How to Find Optimal Admissible Summations Ways.[1]

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April 28, 1953

Consider a set x of numbers x_k, where the location index k ranges over some index set K, and a class of P of admissible subsets p of K. For each set p we study the sum

$$\beta(p) = \sum_{k \in p} x_k .$$

We shall try to find methods for determining optimal sets $\check{p} \in P$ or $\hat{p} \in P$ such that the sum

$$\beta(\check{p}) = \check{\beta} = \operatorname*{Min}_{p \in P} \beta(p)$$

and the sum

$$\beta(\hat{p}) = \hat{\beta} = \operatorname*{Max}_{p \in P} \beta(p)$$

The following theorems seem to be useful for this purpose. Formal proofs will here be omitted.

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[1] Research undertaken by the Cowles Commission for research in Economics under contract Nonr-358(01), NR 047-006 with the Office of Naval Research.

Theorem 1. If for every $p \in P$ a prescribed number N_n of terms in $\beta(p)$ must be located in the subset $K_n \subset K$, then the table $x' = x + z$, where $z_k = z_n$ for all $k \in K_n$, will have the same optimal sets as the table x. Furthermore if $x' = -x$, $\hat{p}' = \check{p}$, $\check{p}' = \hat{p}$.

By means of theorem 1 we can transform the problem to find an optimal admissible summations way to a problem to find a minimum sum, where $\underset{k \in K_n}{\text{Min}}\ x_k = 0$ for every n.

<u>Theorem</u> 2. Let B be a set of subsets $b \in K$ such that for every $b \in B$ there exists some $p \in P$ with $b \subset p$, then

$$\underset{p \in P}{\text{Min}}\ \beta(p) = \underset{b \in B}{\text{Min}}\ (\underset{\substack{p \supset b \\ p \in P}}{\text{Min}}\ \beta(p)) = \underset{b \subset B}{\text{Min}}\ (\beta_b + \underset{\substack{p \supset b \\ p \in P}}{\text{Min}}\ \beta_{p-b})$$

By means of theorem (2) the problem to find a admissible minimum path $\check{p}$ can be decomposed into a chain of simpler minimieum problems.

<u>Theorem</u> 3. Let $K(u)$ = the set of locations $k \subset K$ such that $x_k < u$. Let u_o be the smallest u such that there is some $p \subset K(u)$, $p \in P$. Call p_o any p corresponding to u_o. Then either $\beta(p_o) = \underset{p \in P}{\text{Min}}\ \beta(p)$ or $\beta(p_o) - \nu u_o \geq \beta_b$ for some $b \subset K(u_o)$, $b \subset p$ and $\nu \geq 0$, where ν denotes the number of terms that has to be added to β_b before we get a sum $\beta(p)$ for some $p \supset b$, $p \in P$. If no such b exists $\beta(p_o)$ is surely equal to the minimum sum $\check{\beta}$.

In the large class of problems where the admissible set P of the sets p is restricted only by the requirement that at least $\check{N}_n$ and at most $\hat{N}_n$ terms belongs to K_n and these subsets exhaust K without overlapping we always have $\beta(p_o) = \underset{p \in P}{\text{Min}}\ \beta(p)$. In more complicated cases it is usually easy to see if any b $\subset K(u_0)$ giving $\beta_b \leq \beta(p_0) = V_0$ exists at all and if some b exists how these sets have to be ex[illegible] so that we get a min $\beta(p)$ that perhaps is smaller than $\beta(p_0)$.

To give an example, which shows how these theorems can be used for solving an optimal summation way problem we will numerically solve a quadratic assignment problem.

The table x we assume to be as follows.

$x =$

j / i	ρ / λ	1				2				3				4			
		1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	1	1	-	-	-	o	o	o	o	-	o	o	o	-	o	o	o
	2	-	0	-	-	o	-	o	o	o	-	o	o	o	-	o	o
	3	-	-	2	-	o	o	-	o	o	o	-	o	o	o	-	o
	4	-	-	-	[3]	o	o	o	-	o	o	o	-	o	o	o	-
2	1	-	5	3	[2]	[0]	-	-	-	-	o	o	o	-	o	o	o
	2	3	-	4	1	-	6	o	-	o	-	o	-	o	o	o	-
	3	2	0	-	7	-	-	7	-	o	o	-	o	o	o	-	o
	4	1	9	6	-	-	-	-	8	o	o	o	-	o	o	o	-
3	1	-	7	8	9	-	0	1	4	9	-	-	-	-	o	o	o
	2	3	-	6	3	6	-	5	3	-	6	-	-	o	-	o	o
	3	1	6	-	[4]	[1]	6	-	7	-	-	[0]	o	o	o	-	o
	4	0	5	7	-	4	3	1	-	-	-	-	3	o	o		-
4	1	-	7	8	1	-	1	8	7	-	6	1	0	4	-	-	-
	2	0	-	6	[3]	[3]	-	3	5	7	-	[1]	5	-	[5]	-	-
	3	1	5	-	4	5	6	-	2	8	1	-	3	-	-	0	-
	4	7	8	9	-	7	0	1	-	6	7	8	-	-	-	-	2

