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Nonnegative Square Matrices

Gerard Debreu and I.N. Herstein

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Square matrices, all of whose elements are nonnegative, have played an important role in the probabilistic theory of finite Markov chains (See [4] and the references there given) and, more recently, in the study of linear models in economics [2], [3], [8], [9], [12] to [17] and [20].

The properties of such matrices were first investigated by Perron [18], [19], and then very thoroughly by Frobenius [5], [6], [7]. Lately Wielandt [22] has given notably more simple proofs for the results of Frobenius.

In Section 1 we study nonnegative indecomposable matrices from a different point of view (that of the Brouwer fixed point theorem); a concise proof of their basic properties is thus obtained. In Section 2 properties of a general nonnegative square matrix $A$ are derived from those of nonnegative indecomposable matrices. In Section 3 theorems about the matrix $A1-A$ are proved; they cover in a unified manner a number of results recurrently used in economics. In Section 4 a systematic study of the convergence of $A^p$ when $p$ tends to infinity ($A$ is a general complex matrix) is linked to combinatorial properties of nonnegative square matrices.

Unless otherwise specified, all matrices considered will have real elements. We define for $A = (a_{ij})$, $B = (b_{ij})$: 
$A \preceq B$ if $a_{ij} \preceq b_{ij}$ for all $i, j$

$A \preceq B$ if $A \preceq B$ and $A \neq B$

$A < B$ if $a_{ij} < b_{ij}$ for all $ij$.

Primed letters denote transposes.

When $A$ is an $n \times n$ matrix, $A_T = TAT^{-1}$ denotes the transform of $A$ by the nonsingular $n \times n$ matrix $T$.

1. **Nonnegative Indecomposable Matrices**

An $n \times n$ matrix $A$ ($n \geq 2$) is said to be indecomposable if for no permutation matrix $\pi$, does $A^2 = \pi A \pi^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ where $A_{11}, A_{22}$ are square.

**Theorem I.** Let $A \geq 0$ be indecomposable. Then

1. $A$ has a characteristic root $r > 0$ such that
2. to $r$ can be associated an eigen-vector $x_o > 0$
3. if $\alpha$ is any characteristic root of $A$, $|\alpha| \leq r$
4. $r$ increases when any element of $A$ increases
5. $r$ is a simple root.

**Proof.** 1. a) If $x \geq 0$, then $Ax \geq 0$. For if $Ax = 0$ $A$ would have a column of zeros, and so would not be indecomposable.

1. b) $A$ has a characteristic root $r > 0$.

Let $S = \{x \in \mathbb{R}^n | x \geq 0, \Sigma x_1 = 1\}$ be the fundamental simplex in the Euclidean $n$-space, $\mathbb{R}^n$. If $x \in S$, we define $T(x) = \frac{1}{\rho(x)} Ax$ where $\rho(x) > 0$ is so determined that $T(x) \in S$ (by 1.a) such a $\rho$ exists for every $x \in S$). Clearly $T(x)$ is a continuous transformation of $S$ into itself, so, by the Brouwer fixed-point theorem (see for example [11]), there is an $x_o \in S$ with $x_o = T(x_o) = \frac{1}{\rho(x_o)} Ax_o$. Put $r = \rho(x_o)$.
2. \( x_0 > 0 \). Suppose that after applying a proper \( \mathcal{P} \), \( \dot{x}_0 = \left( \begin{array}{c} \xi \\ \xi \end{array} \right) \), \( \xi > 0 \).

Partition \( A \mathcal{P} \) accordingly. \( A \mathcal{P} \xi = r \xi \) yields

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
\xi \\
0
\end{pmatrix}
= 
\begin{pmatrix}
r \\
0
\end{pmatrix},
\]

thus \( A_{21} \xi = 0 \), so \( A_{21} = 0 \), violating the indecomposability of \( A \).

If \( M = (m_{ij}) \) is a matrix, we henceforth denote by \( M^* \) the matrix \( M^* = (|m_{ij}|) \).

