# Nonnegative Square Matrices[1]

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Square matrices, all of whose elements are nonnegative, have played an important role in the probabilistic theory of finite Markov chains (See [4] and the references there given) and, more recently, in the study of linear models in economics [2], [3], [8], [9], [12] to [17] and [20].

The properties of such matrices were first investigated by Perron [18], [19], and then very thoroughly by Frobenius [5], [6], [7]. Lately Wielandt [22] has given notably more simple proofs for the results of Frobenius.

In Section 1 we study nonnegative indecomposable matrices from a different point of view (that of the Brouwer fixed point theorem); a concise proof of their basic properties is thus obtained. In Section 2 properties of a general nonnegative square matrix A are derived from those of nonnegative indecomposable matrices. In Section 3 theorems about the matrix $sI-A$ are proved; they cover in a unified manner a number of results recurringly used in economics. In Section 4 a systematic study of the convergence of $A^p$ when p tends to infinity (A is a general complex matrix) is linked to combinatorial properties of nonnegative square matrices.

Unless otherwise specified, all matrices considered will have <u>real</u> elements. We define for $A = (a_{ij})$, $B = (b_{ij})$:

$A \leqq B$ if $a_{ij} \leqq b_{ij}$ for all i, j

$A \leq B$ if $A \leqq B$ and $A \neq B$

$A < B$ if $a_{ij} < b_{ij}$ for all ij .

Primed letters denote transposes.

When A is an n.n matrix, $A_T = TAT^{-1}$ denotes the transform of A by the nonsingular n.n matrix T.

## 1. Nonnegative Indecomposable Matrices

An n.n matrix A ($n \geqq 2$) is said to be indecomposable if for no permutation matrix[2/] $\pi$, does $A_\pi = \pi A \pi^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ where $A_{11}$, $A_{22}$ are square.

Theorem I. Let $A \geqq 0$ be indecomposable. Then

1. A has a characteristic root $r > 0$ such that
2. to r can be associated an eigen-vector $x_o > 0$
3. if $\alpha$ is any characteristic root of A, $|\alpha| \leqq r$
4. r increases when any element of A increases
5. r is a simple root.

Proof. 1. a) If $x \geq 0$, then $Ax \geq 0$. For if $Ax = 0$ A would have a column of zeros, and so would not be indecomposable.

1. b) A has a characteristic root $r > 0$.

Let $S = \left\{ x \in R^n \mid x \geq 0, \Sigma x_i = 1 \right\}$ be the fundamental simplex in the Euclidean n-space, $R^n$. If $x \in S$, we define $T(x) = \frac{1}{\rho(x)} Ax$ where $\rho(x) > 0$ is so determined that $T(x) \in S$ (by 1.a) such a $\rho$ exists for every $x \in S$). Clearly T(x) is a continuous transformation of S into itself, so, by the Brouwer fixed-point theorem (see for example [11]), there is an $x_o \in S$ with $x_o = T(x_o) = \frac{1}{\rho(x_o)} Ax_o$. Put $r = \rho(x_o)$.

2. $\underline{x_o > 0.}$ Suppose that after applying a proper $\pi$, $\tilde{x}_o = \begin{pmatrix}\xi \\ o\end{pmatrix}$, $\xi > 0$.

Partition $A_\pi$ accordingly. $A_\pi \tilde{x}_o = r\tilde{x}_o$ yields $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\xi \\ 0\end{pmatrix} = \begin{pmatrix} r\ \xi \\ 0\end{pmatrix}$,

thus $A_{21}\ \xi = 0$, so $A_{21} = 0$, violating the indecomposability of A.

If $M = (m_{ij})$ is a matrix, we henceforth denote by $M^*$ the matrix $M^* = (|m_{ij}|)$.

3 - 4. <u>If</u> $0 \leq B \leq A$, <u>and if</u> $\beta$ <u>is a characteristic root of</u> B, <u>then</u> $|\beta| \leq r$. Moreover $|\beta| = r$ implies $B = A$.

