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Nonnegative, Indecomposable Matrices[1]

Gerard Debreu and I. N. Herstein

All matrices considered will have real elements. We define, for $A = (a_{ij})$, $B = (b_{ij})$

$$A \geqq B \quad \text{if} \quad a_{ij} \geqq b_{ij} \quad \text{for all } i, j$$

$$A \geq B \quad \text{if} \quad A \geqq B \quad \text{but } A \neq B$$

$$A > B \quad \text{if} \quad a_{ij} > b_{ij} \quad \text{for all } i, j.$$

An nxn matrix A is said to be indecomposable if for no permutation matrix P does PAP^{-1} have the form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ where A_{11}, A_{22} are square submatrices.

Theorem: Let $A \geqq 0$ be indecomposable. Then A has a positive characteristic root r so that

(1) r is a simple root

(2) if α is any characteristic root of A, $|\alpha| \leqq r$

(3) to r can be associated an eigenvector $x_0 > 0$.

These results of Frobenius[2] have been proved recently in a drastically simplified fashion by Wielandt[3]. A further simplification is given here.

Proof:

a) If $x \geq 0$ then $Ax \geq 0$. For if $Ax = 0$, then trivially A would have a column of zeros, and so would not be indecomposable.

b) A <u>has a positive characteristic root</u> r. Let $S = \left\{ x \in R^n \mid x \geq 0, \sum_i x_i = 1 \right\}$ be the fundamental simplex in the Euclidean n-space. If $x \in S$, we define $T(x) = \frac{1}{\rho(x)} Ax$ where $\rho > 0$ is so determined that T(x) is also in S. (By (a) such a ρ exists for every $x \in S$). Clearly T(x) is a continuous transformation of S into itself, so has a fixed point x_o (Brouwer),

$x_o = T(x_o) = \frac{1}{\rho(x_o)} Ax_o$. Put $r = \rho(x_o)$.

c) $\underline{x_o > 0}$. Suppose $x_o = \begin{bmatrix} \xi \\ 0 \end{bmatrix}$, $\xi > 0$. Partition A accordingly.

$Ax_o = rx_o$ yields $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \begin{bmatrix} r\xi \\ 0 \end{bmatrix}$, thus $A_{21}\xi = 0$, so $A_{21} = 0$,

violating the indecomposability of A.

If $M = (m_{ij})$ is a matrix we henceforth denote by M^* the matrix $M^* = (|m_{ij}|)$.

d) <u>If</u> $0 \leqq B \leqq A$, A <u>indecomposable</u>, <u>and if</u> β <u>is a characteristic root of</u> B, <u>then</u> $|\beta| \leqq r$. <u>Moreover</u> $|\beta| = r$ <u>implies</u> B = A.

For $Ax_o = rx_o$, $x_o > 0$. Since β is a characteristic root of B, it is also one of B'. So $\beta y = B'y$. The taking of absolute values and use of the triangle inequality, yields

(i) $|\beta| y^* \leqq B'y^* \leqq A'y^*$. So

(ii) $|\beta| x_o'y^* \leqq x_o'A'y^* = rx_o'y^*$. Since $x_o > 0$,

$x_o'y^* > 0$ so $|\beta| \leqq r$. If $|\beta| = r$ then (i) and (ii) lead to

$$ry^* = B'y^* = A'y^*.$$

So $y^* > 0$; hence $B' = A'$.

Putting B = A we obtain that if α is a characteristic root of A, $|\alpha| \leqq r$.

e) r <u>is a simple root of</u> $\Phi(r) = \det(rI-A) = 0$.

$\Phi'(r)$ is the sum of the principal minors of det $(rI-A)$. Let A_1 be the matrix obtained by deleting the first row and column of A.

$B = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix} < A$ since A is indecomposable. So by (d) any characteristic root β of B satisfies $|\beta| < r$. Det $\begin{pmatrix} r & 0 \\ 0 & rI-A_1 \end{pmatrix} > 0$, whence

det $(rI-A_1) > 0$ and the result is proved.

References

1. Research undertaken under contract between the Cowles Commission for Research in Economics and the RAND Corporation.

2. G. Frobenius, "Uber Matrizen aus nicht negativen Elementen," Sitzungsber Preus. Akad. Wiss., Berlin, 1912, p. 456-477.

3. Wielandt, Helmut, "Unzerlegbare, nicht-negative Matrizen," Math. Zeitschr. Vol. 52 (1950), p. 642-648, and its translation Cowles Commission Discussion Paper, Mathematics No. 413.