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The Duality Theorem in Linear Programming

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The Duality Theorem in Linear Programming

The fundamental theorems of linear programming concern the existence of a dual program and the role played by it. The statement of these theorems involves two vector spaces, each supplied with a partial ordering, and a linear transformation between them. (See §1.) The standard results have been given under the condition that the ordered vector spaces in question are ordinary Euclidean spaces. Various applications in economics call for a generalization of these results to infinite-dimensional spaces, for example, when one replaces an n-tuple of numbers by a numerical function of a continuous variable.

§1 - The material of linear programs

The term linear program could be understood here to mean the maximization (or minimization) of a linear form in a convex set. Actually, the operation minimax introduced by von Neumann should be considered as a member of the same species, so we will here refer to minimaxizations in a convex set as a program.

To a linear program, namely

(I') Maximize $\sum_j a_j x_j$ subject to the constraints

$$\begin{cases} x_j \geq 0 & (1 \leq j \leq n) \\ \sum_j t_{ij} x_j \leq b_i & (1 \leq i \leq m) \end{cases}$$

one associates another (called the dual program):

(II') Minimize $\sum_i y_i b_i$ subject to the constraints

$$\begin{cases} y_i \geq 0 & (1 \leq i \leq m) \\ \sum_i y_i t_{ij} \geq a_j & (1 \leq j \leq n) \end{cases}$$

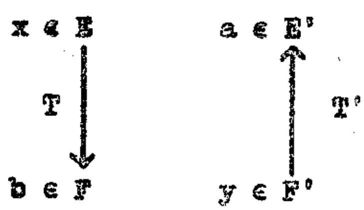
One also associates the following two-person game:

(III') Find a saddlepoint (max in x and min in y) of

$$\varphi(x,y) = \sum_j a_j x_j + \sum_i y_i b_i - \sum_{ij} y_i t_{ij} x_j$$

subject to the constraints $x_j \geq 0$ $(1 \leq j \leq n)$, $y_i \geq 0$ $(1 \leq i \leq m)$.

If the coordinates are boiled out and the finite dimensionality dropped, one gets the following setup: We have two ordered vector spaces E, F; E (resp. F) is in duality with a vector space E' (resp. F'); there is a linear transformation T of E into F which admits an adjoint T' mapping F' into E', and two elements $a \in E'$, $b \in F$.



For brevity denote by K (resp. K') the convex set in E (resp. F') defined by $x \geq 0$, $Tx \leq b$ (resp. $y \geq 0$, $T'y \geq a$).

The three programs become

(I) Maximize $\langle x,a \rangle$ subject to $x \in K$.

(II) Minimize $\langle b,y \rangle$ subject to $y \in K'$.

(III) Find saddlepoint of $\langle x,a \rangle + \langle b,y \rangle - \langle Tx,y \rangle$ subject to $x \geq 0 \in E$, $y \geq 0 \in F'$.

Let us now state the well-known facts (which are known to hold in the finite-dimensional case):

(D 1) For $x \in K$ and $y \in K'$ one has $\langle x,a \rangle \leq \langle Tx,y \rangle \leq \langle b,y \rangle$.

(D 2) If any one of the three programs admits a solution, then all three do.

(D 3) The solutions (x,y) of (III) consist of the solutions x of (I) and y of (II).

(D 4) If the programs admit solutions, then they all have the same scalar value.

It remains to see whether these results still hold in any but the finite-dimensional case.

The above description must be sharpened. For one thing, we will need a topology on E and F, and hence we assume that each of these vector spaces has a locally convex topology, and that E' and F' are their duals (for these topologies), and finally that T is continuous.

For another thing, the orderings on these vector spaces are indispensable, for they are needed to define K and K' and to make sense out of (III).

One should recall how an ordering is given to a vector space. First, for each set $X \subset E$ one notes by $X^{\oplus}$ the set $\subset E'$ made up of the $y \in E'$ such that $\langle x,y \rangle \geq 0$ for each $x \in X$. Dually for a set $Y \subset E'$, $Y^{\oplus}$ is $\subset E$.

Now, an ordering is defined in E by choosing in E a closed convex cone. (See appendix.) P being the cone chosen in E, the elements of P are called the positive elements of E, and the order relation $x \leq y$ is the relation $y - x \in P$.

The dual space E' is ordered by taking as its positive cone the cone $P^{\oplus}$.