A' is indecomposable and therefore has a characteristic root $r_1 > 0$ with an eigen-vector $x_1 > 0$: $A'x_1 = r_1 x_1$. Moreover $\beta y = By$. Taking absolute values and using the triangle inequality, we obtain

(i) $|\beta| y^* \leq By^* \leq Ay^*$. So

(ii) $|\beta| x'_1 y^* \leq x'_1 A y^* = r_1 x'_1 y^*$ .

Since $x_1 > 0$, $x'_1 y^* > 0$, thus $|\beta| \leq r_1$.

Putting $B = A$ one obtains $|\alpha| \leq r_1$. In particular $r \leq r_1$ and since, similarly, $r_1 \leq r$, $r_1$ is equal to r.

Going back to the comparison of B and A and assuming that $|\beta| = r$ one gets from (i) and (ii)

$$ry^* = By^* = Ay^* .$$

From $ry^* = Ay^*$, application of 2 gives $y^* > 0$.

Thus $By^* = Ay^*$ together with $B \leq A$ yields $B = A$.

5. a) <u>If</u> B <u>is a principal submatrix of</u> A <u>and</u> $\beta$ <u>a characteristic root of</u> B, $|\beta| < r$.

$\beta$ is also a characteristic root of the n.n matrix $\bar{B} = \begin{pmatrix} B & 0 \\ 0 & 0\end{pmatrix}$ . Since A is indecomposable, $\bar{B} \leq A_\pi$ for a proper $\pi$ and $|\beta| < r$ (by 3 - 4).

5. b) $r$ is a simple root of $\Phi(t) = \det(tI-A) = 0$.
$\Phi'(r)$ is the sum of the principal $(n-1)\cdot(n-1)$ minors of $\det(rI-A)$.
Let $A_1$ be one of the principal $(n-1)\cdot(n-1)$ submatrices of $A$. By 5.a) $\det(tI-A_1)$ cannot vanish for $t \geq r$, whence $\det(rI-A_1) > 0$ and $\Phi'(r) > 0$.[3]

With a proof practically identical to that of 3 - 4, one obtains the more general result:

If $B$ is a complex matrix such that $B^* \leq A$, $A$ indecomposable, and if $\beta$ is a characteristic root of $B$, then $|\beta| \leq r$. Moreover $|\beta| = r$ implies $B^* = A$.

More precisely if $\beta = re^{i\varphi}$, $B = e^{i\varphi}\, DAD^{-1}$ where $D$ is a diagonal matrix such that $D^* = I$. A proof of this last fact is given in ([22] p. 646, lines 4 - 11).

From this can be derived

Theorem II. Let $A \geq 0$ be indecomposable. If the characteristic equation $\det(tI-A) = 0$ has altogether $k$ roots of absolute value $r$, the set of $n$ roots (with their orders of multiplicity) is invariant under a rotation about the origin through an angle of $\frac{2\pi}{k}$, but not under rotations through smaller angles. Moreover there is a permutation matrix $\pi$ such that

$$(1) \quad \pi A \pi^{-1} = \begin{bmatrix} 0 & A_{12} & 0 & \cdot & 0 \\ 0 & 0 & A_{23} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdot & 0 \end{bmatrix}$$

with square submatrices on the diagonal.

Again the reader is referred to the excellent proof of Wielandt [22, p. 646 - 647].[4]

If $k = 1$, the indecomposable matrix $A = 0$ is said to be primitive.

## 2. Nonnegative Square Matrices

If A is an n·n matrix, there clearly exists a permutation matrix $\Pi$ such that

$$\Pi A \Pi^{-1} = \begin{bmatrix} A_1 & & * \\ & A_2 & \\ 0 & & \ddots \\ & & & A_H \end{bmatrix}$$

where the $A_h$ are square submatrices on the diagonal and every $A_h$ is either indecomposable or a 1·1 matrix.