3 - 4. If \( 0 \leq B \preceq A \), and if \( \beta \) is a characteristic root of \( B \), then \( |\beta| \leq r \). Moreover \( |\beta| = r \) implies \( B = A \).

\( A^* \) is indecomposable and therefore has a characteristic root \( r_1 > 0 \)

with an eigen-vector \( x_1 > 0 \): \( A^* x_1 = r_1 x_1 \). Moreover \( \beta y = By \). Taking absolute values and using the triangle inequality, we obtain

\[
\begin{align*}
(1) & \quad |\beta| y^* = By^* \leq Ay^*. \quad \text{So} \\
(2) & \quad |\beta| x_1^* y^* = x_1^* Ay^* = r_1 x_1^* y^*. \\
& \text{Since } x_1 > 0, x_1^* y^* > 0, \text{ thus } |\beta| \leq r_1.
\end{align*}
\]

Putting \( B = A \) one obtains \( |\beta| \leq r_1 \). In particular \( r = r_1 \) and since, similarly, \( r_1 \leq r, r_1 \) is equal to \( r \).

Going back to the comparison of \( B \) and \( A \) and assuming that \( |\beta| = r \) one gets from (1) and (2)

\[
ry^* = By^* = Ay^*.
\]

From \( ry^* = Ay^* \), application of 2 gives \( y^* > 0 \).

Thus \( By^* = Ay^* \) together with \( B \preceq A \) yields \( B = A \).

5. a) If \( B \) is a principal submatrix of \( A \) and \( \beta \) a characteristic root

of \( B, |\beta| < r \).

\( \beta \) is also a characteristic root of the n-n matrix \( B = \begin{pmatrix} B & 0 \\
0 & 0 \end{pmatrix} \). Since \( A \)

is indecomposable, \( B \preceq A \mathcal{P} \) for a proper \( \mathcal{P} \) and \( |\beta| < r \) (by 3 - 4).
5. b) \( r \) is a simple root of \( \Phi(t) = \det(tI-A) = 0 \).
\( \Phi'(r) \) is the sum of the principal \((n-1) \times (n-1)\) minors of \( \det(tI-A) \).
Let \( A_1 \) be one of the principal \((n-1) \times (n-1)\) submatrices of \( A \). By 5.a)
\( \det(tI-A_1) \) cannot vanish for \( t = r \), whence \( \det(tI-A_1) > 0 \) and \( \Phi'(r) > 0 \).

With a proof practically identical to that of 3 - 4, one obtains the
more general result:

If \( B \) is a complex matrix such that \( B^* = A \), \( A \) indecomposable, and if \( \beta \)
is a characteristic root of \( B \), then \( |\beta|^2 = r \). Moreover \( |\beta| = r \) implies \( B^* = A \).

More precisely if \( \beta = re^{i\psi} \), \( B = e^{i\psi} D A D^{-1} \) where \( D \) is a diagonal matrix
such that \( D^* = I \). A proof of this last fact is given in ([22] p. 646, lines
4 - 11).

From this can be derived

**Theorem II.** Let \( A \neq 0 \) be indecomposable. If the characteristic equation
\( \det(tI-A) = 0 \) has altogether \( k \) roots of absolute value \( r \), the set of \( n \) roots
(with their orders of multiplicity) is invariant under a rotation about the
origin through an angle of \( \frac{2\pi}{k} \), but not under rotations through smaller
angles. Moreover there is a permutation matrix \( \Pi \) such that

\[
\begin{align*}
\Pi A \Pi^{-1} &= \\
0 & A_{12} & 0 & \ldots & 0 \\
0 & 0 & A_{23} & \ldots & 0 \\
& \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & A_{k-1,k} \\
A_{k1} & 0 & 0 & \ldots & 0
\end{align*}
\]

with square submatrices on

the diagonal.

Again the reader is referred to the excellent proof of Wielandt [22,
p. 646 - 647].