$= \{x_{ij\lambda\rho}\}$

The problem is to find that permutation $p = (1 \rightarrow p(1),\ 2 \rightarrow p(2),\ 3 \rightarrow p(3),\ 4 \rightarrow p(4))$ which gives the minimisum.

$$\sum_{j=1}^{4} \sum_{i=1}^{4} x_{ij\check{p}(i)\check{p}(j)} = x_{\check{p}}$$

The "forbidden" numbers are denoted by ∞. To simplify the problem we assume that the numbers $x_{ij\lambda\rho} = x_{ij\lambda\rho} + x_{ji\rho\lambda}$, for $j < i$, and $x_{ij\lambda\rho} = 0$ for $j > i$. This simplification can be done because $x_{ij\lambda\rho}$ and $x_{ji\rho\lambda}$ will either not at all or both be included in x_p, so we can assume that this preliminary modification of the table x is already done. We also have subtracted the smallest term in each subtable x_{ij} from $x_{ij\lambda\rho}$, giving the zeros in the table: (Theorem 1). Because $x_{iip(i)p(i)}$ and $x_{jjp(j)p(j)}$ will be included in x_p if $x_{ijp(i)p(j)}$ is included we can further reduce the table by adding these terms to the corresponding terms in the subtables $(i = 2,\ j = 1)$ and $(i = 4,\ j = 3)$. The relevant information is now summarized in table x'

$x' =$

i		$j=1$				$j=2$				$j=3$			
1	1												
	2												
	3												
2	1	-	5	5	5								
	2	10	-	12	10								
	3	10	7	-	17								
	4	10	17	16	-								
3	1	-	7	8	9	-	9	1	4				
	2	3	-	6	3	6	-	5	3				
	3	1	6	-	4	1	5	-	7				
	4	0	5	7	-	4	3	1	-				
4	1	-	7	8	1	-	1	8	7	-	16	5	7
	2	0	-	6	3	3	-	3	5	15	-	6	13
	3	1	5	-	4	5	6	-	2	17	7	-	6
	4	7	8	9	-	7	0	7	-	17	15	10	-

Now we can again subtract the smallest term (5) in subtable ($i=2$, $j=1$) and the smallest term (5) in the subtable ($i=4$, $j=3$) from the other numbers in these subtables (Theorem 1). We get then the table x''.

$x'' =$

i		$j=1$				$j=2$				$j=3$			
1	1												
	2												
	3												
	4												
2	1	-	0	0	0								
	2	5	-	7	5								
	3	5	2	-	12								
	4	5	12	11	-								
3	1	-	7	8	9	-	0	1	4				
	2	3	-	6	3	6	-	5	3				
	3	1	6	-	4	1	5	-	7				
	4	0	5	7	-	4	3	1	-				
4	1	-	7	8	1	-	1	8	7	-	11	0	2
	2	0	-	6	3	3	-	3	5	10	-	1	8
	3	1	5	-	4	5	6	-	2	12	2	-	1
	4	7	8	9	-	7	0	1	-	12	10	5	-

The terms in the sum x_p which are admissible are six numbers from this table one from every subtable (2,1), (3,1), (4,1) and one from the subtables (3,2), (4,2), and one from (4,3). The six numbers are located in the intersections of lines reflected at the head diagonal. All intersection points must be admissible, that is, they cannot lie on a diagonal where the terms are (-).

If we now fill out a new table by taking the numbers from x'' in order of size and stop the process when we find an admissible summation way, we get the table:

$x''' =$

-	0	0	[0]								
∘	-	∘	∘								
∘	2	-	∘								
∘	∘	∘	-								
-	∘	∘	∘	-	0	1	4				
3	-	∘	3	∘	-	∘	3				
1	∘	-	[4]	[1]	∘	-	∘				
0	∘	∘	-	4	∘	∘	-				
-	∘	∘	1	-	1	∘	∘	-	∘	0	2
0	-	∘	[3]	[3]	-	3	∘	∘	-	[1]	∘
1	∘	-	4	∘	∘	-	2	∘	2	-	1
∘	∘	∘	-	∘	0	1	-	∘	∘	∘	-

In this table the numbers $x'''_{2114} = 0$, $x'''_{3134} = 4$, $x'''_{4124} = 3$, $x'''_{3231} = 1$, $x'''_{4221} = 3$, $x'''_{4323} = 1$ give a sum = 12. corresponding to the permutation $1 \rightarrow 4$, $2 \rightarrow 1$, $3 \rightarrow 3$, $4 \rightarrow 2$. This minimisum is so small that it hardly can be beaten if we let also terms larger than 4 be included in x_p.