The properties of A will therefore be easily derived from those of the $A_h$. For example $\det(tI-A) = \prod_{h=1}^{H} \det(tI-A_h)$ and Theorem I gives

Theorem I*. If $A \geqq 0$ is a square matrix, then

1. A has a characteristic root $r \geqq 0$ such that
2. to r can be associated an eigen-vector $x_o \geqq 0$
3. if $\alpha$ is any characteristic root of A, $|\alpha| \leqq r$
4. r does not decrease when an element of A increases.

Let $r_h$ be the maximal nonnegative characteristic root of $A_h$, we take $r = \operatorname{Max}_h r_h$; 1 - 3 - 4 are then immediate. To prove 2 we consider a sequence $A_\ell$ of n·n matrices converging to A such that for all $\ell$ $A_\ell > 0$. Let $r_\ell$ be the maximal positive characteristic root of $A_\ell$, $x_\ell > 0$ its associated eigen-vector so chosen that $x_\ell \in S$, the fundamental simplex of $R^n$. Clearly $r_\ell$ tends to r. Let us then select $x_o \in S$ a limit point of the set $(x_\ell)$; thus there is a subsequence $x_{\ell'}$ converging to $x_o \geqq 0$ and for every $\ell'$, $A_{\ell'} x_{\ell'} = r_{\ell'} x_{\ell'}$, therefore $Ax_o = rx_o$.

5 of Theorem I no longer holds, but 5.a) becomes:

If B is a principal submatrix of A and $\beta$ a characteristic root of B, $|\beta| \leqq r$.

The proof, almost identical, now rests on 4 of Theorem I*.[5]

## 3. Properties of sI-A for $s > r$.

In this section $A \geq 0$ is an $n \cdot n$ matrix, $r$ is its maximal nonnegative characteristic root.

Lemma*: If for an $x > 0$, $Ax \leq sx$ (resp. $\geq$), then $r \leq s$ (resp. $\geq$).

If for an $x \geq 0$, $Ax < sx$ (resp. $>$), then $r < s$ (resp. $>$).

The proofs of the four statements being practically identical, we present only that of the first one. Let $x_0 \geq 0$ be a characteristic vector of $A'$ associated with $r$ (2 of Theorem I*): $A'x_0 = rx_0$. $Ax \leq sx$ with $x > 0$, therefore $x_0'Ax \leq sx_0'x$ i.e., $rx_0'x \leq sx_0'x$ and, since $x_0'x > 0$, $r \leq s$.

We now derive two theorems (III* and III) from the study of the equation

(2) $(sI-A)\,x = y$

Theorem III*. $(sI-A)^{-1} \geq 0$ if and only if $s > r$.

Sufficiency. Since $s > r$ (2) has a unique solution $x = (sI-A)^{-1}y$ for every $y$; we show that $y \geq 0$ implies $x \geq 0$.

If $x$ had negative components (2) could be given the form [by proper (identical) permutations of the rows and columns and partition]

$$\begin{bmatrix} sI-A_1 & -A_{12} \\ -A_{21} & sI-A_2 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = y$$

where $x_1 > 0$, $x_2 \geq 0$, $y \geq 0$. Therefore $-(sI-A_1)\,x_1 - A_{12}\,x_2 \geq 0$ i.e., $-(sI-A_1)\,x_1 \geq 0$ i.e., $A_1x_1 \geq sx_1$. From the Lemma* the maximal nonnegative characteristic root of $A_1$, $r_1 \geq s$, a contradiction to the fact that $r \geq r_1$. (See end of Section 2) and $s > r$.

Necessity. Since $(sI-A)^{-1} \geq 0$ to a $y > 0$ corresponds an $x \geq 0$. Therefore from $sx - Ax = y$ follows $Ax < sx$ and, by the Lemma*, $r < s$.