A basic fact is that $P^{\oplus\oplus} = P$ for each closed convex cone $P \subset E$. (More generally, if P is any convex cone in E, $P^{\oplus\oplus}$ coincides with the closure of P. See appendix.) In the language of inequalities this says that in order that an element $x \in E$ be positive, it is necessary and sufficient that $\langle x,y \rangle \geq 0$ for each $y \geq 0$ in E'.

§2 - What is obviously true.

One first notes that, for $x \in K$ and $y \in K'$, one has $\langle x,a \rangle \leq \langle Tx,y \rangle \leq \langle b,y \rangle$.

This is clear, for $\langle b,y \rangle - \langle Tx,y \rangle = \langle b-Tx,y \rangle \geq 0$ as $b - Tx \geq 0$ and $y \geq 0$. Similarly $\langle Tx,y \rangle - \langle x,a \rangle = \langle x,T'y \rangle - \langle x,a \rangle = \langle x,T'y - a \rangle \geq 0$ as $x \geq 0$ and $T'y - a \geq 0$.

One deduces from this inequality that if K' is nonvoid, then the linear functional a is majored in K; and dually if K is nonvoid, then the linear functional b is minored in K'.

Secondly, one notes that the following conditions are equivalent:

(1) $x \in K$, $y \in K'$, and $\langle x,a \rangle = \langle b,y \rangle$.

(2) $x \in K$, $y \in K'$, $\langle x,T'y - a \rangle = 0$ and $\langle Tx - b,y \rangle = 0$.

(3) (x,y) is a solution of (III).

That (1) and (2) are equivalent results immediately from above, as $\langle x,T'y - a \rangle = \langle x,a \rangle - \langle Tx,y \rangle$ and $\langle b - Tx,y \rangle = \langle b,y \rangle - \langle Tx,y \rangle$.

One sees that (2) is equivalent to (3) as follows. To say that (x_o,y_o) is a solution of (III) is to say that for each $x \geq 0$ in E and $y \geq 0$ in F' one has $\varphi(x,y_o) \leq \varphi(x_o,y_o) \leq \varphi(x_o,y)$, i.e. that

$$\begin{cases} 0 \leq \varphi(x_o,y) - \varphi(x_o,y_o) = \langle b - Tx_o, y - y_o \rangle \\ 0 \leq \varphi(x_o,y_o) - \varphi(x,y_o) = \langle x - x_o, T'y_o - a \rangle. \end{cases}$$

Now, for a linear functional f on E, the relation "$f(P) \geq f(x_o)$" is equivalent (because $f(P)$ is a cone in the real line) to "$f \geq 0$ and $f(x_o) \leq 0$", and as $x_o \geq 0$, this is in turn equivalent to "$f \geq 0$ and $f(x_o) = 0$". Thus the above conditions become

$$\begin{cases} b - Tx_0 \geq 0 \quad \text{and} \quad < b - Tx_0, y_0 > \; = \; 0 \\ T'y_0 - c \geq 0 \quad \text{and} \quad < x_0, T'y_0 - c > \; = \; 0, \qquad \text{Q.E.D.} \end{cases}$$

One deduces from this result that if (III) admits a solution, then K and K' are both nonvoid and (I) and (II) admit solutions; and if (x, y) is a solution of (III), then x (resp. y) is a solution of (I) (resp. (II)) and $< x,c > \; = \; < b,y >$. Conversely if (I) and (II) both have solutions and in addition these solutions meet - i.e., $< x,c > \; = \; < b,u >$, - the (III) admits a solution.

§4 - The Minkowski-Farkas lemma.

Minkowski-Farkas lemma[1] - Let E and F be locally convex spaces, T a continuous linear transformation of E into F, Q a closed convex cone in F. Then $[\overset{-1}{T}(Q)]^{\oplus}$ coincides with the closure of $T'(Q^{\oplus})$ in E' for the weak - * topology.

Proof: For each set $Y \subset F'$ one has the identity $\overset{-1}{T}(Y^{\oplus}) = (T'(Y))^{\oplus}$. This is, in fact, elementary algebra: $x \in \overset{-1}{T}(Y^{\oplus}) \iff T(x) \in Y^{\oplus} \iff$ $< T(x), Y > \; \geq 0 \iff < x, T'(Y) > \; \geq 0 \iff x \in (T'(Y))^{\oplus}$.