If \( k = 1 \), the indecomposable matrix \( A = 0 \) is said to be primitive.
2. Nonnegative Square Matrices

If $A$ is an $n \times n$ matrix, there clearly exists a permutation matrix $\Pi$ such that

$$
\Pi A \Pi^{-1} = \begin{bmatrix}
A_1 & * \\
0 & A_2 \\
0 & 0 & \ddots \\
0 & 0 & \ddots & A_H
\end{bmatrix}
$$

where the $A_h$ are square submatrices on the diagonal and every $A_h$ is either indecomposable or a $1 \times 1$ matrix.

The properties of $A$ will therefore be easily derived from those of the $A_h$. For example, $\det(tI - A) = \prod_{h=1}^{H} \det(tI - A_h)$ and Theorem I gives

Theorem I*. If $A \geq 0$ is a square matrix, then

1. $A$ has a characteristic root $r \geq 0$ such that
2. to $r$ can be associated an eigen-vector $x_o \geq 0$
3. if $\alpha$ is any characteristic root of $A, |\alpha| \leq r$
4. $r$ does not decrease when an element of $A$ increases.

Let $r_h$ be the maximal nonnegative characteristic root of $A_h$, we take

$$
r = \max r_h; 1 - 3 - 4$$

are then immediate. To prove 2 we consider a sequence $A_\varepsilon$ of n-n matrices converging to $A$ such that for all $\varepsilon$, $A_\varepsilon > 0$. Let $r_\varepsilon$ be the maximal positive characteristic root of $A_\varepsilon$, $x_\varepsilon > 0$ its associated eigen-vector so chosen that $x_\varepsilon \in S$, the fundamental simplex of $R^n$. Clearly $r_\varepsilon$ tends to $r$. Let us then select $x_o \in S$ a limit point of the set $(x_\varepsilon)$; thus there is a subsequence $x_{\varepsilon'}$ converging to $x_o \geq 0$ and for every $\varepsilon'$,

$$
A_{\varepsilon'} x_{\varepsilon'} = r_{\varepsilon'} x_{\varepsilon'}, \text{ therefore } A x_o = r x_o.
$$

5 of Theorem I no longer holds, but 5.a) becomes:

If $B$ is a principal submatrix of $A$ and $\beta$ a characteristic root of $B, |\beta| \leq r$.

The proof, almost identical, now rests on 4 of Theorem I*. 5/
3. Properties of \( sI-A \) for \( s > r \).

In this section \( A \neq 0 \) is an n x n matrix, \( r \) is its maximal nonnegative characteristic root.

**Lemma**\(^*\): If for an \( x > 0 \), \( Ax \preceq sx \) (resp. \( x \preceq s \)), then \( r \preceq s \) (resp. \( r = s \)).

If for an \( x = 0 \), \( Ax < sx \) (resp. \( x < s \)), then \( r < s \) (resp. \( r = s \)).

The proofs of the four statements being practically identical, we present only that of the first one. Let \( x_0 \geq 0 \) be a characteristic vector of \( A' \) associated with \( r \) (2 of Theorem I\(^*\)) \( A'x_0 = rx_0 \). \( Ax \preceq sx \) with \( x \geq 0 \), therefore \( x_0A'x_0 \preceq sx_0'x_0 \) i.e., \( rx_0'x_0 \preceq sx_0'x_0 \) and, since \( x_0'x_0 > 0 \), \( r \leq s \).

We now derive two theorems (III\(^*\) and III) from the study of the equation

\[
(sI-A)x = y
\]

**Theorem III\(^*\).** \( (sI-A)^{-1} \geq 0 \) if and only if \( s > r \).

**Sufficiency.** Since \( s > r \) (2) has a unique solution \( x = (sI-A)^{-1}y \) for every \( y \); we show that \( y \geq 0 \) implies \( x \geq 0 \).