If A is indecomposable these results can be sharpened to the

Lemma: Let A be indecomposable

<u>if for an</u> $x \geqq 0$, $Ax \leqq sx$ (<u>resp.</u> $\geqq$), <u>then</u> $r \leqq s$ (<u>resp.</u> $\geqq$).

<u>if for an</u> $x \geq 0$, $Ax \leq sx$ (<u>resp.</u> $\geq$), <u>then</u> $r < s$ (<u>resp.</u> $>$).

The proofs, practically identical to those of the Lemma*, use a <u>positive</u> characteristic vector of A' associated with r. One of these statements indeed has already been proved in 3 - 4 of Theorem I.

<u>Theorem III</u>. <u>Let</u> A <u>be indecomposable</u>. $(sI-A)^{-1} > 0$ <u>if and only if</u> $s > r$.

<u>Sufficiency</u>. We show that $y \geq 0$ implies $x > 0$. It is already known (from the proof of sufficiency of Theorem III*) that $x \geq 0$. If x had zero components, (2) could be given the form

$$\begin{bmatrix} sI-A_1 & -A_{12} \\ -A_{21} & sI-A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y$$

where $x_1 = 0$, $x_2 > 0$, $y \geq 0$. Therefore $-A_{12}\, x_2 \geqq 0$, and, since $x_2 > 0$, $A_{12} = 0$ violating the indecomposability of A.

The <u>Necessity</u> has already been proved since $(sI-A)^{-1} > 0$ implies $(sI-A)^{-1} \geq 0$.[6]

<u>Theorem IV</u>. <u>The principal minors of</u> sI-A <u>of orders</u> 1, ..., n <u>are all positive if and only if</u> $s > r$.

<u>Sufficiency</u>. det (tI-A) cannot vanish for $t > r$, thus det $(sI-A) > 0$ for $s > r$. Similarly, the maximal nonnegative characteristic root of a principal submatrix of A is not larger than r (see end of Section 2), it is therefore smaller than s, and the corresponding minor of sI-A is positive.

<u>Necessity</u>. The derivative of order m ($< n$) of det (tI-A) with respect to t, for $t = s$, is a sum of principal minors of order (n-m).(n-m) of sI-A and thus is positive. As its derivatives of all orders (0, 1, ..., n-1, n) are positive for $t = s$, the polynomial det (tI-A) can vanish for no $t \geqq s$ i.e., $s > r$.[7]

Since a square matrix with nonpositive (resp. negative) off-diagonal elements can always be given the form sI-A where $A \geqq 0$ (resp. $> 0$), results such as those of Chipman [2], [3], Goodwin [8], Hawkins and Simon [9], Metzler [12] to [15], Morishima [16],8/ Mosak [17], Solow [20] are all contained in the above.

## 4. Convergence of $A^p$ 9/

Theorem V. Let $A$ be a n·n complex matrix. The sequence $A, A^2, \ldots, A^p, \ldots$ of its powers converges if and only if

1. any characteristic root $\alpha$ of $A$ satisfies either $|\alpha| < 1$ or $\alpha = 1$
2. when the second case occurs the order of multiplicity of the root 1 equals the dimension of the eigen-vector space associated with that root.

There is a nonsingular complex matrix $T$ such that

$$A_T = TAT^{-1} = \begin{bmatrix} J_1 & & & & 0 \\ & \ddots & & & \\ & & J_i & & \\ & & & \ddots & \\ 0 & & & & J_q \end{bmatrix} \quad \text{where } J_i = \begin{bmatrix} \alpha_\iota & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ 0 & & & \alpha_\iota & 1 \\ & & & \ddots & \ddots \\ & & & & \ddots \;\; 1 \\ & & & & \alpha_\iota \end{bmatrix}$$

is a square matrix on the diagonal and $\alpha_\iota$ a characteristic root of $A$. To every root $\alpha_\iota$ corresponds at least one $J_i$. (For this reduction of $A$ to its Jordan canonical form see for example [21]).

