation (13) is always solvable under the given conditions when $\rho = G-1$. Hence ρ is always identifiable. But to determine ρ is in general very difficult.

2. Determination of ρ.

In point of principle ρ may be determined by trying a series of different values of ρ, and see if each of them is a possible value, when no account is taken of Condition (11*). Suppose that we try the hypothesis $\rho \leq \rho_0$. If the hypothesis is true, the matrix $\mu_{yy} - \Delta$ must have rank $\leq \rho_0$. This gives a system of equations for determining Δ and we may determine the set $\mathcal{H}_{\rho_0}$ of Δ's satisfying these equations and also satisfying the inequalities resulting from the requirement that both Δ and $\mu_{yy}-\Delta$ have to be non-negative definite. Of course we need not determine the set $\mathcal{H}_{\rho_0}$ itself if we can only find out if it is empty or not. The true value of ρ will be the smallest value of ρ_0 for which the set $\mathcal{H}_{\rho_0}$ is not empty.

In the future we may possibly find necessary and sufficient conditions that $\rho \leqq \rho_0$ which will make it easier to find ρ . So far such necessary and sufficient conditions are only known in the cases $\rho_0 = n-2$ and $\rho_0 = 1$. We have the following theorems:

Theorem 1: A necessary and sufficient condition that $\rho \leqq n-2$, is that the adjoint of the covariance matrix μ_{yy} has signs which are not compatible.

Theorem 2: Necessary and sufficient conditions that $\rho = 1$ are: (a) Excepting the diagonal elements, the rows of μ_{yy} are proportional; (b) Selecting a set of 3-rowed principal minors of μ_{yy} covering all diagonal elements of this matrix, any adjoint of a threerowed matrix of this set has signs which are not compatible.

Theorem 1 is a restatement of Theorem 14 in my paper in Econometrica 1941. Theorem 2a is the well-known tetrad condition. The rest of the theorem follows from Theorem 1.

A necessary but not sufficient condition that $\rho = \rho_0$ when $\rho_0 < n-2$ is the following: The adjoints of all principal ($\rho_0 + 2$)-rowed minor matrices of μ_{yy} have signs which are not compatible.

3. Identifiability of Δ .

From now on we shall suppose that ρ is known. If Π is given, μ_{zz} may generally be determined by the linear equations resulting from equating the non-diagonal elements in the matrices μ_{yy} and $\Pi\mu_{zz}\Pi'$. And when Π and μ_{zz} are known also Δ may be determined. The identifiability of Δ is therefore necessary for the identifiability of Π and $\mu_{\zeta\zeta}$ and we may first discuss the identifiability of Δ and thereafter discuss the identifiability of Π when Δ is known.

We shall first discuss the identifiability of Δ when we do not use any assumptions about zeros in Π. We have G elements of Δ to determine (these elements correspond to the uniquenesses of Thurstone) and $\frac{(G-\varrho)(G-\varrho+1)}{2}$ equations for the determination of them (see for instance Holzinger and Harman 1941, p. 65. Comp. Thurstone 1935, p. 76). From this fact we should expect lack of identifiability, when

$$\varrho > \frac{2G-1-\sqrt{8G+1}}{2} \tag{15}$$

identifiability when

$$\varrho = \frac{2G-1-\sqrt{8G+1}}{2} \tag{16}$$

and overidentifiability when

$$\varrho < \frac{2G-1-\sqrt{8G+1}}{2} \tag{17}$$

The equations determining the uniquenesses are however not linear and may have more than one admissible solution if equation (16) holds good. This has been shown by Wilson and Worcester. (For references see Thurstone 1947, p. 309). Thurstone has pointed out that we shall always have identification of the uniquenesses when

$$\varrho \leqq \frac{n-1}{2} \tag{18}$$

(Thurstone 1947, p. 309).

The matrix Δ may be identifiable on account of assumptions (10) even if the inequality (15) holds good. In order to give a complete discussion of necessary and sufficient conditions for identifiability of Δ, we should have to consider this possibility, which would give rise to a more complicated analysis. Since the case where inequality (15) holds good, does not seem to have any importance in practice, we shall renounce on this discussion. We shall only remark that if the assumptions (10) shall be of any use in identifying Δ , they must overidentify the matrix Π (or at least part of it) for given Δ .