If \( x \) had negative components (2) could be given the form [by proper (identical) permutations of the rows and columns and partition]

\[
\begin{bmatrix}
  sI-A_1 & A_{12} \\
  A_{21} & sI-A_2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = y
\]

where \( x_1 > 0 \), \( x_2 \geq 0 \), \( y \geq 0 \). Therefore \(-(sI-A_1)x_1 - A_{12}x_2 \geq 0 \)

i.e., \(-(sI-A_1)x_1 \geq 0 \) i.e., \( A_1x_1 \geq sx_1 \). From the Lemma\(^*\) the maximal nonnegative characteristic root of \( A_1, r_1 \geq s \), a contradiction to the fact that \( r \leq r_1 \). (See end of Section 2) and \( s > r \).

**Necessity.** Since \( (sI-A)^{-1} \geq 0 \) to a \( y \geq 0 \) corresponds an \( x \geq 0 \). Therefore from \( sx - Ax = y \) follows \( Ax < sx \) and, by the Lemma\(^*\), \( r < s \).

If \( A \) is indecomposable these results can be sharpened to the

**Lemma:** Let \( A \) be indecomposable
if for an \( x \geq 0 \), \( Ax \leq x \) (resp. \( \geq \)), then \( r \leq s \) (resp. \( \geq \)).

The proofs, practically identical to those of the Lemma⁵, use a positive characteristic vector of \( A \) associated with \( r \). One of these statements indeed has already been proved in 3-4 of Theorem I.

**Theorem III.** Let \( A \) be indecomposable. \((sI-A)^{-1} > 0\) if and only if \( s > r \).

**Sufficiency.** We show that \( y \geq 0 \) implies \( x \geq 0 \). It is already known (from the proof of sufficiency of Theorem III⁵) that \( x \geq 0 \). If \( x \) had zero components, (2) could be given the form

\[
\begin{bmatrix}
  sI-A_1 & -A_{12} \\
  -A_{21} & sI-A_2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = y
\]

where \( x_1 = 0 \), \( x_2 > 0 \), \( y \geq 0 \). Therefore \(-A_{12} x_2 \geq 0\), and, since \( x_2 > 0 \),

\( A_{12} = 0 \) violating the indecomposability of \( A \).

The Necessity has already been proved since \((sI-A)^{-1} > 0\) implies \((sI-A)^{-1} \geq 0 \).⁶

**Theorem IV.** The principal minors of \( sI-A \) of orders 1, ..., \( n \) are all positive if and only if \( s > r \).

**Sufficiency.** \( \det (sI-A) \) cannot vanish for \( t > r \), thus \( \det (sI-A) > 0 \) for \( s > r \). Similarly, the maximal nonnegative characteristic root of a principal submatrix of \( A \) is not larger than \( r \) (see end of Section 2), it is therefore smaller than \( s \), and the corresponding minor of \( sI-A \) is positive.

**Necessity.** The derivative of order \( m \) (< \( n \)) of \( \det (sI-A) \) with respect to \( t \), for \( t = s \), is a sum of principal minors of order \((n-m)(n-m)\) of \( sI-A \) and thus is positive. As its derivatives of all orders \((0, 1, ..., n-1, n)\) are positive for \( t = s \), the polynomial \( \det (sI-A) \) can vanish for no \( t \neq s \) i.e., \( s > r \).⁷

Since a square matrix with nonpositive (resp. negative) off-diagonal elements can always be given the form \( sI-A \) where \( A \geq 0 \) (resp. \( \leq 0 \)), results
such as those of Chipman [2], [3], Goodwin [8], Hawkins and Simon [9], Metzler [12] to [15], Morishima [16], Mosak [17], Solow [20] are all contained in the above.


**Theorem V.** Let $A$ be a n×n complex matrix. The sequence $A, A^2, \ldots, A^p, \ldots$ of its powers converges if and only if

1. any characteristic root $\lambda$ of $A$ satisfies either $|\lambda| < 1$ or $\lambda = 1$

2. when the second case occurs the order of multiplicity of the root $1$ equals the dimension of the eigen-vector space associated with that root.

