COWLES COMMISSION DISCUSSION PAPER: MATHEMATICS NO. 413

Indecomposable, Nonnegative Matrices[1]

Helmut Wielandt (in Mainz)

(Translated from the German by I. N. Herstein)

February 5, 1952

A matrix A consisting of nonnegative elements $a_{\mu\nu}$ $(\mu, \nu = 1, 2, \ldots, n;$ $n \geq 2)$ is called, following Frobenius[2], indecomposable when it neither has the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

(with square submatrices A_{11}, A_{22}) nor can be brought into this form by use of a permutation of the rows and the same permutation of the columns. Frobenius[2], continuing the closely related work of Perron[3] on positive matrices,[4] proved a series of important theorems about the characteristic roots of such a matrix A. We collect his essential results into:

I. <u>The characteristic equation</u>

(1) $\Phi(x) \equiv \det(xE - A) = 0$

<u>possesses a simple, positive root r, which is at least as large, in absolute value, as any other root. A characteristic vector can be chosen for this "maximal-root" r having all its components positive; r is the only characteristic root for which a nonnegative characteristic vector exists.</u>

II. If (1) possesses, altogether, k roots of absolute value r, then these are all simple and have the values $re^{\frac{x\,2\pi i}{k}}$ $(x = 1, 2, \ldots, k)$. The set of n roots of (1) is invariant under a rotation about the origin through an angle of $\frac{2\pi}{k}$, but not under rotations through smaller angles.[5] A has the form

$$(2) \qquad A = \begin{pmatrix} 0 & A_{12} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & A_{23} & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

(with square submatrices on the diagonal) or can be brought into this form by an application of the same permutation of the rows and columns.

In what follows, these theorems will be proved in a new, and considerably shorter, manner and, in parts, generalized (compare Theorem III). We rely on a Maximum-Minimum property of r, unmentioned by Frobenius, which is related to the "Einschliesungssatz" of Collatz.[6]

Proof of I.

(a) Definition of r. To each column of nonnegative elements $x_1, x_2, \ldots, x_n$ (in short: to each vector $\mathfrak{x} \geq 0$) which does not consist of zeros alone, we define the nonnegative number $r_{\mathfrak{x}}$ by

$$r_{\mathfrak{x}} = \min_{\mu} \frac{\sum_{\nu} a_{\mu\nu} x_{\nu}}{x_{\mu}}$$

with the agreement to put this fraction equal to $+\infty$ in case $x_{\mu} = 0$. In other words: $r_{\mathfrak{x}}$ is the largest number for which

$$(3) \qquad A\mathfrak{x} - r_{\mathfrak{x}}\mathfrak{x} \geq 0$$

is true. The numbers $r_{\mathfrak{x}}$ are bounded from above. For if we denote by $\mathfrak{S}$

the vector consisting of all 1's, by $\mathfrak{s}'$ its transpose, and by C the largest element of the row $\mathfrak{s}'A$, then as a consequence of (3)

$$r_{\mathfrak{z}} \leqq \frac{\mathfrak{s}' A \mathfrak{z}}{\mathfrak{s}' \mathfrak{z}} \leqq \frac{C \, \mathfrak{s}' \mathfrak{z}}{\mathfrak{s}' \mathfrak{z}} = C.$$

For some $\mathfrak{z}$, $r_{\mathfrak{z}}$ turns out to be positive, e.g., $\mathfrak{z} = \mathfrak{s}$; for since A is indecomposable it does not possess a row of zeros. Thus the set of all $r_{\mathfrak{z}}$ possesses a finite, positive upper bound r. This is actually taken on, since one need only consider those $\mathfrak{z} \geqq 0$ for which $\mathfrak{s}'\mathfrak{z} = 1$; that is, a closed, bounded set of column vectors. We then have that

$$r = \max_{\mathfrak{z} \geqq 0} \ \min_{\mu} \ \frac{\sum_{\nu} a_{\mu\nu} x_{\nu}}{x_{\mu}}. \tag{4}$$

(b) From the definition of r it follows that: <u>There exist "extreme-vectors" $\mathfrak{z}$, with $\mathfrak{z} \neq 0$, $\mathfrak{z} \geqq 0$, $A\mathfrak{z} - r\mathfrak{z} \geqq 0$, but no vector $\mathfrak{z} \geqq 0$ such that</u> $A\mathfrak{z} - r\mathfrak{z} > 0$.

Before proving that r is the asserted maximal root of Theorem I and that the extreme-vectors are the corresponding eigenvectors, we make the following observations.

