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Maximality ("efficiency") and Lagrangian saddle points

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1.1 By a <u>convex cone</u> (*) in an Abelian topological group G is meant a set $K \subseteq G$ with the properties (**)

(1) $g \in K$, (real) $\lambda \geqq 0$, $\lambda g \in G$ imply $\lambda g \in K$

(2) $g' \in K$, $g'' \in K$ imply $g' + g'' \in K$.

An ordering is then defined with $g' \geqq g''$ equivalent to $g' - g'' \in K$.(***) K is <u>pointed</u> if $g \in K$, $-g \in K$ imply $g = 0$.

1.2 Let X^+, Y^+, Z^+ be closed convex cones in the Abelian topological group X, Y, Z respectively, Y^+ being pointed. x^*, y^*, z^* will denote additive continuous functionals on the respective spaces. The terms "non-negative" and "strictly positive" applied to such functionals have been defined in [2].

1.3 f and g are functions from X to Y and Z respectively.

1.4 We define the <u>constraint set</u> in X

$$X^c \equiv \left\{ x: \ x \geqq 0 , \ g(x) \geqq 0 \right\};$$

its image

$$Y^c \equiv f(X^c)$$

will be called the constraint set in Y. ("In X" or "in Y" usually omitted.)

(*) See bottom of page.

(**) If m, n are integers and G an Abelian group, $\frac{m}{n} \cdot g' = g''$ means $m \cdot g' = n \cdot g''$ where $m \cdot g = g + g + \ldots + g$ (<u>m</u> terms).

(***) When G is a vector space, $\lambda g \in G$ necessarily if $G \in K$ hence K is a convex cone in the usual sense of the term.

(*) It remains to be verified whether K is convex in the sense of the definition in [1]. If not, it might be preferable to define a convex cone as a set (a) possessing the property (1) in 1.1 and (b) convex in the sense of the definition in [1].

1.5 We call $\bar{x}$ <u>maximal</u> in X if

$$\bar{x} \in X^c$$

and

$$x \in X^c, \quad f(x) \geqq f(\bar{x}) \quad \text{imply} \quad f(x) = f(\bar{x}).$$

If $\bar{x}$ is maximal and $\bar{y} = f(\bar{x})$, $\bar{y}$ is said to be maximal in Y. ("In X" or "in Y" usually omitted.)

1.6 The <u>Lagrangian</u> <u>expression</u> is defined as

(1.0) $$\emptyset(x, z^*; y^*) = y^* \left[f(x) \right] + z^* \left[g(x) \right].$$

(x_o, z^*_o) is a <u>non-negative saddle point</u> ("non-negative" often omitted) of $\emptyset(\cdot, \cdot; y^*_o)$ if (with y^*_o strictly positive and $z^*_o \geqq 0$)

(1.1) $$\emptyset(x, z^*_o; y^*_o) \leqq \emptyset(x_o, z^*_o; y^*_o) \quad \text{for all } x \geqq 0$$

and

(1.2) $$\emptyset(x_o, z^*_o; y^*_o) \leqq \emptyset(x_o, z^*; y^*_o) \quad \text{for all } z^* \geqq 0.$$

1.7 We shall call a group G <u>regular</u> (*) if, given a fixed g_o (and denoting by g^* additive continuous functionals on G),

(2) (2.1) $$g^*(g_o) \geqq 0 \quad \text{for all } g^* \geqq 0$$

implies

(2.2) $$g_o \geqq 0.$$

It is known that a Banach space is regular (cf. Krein and Rutman, Cor. 1.3, p. 16, [3]). I do not know what the situation is in more general cases. (**)

(*) This is a term coined for the present purpose and is not in general usage.

(**) It would seem that the validity of the theorem asserting the existence of a hyperplane bounding a convex set with interior and going through a given non-interior point (cf. [4]) is sufficient for our "regularity."

1.8 (x_o, z^*_o) is a <u>non-negative quasi-saddle point</u> ("non-negative" often omitted) of $\phi(\cdot, \cdot; y^*_o)$ if (with y^*_o strictly positive and $z^*_o \geqq 0$)

(1.1) $$\phi(x, z^*_o; y^*_o) \leqq \phi(x_o, z^*_o; y^*_o) \quad \text{for all } x \geqq 0$$

and

(1.2') $$z^*_o \left[g(x_o) \right] = 0 .$$

2.1 <u>Lemma</u>. A quasi-saddle point is necessarily a saddle point. A saddle point is a quasi-saddle point if Z is regular. Proof given elsewhere.

2.2 In view of the preceding Lemma we find it convenient to deal with quasi-saddle points; this makes for simpler formulation since it is then unnecessary to distinguish cases according as to whether Z is regular.