One thus has $[\overset{-1}{T}(Q)]^{\oplus} = [\overset{-1}{T}(Q^{\oplus\oplus})]^{\oplus} = [T'(Q^{\oplus})]^{\oplus\oplus} = \overline{T'(Q^{\oplus})}$ (closure of $T'(Q^{\oplus})$ in E' for the weak - * topology), because $T'(Q^{\oplus})$ is a convex cone, and when E' is given the weak - * topology, E is its dual.
Q.E.D.

Remarks - 1. Let E be a Hilbert space, T a continuous operator on E, Q a closed convex cone in E, then as for convex sets the weak - * topology and the ordinary topology yield the same closure, $[\overset{-1}{T}(Q)]^{\oplus}$ coincides with the closure of $T'(Q^{\oplus})$ in E (for the ordinary topology).

2. In particular in ordinary n-dimensional Euclidean space, $[\overset{-1}{T}(Q)]^{\oplus} = \overline{T'(Q^{\oplus})}$, where the closure is the ordinary one.

§5 - Hurwicz's theorem

The only result wanting is to show that, if Program I admits a solution, then Program III admits a solution: this is the nontrivial part of the question. It is answered by the following theorem, which is a sharpened version of a theorem of Hurwicz[2].

Theorem 1 - Let E and F be ordered locally convex spaces, T a continuous linear transformation of E into F, $b \in F$. One supposes that the set K: $x \geq 0$, $Tx \leq b$ in E is nonvoid. In order that, for each $a \in E'$ such that $\sup_{x \in K} \langle x,a \rangle < \infty$ there exist a $y \in K'$ such that $\sup_{x \in K} \langle x,a \rangle = \langle b,y \rangle$ (where K' is the set of $y \geq 0$, $t'y \geq a$), it is necessary and sufficient that the transpose of the operator

$$(\lambda,x) \longrightarrow (\lambda,x, \lambda b - Tx)$$

transforms the positive cone (of $R \times E' \times F'$) into a closed set (for the weak - * topology of $R \times E'$).

One first notes that the relation $\langle b,z \rangle = \sup_{x \in K} \langle x,a \rangle$ is equivalent to "for each $\alpha \geq \langle K,a \rangle$ one has $\alpha \geq \langle b,z \rangle$" because $z \in K'$.

Secondly, denoting by U the transformation $(\lambda,x) \longrightarrow (\lambda,x, \lambda b - Tx)$ of $R \times E$ into $R \times E \times F$, one has by the Minkowski-Farkas lemma $[\overset{-1}{U}("P")]^{\oplus} = U'("P")$.

The theorem thus results from the following two lemmas. (In their statement, "P" refers to the appropriate positive cone - although this is an abuse of language it is unambiguous.)

Lemma 1 - In order that the number α be $\geq \langle K,a \rangle$, it is necessary and sufficient that $(\alpha, -a) \in [\overset{-1}{U}("P")]^{\oplus}$. (K assumed nonvoid.)

We simply have to show the equivalence of the two relations

(1) $x \geq 0$, $b - Tx \geq 0$ imply $\alpha - \langle x,a \rangle \geq 0$.

(2) $\lambda \geq 0$, $x \geq 0$, $\lambda b - Tx \geq 0$ imply $\lambda\alpha - \langle x,a \rangle \geq 0$,

because $\langle(\lambda,x), (\alpha, -a)\rangle = \alpha\lambda - \langle x,a \rangle$. Now (1) is a special case of (2) where $\lambda = 1$. Conversely, suppose that (1) holds, and suppose that $\lambda \geq 0$, $x \geq 0$, $\lambda b - Tx \geq 0$. If $\lambda > 0$, then upon dividing out by λ we immediately get $\lambda\alpha - \langle x,a \rangle \geq 0$. On the other hand, suppose that $\lambda = 0$. For each $\epsilon > 0$ and any $k \in K$ one has $\epsilon^{-1} x + k \geq 0$ and $T(\epsilon^{-1} x + k)$ $= Tk + \epsilon^{-1} Tx \leq Tk \leq b$; hence by (1), one has $\langle \epsilon^{-1} x + k, a \rangle \leq \alpha$, i.e., $-\epsilon\alpha + \langle x + \epsilon k, a \rangle \leq 0$. As ϵ was arbitrary, it follows by continuity that $\langle x, a \rangle \leq 0$, Q.E.D.

<u>Lemma 2</u> - <u>Let</u> α <u>be a given number.</u> <u>In order that there exist</u> $z \in K'$ <u>such that</u> $\langle b, z \rangle \leq \alpha$, <u>it is necessary and sufficient that</u> $(\alpha, -a) \in U'$ ("P").