There is a nonsingular complex matrix $T$ such that

$$A_T = TAT^{-1} = \begin{bmatrix}
J_1 & 0 & \cdots & 0 \\
0 & J_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & J_q
\end{bmatrix}$$

where $J_1 = \begin{bmatrix}
\alpha_1 & 1 & 0 & \cdots & 0 \\
0 & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \alpha_1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & \alpha_1
\end{bmatrix}$

is a square matrix on the diagonal and $\alpha_1$ a characteristic root of $A$. To every root $\alpha_1$ corresponds at least one $J_1$. (For this reduction of $A$ to its Jordan canonical form see for example [21]).

Since

$$TAP_T^{-1} = \begin{bmatrix}
J_1^p & 0 & \cdots & 0 \\
0 & J_1^p & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & J_q^p
\end{bmatrix}, \quad A^p \text{ converges}
$$

if and only if every one of the $J_1^p$ converges. Let us therefore study one of them; for this purpose we drop the subscripts 1 and $\xi$. 
$J$ is a $k$-$k$ matrix of the form $J = \alpha I + M$ where $M = (m_{st})$:

\[
m_{st} = 1 \text{ if } t = s + 1, \quad m_{st} = 0 \text{ otherwise.}
\]

\[
J^p = \alpha P I + P \frac{P-1}{1} M + \ldots + P \frac{p-k+1}{k-1} M^{k-1}
\]

It is easily seen that for $M^h$, $m_{st}^{(h)} = 1$ if $t = s + h$ and $m_{st}^{(h)} = 0$ otherwise. Thus $M^h = 0$ if $h \geq k$; also the nonzero elements of $M^h$ and $M^{h^*}$ ($h \neq h^*$) never occur in the same place so $J^p$ converges if and only if every term of the right-hand sum does.

The first term shows that necessarily either $|\alpha| < 1$ or $\alpha = 1$.

If $|\alpha| < 1$, every term tends to zero and $J^p$ converges.

If $\alpha = 1$ no term other than the first one converges and necessarily $k = 1$ i.e., $J = (1)$; clearly $J^p$ converges in this case.

We wish however to obtain for this necessary and sufficient condition of convergence an expression independent of a reduction to Jordan canonical form.

Consider then an arbitrary $n$-$n$ complex matrix $A$ and let $\mathcal{G}$ be the set of $i$ for which $J_i$ corresponds to the root $1$. The equation $A x = x$, in which $x$ is partitioned in the same way as $A$, yields $J_i x_i = x_i$ for all $i$ i.e.,

if $i \notin \mathcal{G}$, $x_i = 0$

if $i \in \mathcal{G}$ all components of $x_i$ but the first one equal zero.

Thus the dimension of the eigenvector space associated with the root 1 equals the number of elements of $\mathcal{G}$. This number, in turn, equals the order of multiplicity of the root 1 if and only if $J_i = [1]$ for all $i \in \mathcal{G}$.

We now assume that the limit $C$ exists and give its expression. If 1 is not a characteristic root of $A$, $C = 0$. Let therefore 1 be a root of $A$ of order $\mu$. Thus $x$ (resp. $y$), an eigenvector of $A$ (resp. $A^*$) associated with the root 1, has the form $x = X \xi$ (resp. $y = Y \eta$) where $X$ (resp. $Y$) is a
matrix of rank \( \mu \) and \( \xi \) (resp. \( \eta \)) is a \( \mu \cdot 1 \) matrix. For an arbitrary \( x \) the relation \( A^p x = A^{p-1} x \) gives in the limit \( A^\infty x = Cx \) i.e., \( Cx = X \xi(x) \).

To determine \( \xi(x) \) we remark that \( Y^p = Y^p A \) i.e., by iteration \( Y^p = Y^{p-1} A \), and therefore \( Y^p = Y^p C \); thus \( Y^p x = Y^p C x = Y^p X \xi(x) \). \( Y^p X \) is a nonsingular matrix i.e., \( \xi(x) = (Y^p X)^{-1} Y^p x \). Finally for all \( x \), \( Cx = X(Y^p X)^{-1} Y^p x \) i.e., \( C = X(Y^p X)^{-1} Y^p \).

Corollary. Let \( A \geq 0 \) be indecomposable and \( \lambda \) be its maximal positive characteristic root. The sequence \( A^p \) converges if and only if \( A \) is primitive.