Since

$$TA^pT^{-1} = \begin{bmatrix} J_1^p & & & & 0 \\ & \ddots & & & \\ & & J_i^p & & \\ & & & \ddots & \\ 0 & & & & J_q^p \end{bmatrix}, \qquad A^p \text{ converges}$$

if and only if every one of the $J_i^p$ converges. Let us therefore study one of them; for this purpose we drop the subscripts $i$ and $\iota$.

J is a k·k matrix of the form $J = \alpha I + M$ where $M = (m_{st})$:
$m_{st} = 1$ if $t = s + 1$, $m_{st} = 0$ otherwise.

$$J^p = \alpha^p I + \binom{p}{1} \alpha^{p-1} M + \dots \binom{p}{k-1} \alpha^{p-k+1} M^{k-1}$$

It is easily seen that for $M^h$, $m_{st}^{(h)} = 1$ if $t = s + h$ and $m_{st}^{(h)} = 0$ otherwise. Thus $M^h = 0$ if $h \geq k$; also the nonzero elements of $M^h$ and $M^{h'}$ $(h \neq h')$ never occur in the same place so $J^p$ converges if and only if every term of the right-hand sum does.

The first term shows that necessarily either $|\alpha| < 1$ or $\alpha = 1$.
If $|\alpha| < 1$, every term tends to zero and $J^p$ converges.
If $\alpha = 1$ no term other than the first one converges and necessarily $k = 1$ i.e., $J = [1]$; clearly $J^p$ converges in this case.

We wish however to obtain for this necessary and sufficient condition of convergence an expression independent of a reduction to Jordan canonical form.

Consider then an arbitrary n·n complex matrix A and let $\mathcal{J}$ be the set of i for which $J_i$ corresponds to the root 1. The equation $A_T x = x$, in which x is partitioned in the same way as $A_T$, yields $J_i x_i = x_i$ for all i i.e.,
if $i \notin \mathcal{J}$, $x_i = 0$
if $i \in \mathcal{J}$ all components of $x_i$ but the first one equal zero.
Thus the dimension of the eigen-vector space associated with the root 1 equals the number of elements of $\mathcal{J}$. This number, in turn, equals the order of multiplicity of the root 1 if and only if $J_i = [1]$ for all $i \in \mathcal{J}$.

We now assume that the limit C exists and give its expression. If 1 is not a characteristic root of A, C = 0. Let therefore 1 be a root of A of order $\mu$. Thus x (resp. y), an eigen-vector of A (resp. A'), associated with the root 1, has the form $x = X \xi$ (resp. $y = Y \eta$) where X (resp. Y) is a

$n \cdot \mu$ matrix of rank $\mu$ and $\xi$ (resp. $\eta$ ) is a $\mu \cdot 1$ matrix. For an arbitrary x the relation $AA^p x = A^{p+1}x$ gives in the limit $ACx = Cx$ i.e., $Cx = X\,\xi(x)$. To determine $\xi(x)$ we remark that $Y' = Y'A$ i.e., by iteration $Y' = Y'A^p$, and therefore $Y' = Y'C$; thus $Y'x = Y'Cx = Y'X\,\xi(x)$. $Y'X$ is a nonsingular[10] $\mu \cdot \mu$ matrix i.e., $\xi(x) = (Y'X)^{-1}\,Y'x$. Finally for all x, $Cx = X(Y'X)^{-1}\,Y'x$ i.e., $C = X\,(Y'X)^{-1}\,Y'$.

<u>Corollary</u>. <u>Let</u> $A \geqq 0$ <u>be indecomposable and</u> 1 <u>be its maximal positive characteristic root. The sequence</u> $A^p$ <u>converges if and only if</u> A <u>is primitive</u>.

The necessity is obvious. The sufficiency follows from the fact that 1 is a simple root.