U' being the adjoint of U, use of the identity $\langle m,U'n\rangle = \langle Um,n \rangle$ shows that $U'(\lambda, y, z) = (\lambda + \langle b,z \rangle, y - T'z)$. Now, the relation "$(\alpha, -a) \in U'$("P")" is thus equivalent to "there exists $(\lambda, y, z) \geq 0$ in $R \times E' \times F'$ such that $\alpha = \lambda + \langle b,z \rangle$, and $-a = y - T'z$," which becomes, upon eliminating λ and y, "there exists $z \geq 0$ in F' such that $\alpha \geq \langle b,z \rangle$ and $a \geq T'z$, Q.E.D.

<u>§6 - The duality theorem.</u>

From theorem 1 and the elementary results of §2 we deduce

<u>Theorem 2 (Duality theorem)</u>[3] - <u>Let</u> E, F <u>be ordered locally convex spaces</u>, T <u>a continuous linear transformation</u> of E <u>into</u> F, $a \in E'$, $b \in F$, <u>satisfying the following condition:</u>

(H) The transpose of the operator $(\lambda, x) \longrightarrow (\lambda, x, \lambda b - Tx)$ transforms the positive cone (of $R \times E' \times F'$) into a closed set (for the weak - * topology on $R \times E'$). And dually the transpose of the operator $(\lambda, y) \longrightarrow (\lambda, y, T'y - \lambda a)$ transforms the positive cone (of $R \times F \times E$) into a closed set (in $R \times F$).

Denoting by K (resp. K') the convex set $x \geq 0$, $Tx \leq b$ in E (resp. $y \geq 0$, $T'y \geq a$ in F'), one has the following three linear programs:

(I) Maximize $\langle x, a \rangle$ subject to $x \in K$

(II) Minimize $\langle b, y \rangle$ subject to $y \in K'$

(III) Find saddlepoint of $\langle x, a \rangle + \langle b, y \rangle - \langle Tx, y \rangle$ subject to $x \geq 0$ in E, $y \geq 0$ in F. (Max in x, min in y.)

It results then that

(1) For $x \in K$ and $y \in K'$ one has $\langle x, a \rangle \leq \langle Tx, y \rangle \leq \langle b, y \rangle$.

(2) K assumed nonvoid, in order that a be majored in K, it is necessary and sufficient that K' be nonvoid; and dually K' assumed nonvoid in order that b be minored in K' it is necessary and sufficient that K be nonvoid.

(3) If any one of the three programs admits a solution, then all three do.

(4) The solutions (x,y) of (III) consist of the solutions x of (I) and y of (II).

(5) If the programs admit solutions, then they all yield the same scalar value.

One sees now why the duality theorem holds in the original setup (page 1), because here all the convex cones in question are polyhedral (i.e., finitely-generated), a property which implies closure and which is preserved by linear transformations.

Question: Is it true that $\sup_{x \in K} \langle x,a \rangle = \inf_{y \in K'} \langle b,y \rangle$, even when hypothesis (H) is removed altogether? (Conclusion (2) of Th.2 would follow automatically from this. It would also follow that K and K' can never be void simultaneously.)

§6 - Extension of Positive Functionals

E everywhere denotes an ordered locally convex space, P, the cone of the positive elements of E.

V being a subspace of E, a functional f defined in V is called positive when $f(x) \geq 0$ for each $x \in V \cap P$.

Whenever it is a question of the topology on E', the weak - * topology is to be understood.

Def. 1 - A subspace V of E is called of type ε when each positive continuous functional on V can be extended to a positive continuous functional on E; of type Q when the image of P in the quotient space E/V is closed.

Prop. 1 - In order that a closed subspace V be of type ε it is necessary and sufficient that $V^{\perp}$ be of type Q, and vice versa.

Both are equivalent to the condition $(V \cap P)^{\oplus} = V^{\oplus} + P^{\oplus}$, which proves the proposition.

Th. 3 - Let E be an ordered locally convex space, E' its dual. If each closed subspace of E is of type ε (resp. Q) then each closed subspace of E' is of type ε (resp. Q), and conversely.

Another way of stating Th. 1 is that if a closed subspace of E is of type Q then each closed subspace of E is of type ε. All the rest follows by duality and by prop. 1.