3. <u>Theorem 1</u>. If (x_o, z^*_o) is a quasi-saddle point of $\phi(\cdot, \cdot; y^*_o)$, then x_o is maximal. (Note that, by definition, y^*_o is strictly positive and $z^*_o \geqq 0$.) Proof given elsewhere.

4.1 The question now arises as to conditions under which, given that $\bar{x}$ is maximal, there exist y^*_o strictly positive and $z^*_o \geqq 0$ such that the Lagrangian expression $\phi(\cdot, \cdot; y^*_o)$ has a quasi-saddle point at $(\bar{x}, z^*_o)$.

4.2 Consider the "direct product" space

(3.1) $$W \equiv Y \times Z \quad (w = (y, z))$$

and let

(3.2) $$h(x) \equiv (f(x), g(x)),$$

(3.3) $$w^*(w) \equiv (y^*(y), z^*(z)) .$$

Then the Lagrangian expression may be written as

(4.1) $$\phi(x, z^*; y^*) = w^* \left[h(x) \right]$$

and (1.1) becomes

(4.2) $$w_o^* \left[h(x) \right] \leqq w_o^* \left[h(x_o) \right] \quad \text{for all } x \geqq 0 ,$$

where, for $w = (y, z)$,

(4.3) $$w_o^*(w) \equiv (y_o^*(z), z_o^*(z)) .$$

Furthermore, with the usual conventions concerning ordering,

(5.1) $$w_o^* \geqq 0$$

and

(5.2) w_o^* is strictly positive on $\left\{ w: w = (y, 0), \; y \in Y^+ \right\}$.

As for the condition (1.2'), it may be written as

(6) $$w_o^*(w) = 0 \quad \text{for } w = (0, g(x_o)) .$$

4.3 Let us write

(7.1) $$\bar{w} = h(\bar{x}) .$$

If we let $f(\bar{x}) = 0$ (for the sake of simplicity), we have

(7.2) $$\bar{w} = (0, \bar{z}) , \quad \bar{z} \geqq 0 ,$$

Also, define the following sets:

(8.1) $$h(X^+) \equiv \left\{ w: w = h(x), \; x \geqq 0 \right\}$$

(8.2) $$W^+ \equiv \left\{ w: w \geqq 0 \right\}$$

(8.3) $$W_y^+ \equiv \left\{ w: w = (y, 0), \; y \in Y^+ \right\} .$$

Then, in the light of 4.1, the proposition of 4.1 can be stated as requiring that there exist a w_o^* with the following properties:

(9.1) $$w_o^*(w) \leqq 0 \quad \text{for } w \in h(X^+),$$

(9.2) $$w_o^*(w) \geqq 0 \quad \text{for } w \in W^+ ,$$

(9.3) w_o^* strictly positive on W_y^+

(9.4) $$w_o^*(\bar{w}) = 0 .$$

Now let

(10) $$W_1 \equiv \left\{ w: -w \in h(X^+) \right\} .$$

Then the requirements of (9) may be stated geometrically as requiring that there exist a hyperplane, say H_o, through $\bar{w}$, bounding $W_1 \cup W^+$ and not intersecting W_y^+.

Sufficient conditions for the existence of an H_o bounding $W_1 \cup W^+$ and going through $\bar{w}$ are well known if W is a linear topological space. Essentially it is then required that $\bar{w}$ should not belong to the interior, if any, of the convex hull C of $W_1 \cup W^+$ and that W be finite-dimensional or C have interior. (*) Suppose it is assumed that W is finite-dimensional or that Y^+, Z^+ (hence W^+ and also C) have interiors.

That $\bar{w} \notin C$ can be shown to follow from (7) and the definition of maximality if $h(x)$ is assumed concave.

However, stronger assumptions are required to satisfy (9.3). This is not surprising since, even in Banach spaces, a strictly positive functional may not exist for every closed convex cone. The separability of Y is a sufficient condition that such a functional should exist in Y. But additional restrictions on the set W_1 are necessary as can be shown by examples.

When we deal with groups other than linear spaces, there is a need for a generalization of the bounding hyperplane theorem, but I do not know whether any work has been done in this direction. Of course, some additional restriction on the nature of the group might be required. (**)

(*) w_o^* characterizing H_o will be continuous in whatever topology the term "interior" is interpreted.

(**) One line of approach might be through "embedding" groups in linear spaces. On the other hand I have the (very superficial !) impression that the proof given by Dieudonné for linear spaces ([4]) might be suitable as a starting point for generalization to additive groups.

References

[1] L. Hurwicz Decentralized Resource Allocation, CCDP Econ. No. 2070, p. 24.

[2] L. Hurwicz Lagrangian Saddle Points in Banach Spaces (Summary of Results), CCDP Math. No. 418, p. 1.

[3] See Reference 4 in CCDP 415

[4] J. Dieudonné Sur le Théorème de Hahn-Banach, Revue Scientifique (Revue Rose Illustrée), 79e Année, 1941.