To see if any other promising ways exist (Theorem 3) we can study which is the smallest admissible partial sum of at most five terms that can be found from x'''. These sum must be smaller or equal to $12 - 5 = 7$ if it contains 5 terms and $\leq 12 - 10 = 2$ if it contains 4 admissible terms if the corresponding sum shall be ≤ 12. We see without difficulty that no such partial sums exist. The permutation $p = (1 \rightarrow 4,\ 2 \rightarrow 1,\ 3 \rightarrow 3,\ 4 \rightarrow 2)$ gives thus the minimisum path. The corresponding minimisum in the table x is $\underline{12 + 10 = 22}$. In this case we did not need theorem 2 to solve the problem. By means of theorem 2 we can find the minimum sum, for instance, as follows:

Taking as the first term in x_p'' a term from the subtable $(i = 2,\ j = 1)$ we get:

$$x_p'' = \text{Min} \left\{ s_0,\ 2 + s_2,\ 5 + s_5,\ 7 + s_7,\ 11 + s_{11},\ 12 + s_{12} \right. = 12$$

$$s_0 = \text{Min} \left\{ \begin{array}{l} \beta_{01} = \text{Min}\ (\underline{4 + 1 + 3 + 3 + 1}\ ,\ 3 + 6 + 4 + 5 + 2) = 12 \\ \beta_{02} = \text{Min}\ (6 + 6 + 9 + \ldots\ ,\ 7 + 4 + 6 + \ldots\ ) > 17 \\ \beta_{03} = \text{Min}\ (6 + 1 + 8 + 7 + 5\ ,\ 5 + 4 + 5 + 5 + 1) > 17 \end{array} \right\} = 12$$

$$2 + s_2 = +2 + \beta_{21} = 2 + \text{Min}\ (7 + 1 + 8 + \ldots\ ,\ 5 + 1 + 7 + 8) > 17$$

$$5 + s_5 = 5 + \text{Min} \left\{ \begin{array}{l} \beta_{51} = \text{Min}\ (1 + 5 + 7 + \ldots\ ,\ \underline{0 + 3 + 1 + 6} + 1\ ) = 16 \\ \beta_{52} = \text{Min}\ (3 + 5 + 7 + \ldots\ ,\ 0 + 1 = 0 + 3 + 8\ ) = 17 \\ \beta_{53} = \text{Min}\ (\underline{3 + 3 + 1 + 2 +} 2\ ,\ 1 + 7 + 0 + 5 + 1\ ) = 16 \\ \beta_{54} = \text{Min}\ (9 + 0 + 4 + 6 + \ldots\ ,\ \underline{4 + 5 + 1 + 1 + 0}\ ) = 16 \end{array} \right.$$

$7 + s_7 = 7 + \beta_{71} = 7 + \text{Min}\ (8 + 0 + 9 + 0 + 12\ ,\ 7 + 3 + 8 + 1 + 2) = 28$

$11 + s_{11} = 11 + s_{11,1} = 11 + \text{Min}\ (8 + 4 + \qquad ,\ 6 + 3 \qquad) > 17$

$$12 + s_{12} = 12 + \text{Min} \begin{Bmatrix} \beta_{12,1} = (7 + \ldots \qquad , 6 + \qquad) \\ \beta_{12,2} = (9 + \ldots \qquad , 3 + 3 + \qquad) \end{Bmatrix} > 17$$

The minimum sum is thus $12 = 0 + 4 + 1 + 3 + 3 + 1$., There are in all three next smallest sums = 16.

<u>**Some concluding remarks about the case when the numbers x_k are subject to random variations.**</u>

If the numbers x_k in the table x are estimates or rounded numbers, it is not possible to say with certainty that the set of optimal admissible summation ways corresponding to the error free table $\bar{\bar{x}}$ is the same as for the approximative table x. In this case two sums $\beta(p)$ and $\beta(p')$ are not significantly different if the square of their difference is not at least for instance four times larger than the random variance $2N'\varepsilon^2$ due to estimation-errors. Here N' denotes the number of terms in $\beta(p)$ not identical with a term in $\beta(p')$ and ε^2 - the expected value of $(x_k - \bar{\bar{x}}_k)^2$ for admissible terms x_k in $\beta(p)$.