Thus suppose that each closed subspace of E is of type Q, and let V be a closed subspace of E, f a positive continuous functional in V. The set H in V where $f = 0$ is a closed subspace of E, hence the canonical image $\varphi(P)$ of P in E/H is closed. As f is positive in V, the set H_1 in V where $f = -1$ is disjoint from P, and hence $\varphi(H_1)$ (which is a point in E/H) does not belong to $\varphi(P)$. Therefore there exists a continuous functional g in E/H which is positive in $\varphi(P)$ and $= -1$ at $\varphi(H_1)$; $g \circ \varphi$ is then a positive functional in E extending f. Q.E.D.

Let us say that a cone P in a locally convex space E is "completely closed" when it is closed, and in addition, for each closed subspace V of E, the image of P in E/V is closed. It is clearly the same to say that each closed subspace of E is of type Q (resp. type ϵ). One can then state the following corollaries to theorem 1:

Corollary 1 - In a space ordered by a polyhedral cone one can always extend positive functionals.

This is because a polyhedral convex cone is always closed and each quotient of it is again polyhedral.

Corollary 2 - If P is a completely closed convex cone in E, then $P^{\oplus}$ is completely closed in E', and conversely.

This is immediate from the Th.

Note: Here is an example of an ordered 3-dimensional vector space where one cannot always extend positive functionals. The space E is the space of number triples R^3, ordered by the cone P defined by the relations $x \geq 0$, $y \geq 0$, $z \geq -\sqrt{xy}$. In the subspace V defined by the relation $y = 0$ the functional $f(x,y,z) = z$ is positive. However, one can verify that no extension of f to all of E is positive.

The question naturally comes up, when can a given positive functional in a subspace V be extended one dimension higher. It is answered by

Prop. 2 - Let V be a subspace of E, e an element of E. One assumes that the set B consisting of the $x \in V$ such that $x \leq e$ is nonvoid. Under this assumption, in order that a continuous functional f defined in V be extendable to a positive continuous functional in $V + (e)$, it is necessary and sufficient that f be majored in B.

Remarks - 1. As a corollary of this prop., one concludes that if f is majored in B, then f is positive (in V).

2. Assuming that f is positive (in V), if $x \in A$ and $y \in B$, then $f(x) \geq f(y)$, where A denotes the set of $x \in V$ such that $x \geq e$.

3. In particular, if A and B are both nonvoid, then each positive functional on V is automatically majored in B and minored in A.

4. Thus the only case not covered by the prop. is where A and B are both void.

The condition of prop. 2 is clearly necessary. It is sufficient. For suppose that f has a finite supremum σ in B.

First of all, one concludes that f is positive, for if one arbitrarily chooses $\epsilon > 0$, $v \geq 0$ in V, and $b \in B$, then $b - \epsilon^{-1}v \in B$, hence $f(b - \epsilon^{-1}v) \leq \sigma$, i.e., $f(v) \geq \epsilon(f(b) - \sigma)$. It follows that $f(V \cap P) \geq 0$, i.e., that f is positive.

The desired extension of f is given by $\bar{f}$: $v + \lambda e \longrightarrow f(v) + \sigma\lambda$. To see that $\bar{f}$ is well-defined and positive, it suffices to show that if $v + \lambda e \geq 0$ $(v \in V)$, then $f(v) + \sigma\lambda \geq 0$.

But in case $\lambda > 0$, one then has $-\lambda^{-1}v \in B$, and so by the definition of σ, $f(-\lambda^{-1}v) \leq \sigma$, i.e., $f(v) + \lambda\sigma \geq 0$.

In case $\lambda = 0$, one has $v \geq 0$ and hence as $f \geq 0$, $f(v) \geq 0$.

Finally, in case $\lambda < 0$, one has then $-\lambda^{-1}v \in A$, hence $f(-\lambda^{-1}v) \geq \sigma$, i.e., $f(v) + \lambda\sigma \geq 0$. The proposition is thus proved.

§7 - The duality theorem revisited.