4. The Identifiability of the Factor Loadings.

When the matrix Δ is known, we know the distribution of η . The vector η will be confined to a ρ-flat (a ρ-flat is a linear space of ρ dimensions) in G-dimensional space. The column vectors of the matrix Π must belong to this ρ-flat, for equation (7) may be written

(19) $$\eta = \sum_k \pi_{\eta k} \zeta_k$$

where $\pi_{\eta k}$ is the k-th column of the matrix Π . It is seen that the vectors $\pi_{\eta k}$ represent a set of reference axes in the ρ-flat. Thus, when we know the uniquenesses, the ρ-flat generated by the column vectors of Π is known, even though the individual vectors are not. Without any further assumptions about Π , the vectors $\pi_{\eta k}$ may be chosen as any set of ρ linearly independent vectors in the ρ-flat. That is, if we have one set of vectors $\pi_{\eta k}$ we may replace them by any set of linear combinations which gives a new matrix

(20) $$\Pi^* = \Pi$$

where is any non-singular ρ-rowed square matrix.

In the following we shall suppose that Δ is already identified and we shall examine conditions for the identification of Π when Δ is known.

If we could assume certain prescribed elements of the factor matrix to be zero or if we could assume linear restrictions connecting prescribed elements in a column of Π , then we could immediately apply the theory of identification developed by Koopmans and Rubin for systems of linear stochastic equations. (Cowles Commission Monograph No. 10.) Suppose that the identification is only due to assumptions that prescribed elements of the factor matrix are zero. Then a necessary and sufficient condition for the identifiability of the column k in the factor matrix is that the column is supposed to contain at least $\rho - 1$ zeros and that the submatrix of Π formed by the rows whose elements in the k-th column are zero, has rank $\rho - 1$.

The assumptions used in factor analysis imply that there are a certain minimum amount of zeros in the factor matrix, but it implies no assumption about a zero value of any individual coefficient. Thurstone uses the term simple structure to describe a structure where each row in the factor matrix has at least one zero. He discusses conditions necessary and sufficient for uniqueness (i.e. identifiability) of the simple structure. He links 3 criteria which he says are almost certain to constitute sufficient and more than necessary conditions for identifiability. (Thurstone 1935, p. 156). Thurstone's criteria are

(21) Each row of Π should have at least one zero.

(22) Each column of Π should have at least ρ zeros.

(23) For every pair of columns of Π there should be at least ρ tests whose entries vanish in one column but not in the other.

It is easy to see that the second criterion is necessary for identification (with the exception that if the criterion is valid for $\rho - 1$ of the columns, it may be sufficient to assume only $\rho - 1$ zeros in the ρ-th column). It is no longer sufficient to assume only $\rho - 1$ zeros in each column as it is in the case when we know which of the coefficients are equal to zero. This may be seen geometrically. Let the row vectors of Π be denoted by $\pi_{1\zeta}$, $\pi_{2\zeta}$, ..., $\pi_{G\zeta}$, and let the components of each vector be interpreted as homogeneous simplex coordinates in ($\rho - 1$)-dimensional space. When we know the uniquenesses we know a configuration of G points, but we do not know the coordinate simplex. If we assume $\rho - 1$ zeros in each column, we shall have $\rho - 1$ points in each coordinate hyperplane. These points are not sufficient to determine the coordinate hyperplanes, because we can lay a hyperplane through any set of $\rho - 1$ of the G