The necessity is obvious. The sufficiency follows from the fact that \( \lambda \) is a simple root.

Let then \( y_0 > 0 \) (resp. \( x_0 > 0 \)) be an eigen-vector of \( A \) (resp. \( A' \)) associated with the root \( \lambda \), the limit \( C \) of \( A^p \) has the simple expression

\[
C = \frac{x_0 y_0'}{y_0' x_0}.
\]

Clearly \( C > 0 \), thus if the indecomposable matrix \( A \geq 0 \) is primitive, there is a positive integer \( m \) such that \( A^p > 0 \) when \( p \geq m \). The converse is an immediate consequence of the decomposition (1) of Theorem II.
1. This paper is a result of the work being done at the Cowles Commission for Research in Economics on the "Theory of Resource Allocation" under sub- contract to The RAND Corporation. Based on Cowles Commission Discussion Paper, Mathematics No. 411, February, 1952.

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2. In any row or column of a permutation matrix one element equals 1, the others equal 0. \( \Pi A \Pi^{-1} \) is obtained by performing the same permutation on the rows and on the columns of \( A \).

3. As an immediate consequence of 4 one obtains:

\[
\min_{i,j} \sum a_{ij} = r = \max_{i,j} \sum a_{ij}
\]

and one equality holds only if all row sums are equal (then they both hold).

This is proved by increasing (resp. decreasing) some elements of \( A \) so as to make all row sums equal to \( \max_{i,j} \sum a_{ij} \) (resp. \( \min_{i,j} \sum a_{ij} \)). A similar result naturally holds for column sums.

4. Decomposition (1) can indeed be completely characterized.

Lemma. Let \( A \) be a square complex matrix such that \( \Pi A \Pi^{-1} \) has form (1) and let \( B_1 = A_{1,1} x_1 \cdots x_{k-1} k \cdots x_{1-1}, k \). For \( \alpha \neq 0 \) to be a characteristic root of \( A \) it is necessary (resp. sufficient) that \( \alpha^k \) be a characteristic root of every (resp. one) \( B_1 \).

After proper partition of \( x \), an eigen-vector of \( A_{11} \) associated with the root \( \alpha \), the equation \( A_{11} x = \alpha x \) becomes

\[
(1') A_{1,1+1} x_{1+1} = \alpha x_1 \quad (i=1, \ldots, k) \quad \text{which implies}
\]

\[
(1'') B_1 x_1 = \alpha^k x_1.
\]

Since no \( x_i \) can vanish (by \( 1' \) they all would), \( \alpha^k \) is a characteristic root of every \( B_1 \).

Conversely let \( \alpha^k \) be a characteristic root of \( B_1 \) and \( x_1 \) an associated eigen-vector, we construct a vector \( x \), whose \( i \)-th component is \( x_1 \), and such that \( A_{11} x = \alpha x \). The \((i-1)\)-th equation (1') determines \( x_{i-1} \); \( x_{i-2}, \ldots, x_1, x_k, \ldots, x_{i+2} x_{i+1} \) are determined in turn; the \( i \)-th equation is redundant because of \( 1'' \).
An immediate consequence of the lemma one finds:

**Theorem.** Let $A \succeq 0$ be indecomposable. For $A$ to have exactly $k$ characteristic roots of maximum absolute value $r$ it is necessary (resp. sufficient) that $A$ can be brought to form (1) where every (resp. one) $B_i$ has no other characteristic root of maximum absolute value $s_i$ than $s_i$ itself.

Naturally $s_i = r^k$ for every $i$.

5. A stochastic non matrix $P$ is defined by $p_{ij} \geq 0$ for all $i, j$ and $\sum_j p_{ij} = 1$ for all $i$. Clearly 1 is a characteristic root of $P$ (take an eigenvector with all components equal). 1 is therefore a root of some of the indecomposable matrices $P_1, P_2, \ldots, P_n$. Suppose that 1 is a root of $P_n$, it follows from Footnote (3) that all row sums of $P_n$ are equal to 1, i.e.,

$$\Pi_{P_n}^{-1} = P_1 P_2 \cdots P_{n-1}$$

This simple remark makes properties of stochastic matrices (the subject of the theory of finite Markov chains; see [14] and its references) ready consequences of the results of this article.