In many cases, when both the number N of terms in $\beta(p)$ and the number N_p of admissible summations ways is large the sums $\beta(p)$ are approximately normally distributed with the mean value

$$E\beta(p) = N \underset{\substack{k \in p \\ p \in P}}{E}\, x_k = N \cdot m$$

and a variance equal to

$$N\,\sigma^2 = N\,(\bar{\sigma}^{\,2} + \varepsilon^2)$$

where

$$\bar{\sigma}^{\,2} = \underset{k \subset p \in P}{E}\,(\bar{x}_k - m)^2\,, \quad \varepsilon^2 = E\,(x_k - \bar{x}_k)^2.$$

The smallest sum $\beta(\check{p}) = \check{\beta}_x$ is usually smaller than the corresponding sum $\check{\beta}_{\bar{x}}$. The order of size of $\check{\beta}_x$ and $\check{\beta}_{\bar{x}}$ is such that

$$\Phi\left(\frac{\check{\beta}_x - N\,m}{\sigma\sqrt{N}}\right) = \Phi\left(\frac{\check{\beta}_{\bar{x}} - N\,m}{\bar{\sigma}\sqrt{N}}\right) = \frac{1}{N_P + 1}$$

The difference $\check{\beta}_{\bar{x}} - \check{\beta}_x$ is thus approximately of the order of size

$$\check{\beta}_{\bar{x}} - \check{\beta}_x \approx (\sigma - \bar{\sigma})\sqrt{N}\,.\;\; \Phi^{-1}\left(\frac{1}{N_P + 1}\right)$$

For large N and N_P and nonnegligible small $\varepsilon/\bar{\sigma}$ we have to be satisfied with approximately optimal summation ways. The set of these ways contains usually so many different $p \in P$ that the probability that the way p_o mentioned in theorem 3 shall be one of them is very large. Usually we will have $\beta(p_o) < \check{\beta}_{\bar{x}}$.

APPENDIX to

COWLES COMMISSION DISCUSSION PAPER: MATHEMATICS NO. 422

How to Find Optimal Admissible Sums

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July 10, 1953

Theorem IV. If $s_p = a_p + b_p$; $p \in P$ is a set of real numbers and $\alpha_1 < \alpha_2 < \dots < \alpha_i < \dots$ denote the different numbers a_p ordered according to size and $a_p = \alpha_i$ for $p \in P_i$, $\beta(\alpha_i) = \underset{p \in P_i}{\text{Min}}\, b_p$, and $\beta_1 < \beta_2 < \dots < \beta_j < \dots$ denote coreespondingly the different numbers b_p ordered according to size and $b_p = \beta_j$ for $p \in P_j'$, and $\alpha(\beta_j) = \underset{p \in P_j'}{\text{Min}}\, a_p$, if the numbers k and k' are so large that the two sets $\underset{i \leq k}{\{}\alpha_i + \beta(\alpha_i)\}$ and $\underset{j \leq k'}{\{}\alpha(\beta_j) + \beta_j\}$ will have a common element $a_{p'} + b_{p'}$, then

$$(s_{\check{p}} = \underset{p \in P}{\text{Min}}\, s_p = \text{Min}\left(\underset{i \leq k}{\text{Min}} \left\{\alpha_i + \beta(\alpha_i)\right\},\ \underset{j \leq k'}{\text{Min}} \left\{\alpha(\beta_j) + \beta_j\right\}\right)$$

<u>Proof.</u> If the common element is $s_{p'} = a_{p'} + b_{p'}$ then

$$s_{\check{p}} = \underset{p \in P}{\text{Min}}\,(a_p + b_p) \equiv a_{\check{p}} + b_{\check{p}} \leq a_{p'} + b_{p'} .$$

If $a_{\check{p}} + b_{\check{p}} \leq a_{p'} + b_{p'}$ then either $a_{\check{p}} \leq a_{p'}$ or $b_{\check{p}} \leq b_{p'} \leq \beta_{k'}$. In the case $a_{\check{p}} < a_{p'} \leq \alpha_k$ the number s_p is included in the set $\underset{i \leq k}{\{}\alpha_i + \beta(\alpha_i)\}$, and if $b_{p}^{*} \leq b_{p'} \leq \beta_{k'}$ in the set $\{_{j<k'} \alpha(\beta_j) + \beta_j \}$. The Theorems II - III follow from Theorem IV as special cases.

This theorem can be used if we like to divide the work to find $\min_{p \in P} (a_p + b_p)$ between different persons. One can order the number a_p according to size, another the numbers b_p according to size, and a coordinator can calculate the corresponding $\alpha(\beta_j)$ and $\beta(\alpha_i)$ and give order to stop the calculation, when a common sum $a_{p'} + b_{p'}$ is found.