Lemma 5 - Let V be a closed subspace of an ordered locally convex space E and e an element of E such that $e \geq 0$ and $V + (e)$ is of type ϵ. One then has

$$[V \cap (P - e)]^{\oplus} = V^{\oplus} + (P - e)^{\oplus}$$

We can assume that $e \notin V$ (else it reduces to prop. 1 of §6). $W = V + (e)$ is then a topological direct sum, i.e., the projection π: $W \longrightarrow V$ is continuous. Suppose that $f \in [V \cap (P - e)]^{\oplus}$; the result f_1 of cutting f down to V is extended by the continuous functional $\bar{f}_1 = f \circ \pi$ in W. Let $v + \lambda e \geq 0$ be any positive element of W. One then has $f(v) \geq 0$; this is clear if $\lambda \leq 0$ for then $v \geq - \lambda e \geq 0$, hence $v \in V \cap (P - e)$; while if $\lambda > 0$ then $\lambda^{-1}v \geq -e$ so $\lambda^{-1}v \in V \cap (P - e)$. As $\bar{f}_1(v + \lambda e) = f(v) \geq 0$, this shows that $\bar{f}_1$ is positive in W. As W is of type ϵ, there exists a positive extension g of $\bar{f}_1$ to all of E. Not only is $g \in P^{\oplus}$ but indeed $g \in (P - e)^{\oplus}$ for $g(e) = \bar{f}_1(e) = 0$. One thus has $f = (f - g) + g$ where $f - g \in V^{\perp} = V^{\oplus}$ and $g \in (P - e)^{\oplus}$, Q.E.D.

The following theorem is a variant form of the duality theorem:

Theorem 4 - Let E be an ordered locally convex space, V a closed subspace of E, $e \in E$, $e' \in E'$. Assume that $V + (e)$ is of type ϵ. The relations $x \in V$, $x \leq e$ (resp. $y \in V^{\perp}$ and $y \leq e'$) define a convex set B in E (resp. B' in E').

(1) For $x \in B$ and $y \in B'$ one has $< x, e' > + < e, y > \leq < e, e' >$.

(2) B assumed nonvoid, in order that e' be majored in B it is necessary and sufficient that B' be nonvoid.

(3) If e' can be maximized in B, then e can be maximized in B' and one has $< x_o, e' > + < e, y_o > = < e, e' >$ where x_o, y_o are the maximizers.

The result (1) is proved as follows: For $x \in B$ and $y \in B'$ one has $0 \leq < e - x, e' - y > = < e, e' > - < x, e' > - < e, y >$ as $< x, y > = 0$.

As for (3), suppose that x_o maximizes e' in B. This says that for each $y \in E$ such that $x_o - y \in V$ and $x_o - y \leq e$, one has $< y, e' > \geq 0$; which is to say (as $x_o \in V$) that $e' \in [V \cap (P + x_o - e)]^{\oplus}$. Now $e - x_o \geq 0$ and $V + (e - x_o) = V + (e)$. Hence by lemma 3, $e' \in V^{\oplus} + (P + x_o - e)^{\oplus} = V^{\perp} + P^{\oplus} \cap (e - x_o)^{\perp}$; i.e., one can decompose $e' = y_o + q$ where $< V, y_o > = 0$, $q \geq 0$, and $< e - x_o, q > = 0$; i.e., $y_o \in B$ and $0 = < e - x_o, e - y_o >$, which shows, in the light of (1), that y_o maximizes e in B'.

Finally, to establish (2), one notes that the condition is easily seen to be sufficient. It is also necessary, for if e' is majored in B, then by prop. 2, §6, e' is positive in V and indeed can be extended to a positive continuous functional in $V + (e)$. As $V + (e)$ is of type ϵ, it can thus be extended all the way: there exists $f \in E'$ such that $f \geq 0$ and $e' - f \in V^{\perp}$. It follows that B' is nonvoid. Q.E.D.

We are now in the position to give a new hypothesis for theorem 2.

Theorem 5 - The hypothesis (H) of the Duality Theorem can be replaced by the following:

(H') The subspace of $E \times F$ consisting of the $(x, \lambda b - Tx)$, as x runs over E and λ over R, is of type ϵ. Dually, the subspace of $E' \times F'$ consisting of the $(T'y - \lambda a, y)$, as y runs over F' and λ over R, is of type ϵ.

Remark: We will see that these subspaces are automatically closed.

The th. is proved as follows. Suppose a is maximized at x_0 in K (terminology of th. 2). Now, let's denote by S the transformation $x \longrightarrow (x, -Tx)$ of E into $E \times F$. The transpose S' of S is then the transformation $(u, v) \longrightarrow u - T'v$ of $E' \times F'$ into E'. Denoting by d the element $(0, -b) \in E \times F$, one has $x \in K$ if and only if $Sx \geq d$.