6. It is worth [9] emphasizing a result obtained in the proof of necessity of Theorem III*.

**Remark.** Let $A \succeq 0$ (resp. $A \succeq 0$ indecomposable) be a square matrix. If for $a y > 0$ (resp. $y \geq 0$), $x \geq 0$, then $(sI-A)^{-1} \succeq 0$ [resp. $(sI-A)^{-1} > 0$].

The proof of indecomposable matrices uses the Lemma instead of the Lemma*.

7. We give a last property useful in economics [14], [15].

**Theorem.** Let $A > 0$ be a square matrix and let $c_{ij}$ be the cofactor of the $i$th row, $j$th column element of $sI-A$. If $s \geq \sum_j a_{ij}$ for all $i$, then $i \neq j$ implies $c_{ii} > c_{ij}$.

Let us define the matrix $B = (b_{pq})$ as follows:

$$b_{pq} = a_{pq} \text{ if } p \neq i; \quad b_{iq} = 0 \text{ if } i \neq q \neq j; \quad b_{ii} = \frac{s}{2} = b_{ij}.$$  

$B$ is indecomposable, moreover $\sum q b_{iq} = s$, $\sum p b_{pq} < s$ for $p \neq i$.

Therefore (see Footnote 3) the maximal positive characteristic root of $B$, $\lambda(B) < s$. Thus det $(sI-B) > 0$; a development according to the $i$th row yields:

$$\frac{s}{2} c_{ii} - \frac{s}{2} c_{ij} > 0.$$  

(ii)
8. Morishima studies square matrices A such that for a permutation matrix \( \Pi \),

\[
\Pi A \Pi^{-1} = A_{\Pi} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\]

where \( A_{11} \geq 0 \) and \( A_{22} \geq 0 \) are square,

\( A_{12} \leq 0 \), \( A_{21} \leq 0 \). The relation

\[
\begin{bmatrix}
I & 0 \\
0 & -I
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & -I
\end{bmatrix}
= \begin{bmatrix}
A_{11} & -A_{12} \\
-A_{21} & A_{22}
\end{bmatrix}
\]

shows how properties of \( A_{\Pi} \) can be immediately derived from those of the nonnegative matrix

\[
A_{\Pi}^S = \begin{bmatrix}
A_{11} & -A_{12} \\
-A_{21} & A_{22}
\end{bmatrix}
\]

In particular \( A_{\Pi} \) and \( A_{\Pi}^S \) have the same characteristic roots.

9. The Cesaro convergence of \( A^P \), i.e., the convergence of \( \frac{1}{p} (A + A^2 + \ldots + A^P) \) can be studied in exactly the same fashion.

10. \( X_T = T X \) (resp. \( Y^T = Y^T T^{-1} \)) plays for \( A_T \) the same role as \( X \) (resp. \( Y \)) does for \( A \). Moreover \( Y^TX = Y^TX_T \). The right-hand matrix is nonsingular for the form taken by the Jordan matrix \( A_T \) in the convergence case implies that the eigen-vector space \( U \) generated by \( X_T \) is identical with the eigen-vector space \( V \) generated by \( Y_T \). Thus \( Y^TX_T \xi = 0 \) implies \( X_T \xi = 0 \) (there is no vector different from zero in \( U \) perpendicular to \( V \), i.e., to \( U \)) therefore \( \xi = 0 \) since the rank of \( X_T \) is \( \mu \).

11. This characterization of a primitive matrix is typical of the purely combinatorial properties of the nonnegative square matrix \( A \) (used for example in the theory of communication networks): the smallest \( m \) satisfying the above condition is independent of the values of the nonzero elements of \( A \) as long as they stay positive.

The development of combinatorial techniques adapted to the treatment of such properties is the subject of [10].
References