Let then $x_o > 0$ (resp. $y_o > 0$) be an eigen-vector of A (resp. A') associated with the root 1, the limit C of $A^p$ has the simple expression

$$C = \frac{x_o\,y'_o}{y'_o\,x_o} \quad .$$

Clearly $C > 0$, thus if the indecomposable matrix $A \geqq 0$ is primitive, there is a positive integer m such that $A^p > 0$ when $p \geqq m$. The converse is an immediate consequence of the decomposition (1) of Theorem II.[11]

# Footnotes

1. This paper is a result of the work being done at the Cowles Commission for Research in Economics on the "Theory of Resource Allocation" under subcontract to The RAND Corporation. Based on Cowles Commission Discussion Paper, Mathematics No. 414, February, 1952.

Acknowledgment is due to staff members and guests of the Cowles Commission, to J.L. Koszul with whom one of us had the privilege of discussing the problem of Section 4, to R. Solow who in particular pointed out to us that Alexandroff and Hopf [1] had already suggested the use of Brouwer's theorem in connection with the problem of Section 1.

2. In any row or column of a permutation matrix one element equals 1, the others equal 0. $\pi A \pi^{-1}$ is obtained by performing the same permutation on the rows and on the columns of A.

3. As an immediate consequence of 4 one obtains:

$$\underset{i}{\text{Min}} \sum_j a_{ij} \leqq r \leqq \underset{i}{\text{Max}} \sum_j a_{ij}$$

and one equality holds only if all row sums are equal (then they both hold).

This is proved by increasing (resp. decreasing) some elements of A so as to make all row sums equal to $\underset{i}{\text{Max}} \sum_j a_{ij}$ (resp. $\underset{i}{\text{Min}} \sum_j a_{ij}$). A similar result naturally holds for column sums.

4. Decomposition (1) can indeed be completely characterized.

Lemma. Let A be a square complex matrix such that $\pi A \pi^{-1}$ has form (1) and let $B_i = A_{i,i+1} \times \ldots \times A_{k-1,k} \times \ldots \times A_{i-1,i}$. For $\alpha \neq 0$ to be a characteristic root of A it is necessary (resp. sufficient) that $\alpha^k$ be a characteristic root of every (resp. one) $B_i$.

After proper partition of x, an eigen-vector of $A_\pi$ associated with the root $\alpha$, the equation $A_\pi x = \alpha x$ becomes

(1') $A_{i,i+1} x_{i+1} = \alpha x_i$ $\quad (i=1, \ldots, k)$ which implies

(1") $B_i x_i = \alpha^k x_i$. Since no $x_i$ can vanish (by (1') they all would), $\alpha^k$ is a characteristic root of every $B_i$.

Conversely let $\alpha^k$ be a characteristic root of $B_i$ and $x_i$ an associated eigen-vector, we construct a vector x, whose $i^{th}$ component is $x_i$, and such that $A_\pi x = \alpha x$. The $(i-1)^{th}$ equation (1') determines $x_{i-1}$; $x_{i-2}, \ldots, x_1, x_k, \ldots, x_{i+2}, x_{i+1}$ are determined in turn; the $i^{th}$ equation is redundant because of (1").

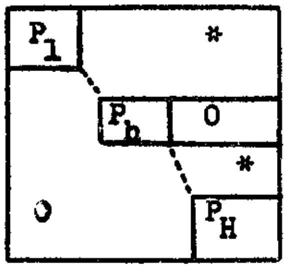
As an immediate consequence of the lemma one finds

Theorem. Let $A \geq 0$ be indecomposable. For $A$ to have exactly $k$ characteristic roots of maximum absolute value $r$ it is necessary (resp. sufficient) that $A$ can be brought to form (1) where every (resp. one) $B_i$ has no other characteristic root of maximum absolute value $s_i$ than $s_i$ itself.

Naturally $s_i = r^k$ for every $i$.