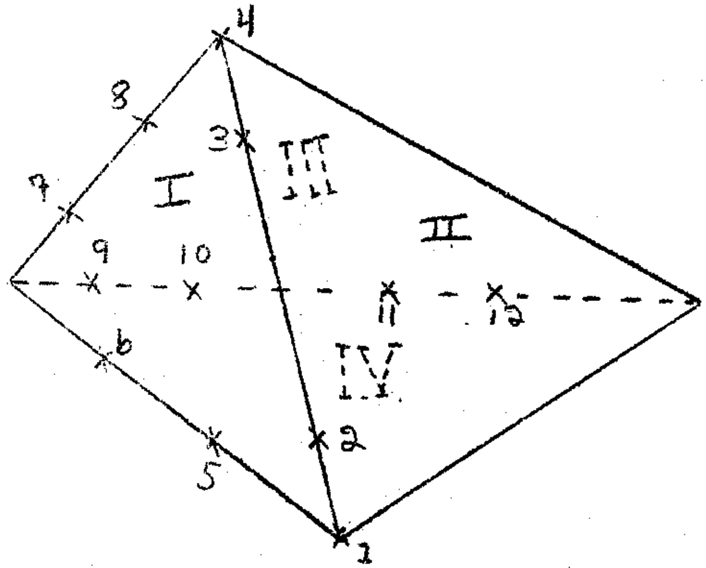
points and we have no possibility of determining which of these hyperplanes are the coordinate hyperplanes. An example with 6 tests and 3 common factors is given in Thurstone 1935, p. 155. We may eliminate some of the possible combinations if we assume that the factor loadings are non-negative (Comp. Thurstone 1935, p. 165), but we still shall have left more than one possible simple structure. In the following we shall use no assumption about non-negative factor loadings. We shall show by an example that Thurstone's 3 criteria are not always sufficient for identification. We choose an example with 12 tests (denoted by arabic numerals) and 4 common factors (denoted by roman numerals).

Matrix of factor loadings:

Common factors

	I	II	III	IV
1	0	0	x	0
2	0	0	x	x
3	0	0	x	x
4	0	0	0	x
5	0	x	x	0
6	0	x	x	0
7	0	x	0	x
8	0	x	0	x
9	x	x	0	0
10	x	x	0	0
11	x	x	0	0
12	x	x	0	0

This matrix may be represented graphically in the following figure.



It is easily seen from the figure that the only coordinate plane which is identified by the set of 12 points is the plane I.

In order that p points in a hyperplane shall identify this hyperplane as a coordinate hyperplane it is necessary that each subset of $p-1$ points uniquely determine a hyperplane.

Let us apply this condition to the matrix Π^* which represents a linear transformation of the true simple structure matrix Π. We may examine if there exists groups of at least ρ row vectors in Π^*, such that the matrix formed by each group has rank $\rho-1$ and also any submatrix of the ρ-rowed matrix obtained by deleting one of the rows has the same rank.

Let one group include the rows $h_1, h_2, \dots, h_\rho$ and let H denote the sequence $h_1, h_2, \dots, h_\rho$ and let $H-h$ denote a subsequence of H obtained by deleting the element h. Let the submatrix of Π^* containing the rows H be denoted by Π^*_H. Then Π^*_H has rank $\rho-1$ and we require that also all submatrices Π^*_{H-h} have rank $\rho-1$.

If there are exactly ρ such groups, then the simple structure is identifiable. If there exists only $\rho-1$ such groups, then these groups will determine

$\rho - 1$ coordinate hyperplanes. The ρ-th coordinate hyperplane will be determined if there are at least $\rho - 1$ points not lying in any of the other hyperplanes.

Let us examine what the additional rank criteria means in terms of zeros in the matrix Π. Consider for instance the rows H of Π whose element in the first column is zero. Each of the other columns must contain at least 2 non-zero elemtnts in these rows, for if it contained only one non-zero element, then there would exist one matrix Π_{H-h} which would be of rank not higher than $\rho - 2$. In the same way it is seen that each combination of q columns must contain at least $q+1$ rows in Π_H with non-zero elements.

If we want criteria which are more nearly necessary and sufficient than Thurstone's criteria, we may drop condition (21) and replace condition (23) by the following condition:

(24) Let us consider the submatrix Π_H of Π consisting of all the rows of which have zeros in the k-th column. Then for $q = 1, 2, \cdots, \rho - 1$, there should for any combination of q columns different from the k-th, exist at least $q+1$ rows of Π_H containing non-zero elements in the q columns.

The preceding discussion presupposes that ρ points never happen by chance to lie in the same hyperplane different from a coordinate hyperplane. Evidently Condition (21) is useful even though it is not necessary for identifiability, for if we have a row with no zero it can never make any contribution to the identification of the matrix Π for given Δ and may be ~~out~~ a hindrance for this determination if it happens to be together with $\rho - 1$ other points in a hyperplane which does not coincide with a coordinate hyperplane.

References

Cowles Commission Monograph No. 10.

Holzinger and Harman (1941): Factor Analysis.

Reiersol (1941), Econometrica, Vol. 9, p. 1.

Thurstone (1935): The Vectors of Mind.

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