Now S' is manifestly onto, so there exists $e' \in E' \times F'$ such that $-a = S'e'$. For $x \in E$, one then has $< x, a > = < x, -S'e' > = -< Sx, e' >$. On the other hand, the relation "$x_0 - x \in K$" is equivalent to "$S(x_0 - x) \geq d$," i.e., (writing $Sx_0 - d = e$), "$Sx \leq e$." Thus, as x_0 maximizes a in K, the relation $Sx \leq e$ implies $< Sx, e' > \leq 0$. Now the image $V = S(E)$ of the whole space by the operator S is a closed vectorial subspace of $E \times F$ (as it is the graph of the operator $-T$). Denoting by B the set in $E \times F$ defined by the relations $y \in V$ and $y \leq e$, we have seen that $y \in B$ implies $< y, e' > \leq 0$, i.e., the element $0 \in E \times F$ maximizes the functional e' in B. Now $V + (e) = V + (d) = \left\{(x, \lambda b - Tx)\right\}_{\lambda \in R,\, x \in E}$, which by hypothesis is of type ϵ. It is also closed, being one dimensional over the closed subspace V. Denote by B' the set in $E' \times F'$ defined by $u \in V^{\perp}$ and $u \leq e'$. Theorem 4 applies, showing that there exists $u \in B'$ such that $< 0, e' > + < e, u > = < e, e' >$.

Now the relation $u \in V^{\perp}$ is equivalent to $S'u = 0$. Writing $e' - u = (z, y)$ in $E' \times F'$, one thus has $z \geq 0$ in E', $y \geq 0$ in F', and $z - T'y = S'(e' - u) = -a$, and $0 = < e, e' - u > = < (x_0, b - Tx_0), (z, y) > = < x_0, z > + < b - Tx_0, y >$. From the latter relation, as $< x_0, z > \geq 0$ and $< b - Tx_0, y > \geq 0$, we deduce $< x_0, z > = 0$ and $< b - Tx_0, y > = 0$. Elminating z from all these relations shows that $y \in K'$ and $< x_0, a > = < y, b >$, ie., that y fills the bill.

It remains to see that, K being nonvoid and a being majored in K, that K' is nonvoid. But here again there exists $x_1 \in E$ (but not necessarily $\in K$ as above) such that "$Sx \leq e_1$" implies "$< Sx, e' > \leq 0$, where $e_1 = Sx_1 - d$. Thus $-e'$ is majored in the set B_1 of the $y \in V$, $y \leq e_1$. As $V + (e_1) = V + (d)$, theorem 4 applies and shows that B_1' is nonvoid, i.e., there exists $u \in E' \times F'$ such that $S'u = 0$, $e' - u = (z,y) \geq 0$; i.e., $y \geq 0$, $z \geq 0$, and $z - T'y = -a$; i.e., $y \in K'$, Q.E.D.

<u>§ Appendix - Convexity, duality, and weak topologies in locally convex spaces.</u>

In this § we shall summarize certain standard results in the theory of locally convex spaces. You will remember that a locally convex space is a vector space supplied with a topology satisfying certain axioms.[4] All the topological vector spaces you meet are locally convex. For example, in increasing order of generality: Euclidean n-dimensional space, Hilbert spaces, Banach spaces. (Indeed, a Banach space and its dual remain locally convex when you switch to the weak and weak-* topologies.)

A set C in a vector space is called a <u>cone</u> when it has the property that, if $x \in C$ and $\lambda > 0$, then $\lambda x \in C$. In order that a cone C be <u>convex</u>, it is necessary and sufficient that $C + C \subset C'$ -- i.e., that $x \in C, y \in C$ imply $x + y \in C$.

Example - One of the most important examples of a convex cone occurs in the Hilbert space L^2, the space of the square-integrable functions on a measure space. It is the set C of the functions f(x) which are ≥ 0 almost everywhere. C is closed, but has void interior, unless the measure space is finite. (The latter case is where L^2 becomes a finite-dimensional Euclidean space and C the set of vectors whose coordinates are ≥ 0.) The importance of C is that the relation "$f(x) \leq g(x)$ a.e." in L^2 is equivalent to the relation "$g - f \in C$".

Fundamental to the subject of convexity is the separation theorem[5]: If K is a closed convex set in a locally convex space E, and p a point of E not belonging to K, then there exists a continuous linear functional f on E and a number λ such that $f(p) < \lambda$ and $f(K) > \lambda$. But the set up is seen its greatest simplicity as follows. The set H consisting of the points x where $f(x) = \lambda$ is a hyperplane, closed if and only if f is continuous. The sets where $f(x) \leq \lambda$, resp. $f(x) \geq \lambda$, are called the two half-spaces corresponding to H. Two sets are called separated by H when one is contained in one half-space and the other in the other half-space; strictly separated when in addition neither meet H. The separation theorem can thus be restated: A closed convex set and a point not in it can be strictly separated by a closed hyperplane. The separation theorem only affirms what our geometric intuition readily tells us about the nature of convex sets.