5. A stochastic n·n matrix $P$ is defined by $p_{ij} \geq 0$ for all $i$, $j$ and $\sum_j p_{ij} = 1$ for all $i$. Clearly 1 is a characteristic root of $P$ (take an eigenvector with all components equal). 1 is therefore a root of some of the indecomposable matrices $P_1, P_2, \ldots, P_H$. Suppose that 1 is a root of $P_h$, it follows from Footnote (3) that all row sums of $P_h$ are equal to 1, i.e.,

$\pi P \pi^{-1} =$




This simple remark makes properties of stochastic matrices (the subject of the theory of finite Markov chains; see [4] and its references) ready consequences of the results of this article.

6. It is worth [9] emphasizing a result obtained in the proof of necessity of Theorem III*.
Remark. Let $A \geq 0$ (resp. $A \geq 0$ indecomposable) be a square matrix. If for a $y > 0$ (resp. $y \geq 0$), $x \geq 0$, then $(sI-A)^{-1} \geq 0$ [resp. $(sI-A)^{-1} > 0$].

The proof of indecomposable matrices uses the Lemma instead of the Lemma*.

7. We give a last property useful in economics [14], [15].
Theorem. Let $A > 0$ be a square matrix and let $C_{ij}$ be the cofactor of the $i^{th}$ row, $j^{th}$ column element of $sI-A$. If $s > \sum_j a_{ij}$ for all $i$, then $i \neq j$ implies $C_{ii} > C_{ij}$.

Let us define the matrix $B = (b_{pq})$ as follows:

$$b_{pq} = a_{pq} \text{ if } p \neq i; \quad b_{iq} = 0 \text{ if } i \neq q \neq j; \quad b_{ii} = \frac{s}{2} = b_{ij}.$$

$B$ is indecomposable, moreover $\sum_q b_{iq} = s$, $\sum_q b_{pq} < s$ for $p \neq i$.
Therefore (see Footnote 3) the maximal positive characteristic root of $B$, $r(B) < s$. Thus $\det(sI-B) > 0$; a development according to the $i^{th}$ row yields:
$\frac{s}{2} C_{ii} - \frac{s}{2} C_{ij} > 0$.

8. Morishima studies square matrices A such that for a permutation matrix $\pi$,

$$\pi A \pi^{-1} = A_\pi = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ where } A_{11} \geqq 0 \text{ and } A_{22} \geqq 0 \text{ are square,}$$

$A_{12} \leqq 0$, $A_{21} \leqq 0$. The relation

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}$$

shows how properties of $A_\pi$ can be immediately derived from those of the <u>non-negative</u> matrix

$$A_\pi^S = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}$$

In particular $A_\pi$ and $A_\pi^S$ have the same characteristic roots.

9. The Cesaro convergence of $A^p$, i.e., the convergence of $\frac{1}{p}(A + A^2 + \ldots + A^p)$ can be studied in exactly the same fashion.

10. $X_T = TX$ (resp. $Y_T' = Y'T^{-1}$) plays for $A_T$ the same role as X (resp. Y') does for A. Moreover $Y'X = Y_T'X_T$. The right-hand matrix is nonsingular for the form taken by the Jordan matrix $A_T$ in the convergence case implies that the eigen-vector space U generated by $X_T$ is identical with the eigen-vector space V generated by $Y_T$. Thus $Y_T'X_T \xi = 0$ implies $X_T \xi = 0$ (there is no vector different from zero in U perpendicular to V, i.e., to U) therefore $\xi = 0$ since the rank of $X_T$ is $\mu$.

11. This characterization of a primitive matrix is typical of the purely combinatorial properties of the nonnegative square matrix A (used for example in the theory of communication networks): the smallest m satisfying the above condition is independent of the values of the nonzero elements of A as long as they stay positive.

The development of combinatorial techniques adapted to the treatment of such properties is the subject of [10].

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