Its converse also holds: let K be a set such that each point $p \in E \cap \complement K$ can be strictly separated from K by a closed hyperplane. Then K is a closed convex set. This is easy to see, for then K coincides with the intersection of all the closed half-spaces which contain K, and so K is an intersection of closed sets.

E being a locally convex space, the set E' of the continuous linear functionals on E can be regarded in a natural way as itself a vector space. It is called the dual. (There are various locally convex topologies which can be given to E'.) Because of the duality inherent in this situation, one writes $\langle x,y \rangle$ instead of $y(x)$ for the value of the functional $y \in E'$ on the point $x \in E$. If x is fixed while y is allowed to vary over E', one sees that x acts as a linear functional on E'. Or we can let both x and y vary, and then we have a bilinear form $(x,y) \quad \langle x,y \rangle$ on $E \times E'$.

Indeed, suppose that we are given two vector spaces E, F (without topology), coupled by a bilinear form $B: E \times F \longrightarrow R$. This coupling evidently makes each element of F act as a linear functional on E, and vice versa. The two spaces E and F are by virtue of this said to be "in duality".

Vector spaces E, F in duality automatically obtain locally convex topologies, called the <u>weak topologies</u> induced by the duality.[6] The weak topology on E induced by its duality with F is defined as the weakest topology rendering continuous all the elements of F. Similarly for F. It results that when E is given this topology, its dual is F. (and dually.)

In the case where we start out with a given topology on E, there arises the dual E' of E. E and E' being in duality, E gets a new topology, the weak topology induced by E'. It is weaker than the original topology on E (hence its name), and unless E is finite dimensional, these two topologies are distinct. However, as already noted, the dual of E for the new topology is still E'. Yet there is a remarkable fact: each convex set closed in E for the original topology is closed in E for the weak topology. This result is easily deduced from the separation theorem.

Similarly E' can be given the weak topology induced by E; this topology is sometimes called the "weak-* topology", and as noted, the dual of E' with this topology is E itself.

E and F being two vector spaces in duality, A a set $\subset E$, one calls the <u>positive polar</u>[7] and notes by $A^{\oplus}$, the set of $y \in F$ such that

$< x,y > \geq 0$ for each $x \in A$. And dually. A positive polar is always a convex cone. Let E be a locally convex space, considered as being in duality with its dual E'. A basic fact is that for each closed convex cone C in E, $C^{\oplus\oplus} = C$.

It is worth while to prove this. As a matter of fact, it follows easily from the separation theorem. If $p \in E \cap \complement C$, the separation theorem says that there exists $f \in E'$ and a number λ such that $f(p) < \lambda < f(C)$. f being linear, it is easy to see that f(C) is a cone in the real line. But there are not very many cones in the real line -- indeed as $\lambda < f(C)$, one must have either $f(C) = [0, \infty[$ or $f(C) = \{0\}$. In either case, $f \in C^{\oplus}$ and $< x,f > < 0$, which shows that $x \notin C^{\oplus\oplus}$, Q.E.D.

As a corollary one deduces the more general result: for each convex cone C in E, $C^{\oplus\oplus} = \bar{C}$ (closure of C). This results because $C^{\oplus} = (\bar{C})^{\oplus}$.

Footnotes

(1) This version is essentially that of L. Hurwicz, CCDP Econ. No. 2109 (theorem V. 3 pp. V. I - V.8).

(2) loc. cit.; it encompasses theorems VI.1 (pp. VI. 1 - VI.4), VI.3 (pp. VI.7 - VI.11), and VII.1 (pp. VII.1 - VII.4).

(3) Given in the finite-dimensional case by A. W. Tucker, "On systems of equated constraints satisfied by solutions of dual linear programs" (summary) O.N.R. Logistics Project, Fine Hall, Princeton University.

(4) see Bourbaki, Éléments de Mathematique, livre V: Espaces vectoriels topologiques.

(5) Bourbaki, loc. cit. (prop. 4, page 74.)

(6) J. Dieudonné, La dualité dans les espaces vectoriels topologiques. Ann. Sci. Ecole Norm. Sup. 59, 107-139 (1942).

(7) I believe this term is due to Tjalling C. Koopmans.