(c) <u>If $\mathfrak{y} \geqq 0$, $\mathfrak{y} \neq 0$, then $(E + A)^{n-1}\mathfrak{y} > 0$.</u> For let $\mathfrak{y}_{\nu} = (E + A)^{n-1}\mathfrak{y}$. Then

$$\mathfrak{y}_{\nu+1} = \mathfrak{y}_{\nu} + A\mathfrak{y}_{\nu} \geqq \mathfrak{y}_{\nu} \geqq 0.$$

Those coordinates of $\mathfrak{y}_{\nu+1}$ which vanish are at most those coordinates which also vanished in $\mathfrak{y}_{\nu}$. However, it can not possibly happen that exactly all those coordinates which are zero in $\mathfrak{y}_{\nu}$ are also zero in $\mathfrak{y}_{\nu+1}$; for then, by a suitable ordering of the coordinates we would have

$$\mathfrak{y}_{\nu} = \begin{bmatrix} p \\ 0 \end{bmatrix}, \qquad p > 0$$

$$\mathfrak{y}_{\nu+1} = \mathfrak{y}_{\nu} + A\mathfrak{y}_{\nu} = \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} p \\ 0 \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix}.$$

Hence it follows that $A_{21}\ p = 0$, whence $A_{21} = 0$, contrary to the assumption that A is indecomposable. Thus if $\mathfrak{y}_\nu$ contains zeros as coordinates, $\mathfrak{y}_{\nu+1}$ contains fewer zeros as coordinates. Now $\mathfrak{y}_0 = \mathfrak{y}$ contains at most n-1 zeros, so $\mathfrak{y}_{n-1}$ contains none at all.

(d) <u>Each extreme-vector $\mathfrak{z}$ is a characteristic vector of A, has eigenvalue r, and is a positive vector.</u>

By definition, $\mathfrak{z} \geqq 0$, $A\mathfrak{z} - r\mathfrak{z} = \mathfrak{y} \geqq 0$. Should $\mathfrak{y} \neq 0$, then by multiplication with $(E + A)^{n-1}$ we would have, for the vector $\mathfrak{x} = (E + A)^{n-1}\mathfrak{z}$ the inequalities

$$\mathfrak{x} > 0, \quad A\mathfrak{x} - r\mathfrak{x} = (E + A)^{n-1}\,\mathfrak{y} > 0,$$

in contradiction to (b). So $\mathfrak{y} = 0$, $A\mathfrak{z} = r\mathfrak{z}$. That $\mathfrak{z} > 0$ follows from $0 < \mathfrak{x} = (1 + r)^{n-1}\mathfrak{z}$.

(e) <u>If α is an arbitrary characteristic root of A then $|\alpha| \leqq r$.</u>

For brevity, for any matrix $M = (m_{\rho\sigma})$, we denote here, and in what follows, the matrix of absolute values, $(|m_{\rho\sigma}|) = M^*$. From $\alpha\mathfrak{x} = A\mathfrak{x}$ and from the triangle inequality, we obtain that

$$|\alpha|\,\mathfrak{x}^* \leqq A\,\mathfrak{x}^*, \text{ so } |\alpha| \leqq r_{\mathfrak{x}^*} \leqq r.$$

Hence r is the maximal characteristic root of A of Theorem I.

(f) <u>If α, a characteristic root of A, possesses a nonnegative characteristic vector $\mathfrak{x}$, then $\alpha = r$.</u>

Since A is nonnegative and indecomposable, then so must A', the transpose of A, also be nonnegative and indecomposable. Since A and A' have the same characteristic roots, r must also be a maximal characteristic root of A'. If $\mathfrak{y}$ is an extreme vector of A' we have

$$\alpha\,\mathfrak{y}'\mathfrak{x} = \mathfrak{y}'(A\mathfrak{x}) = (\mathfrak{y}'A)\mathfrak{x} = r\,\mathfrak{y}'\mathfrak{x}.$$

Since $\mathfrak{y} > 0$, $\mathfrak{y}'\mathfrak{x} \neq 0$, so $\alpha = r$.

There only remains to show that r is a simple root of $\Phi(x)$. We first prove somewhat less:

(g) A possesses, for the characteristic root r, only one linearly independent characteristic vector; its components all have the same sign.

Let $\mathfrak{y}$ be an arbitrary characteristic vector of A belonging to r, $\mathfrak{z}$ a fixed extreme-vector. It suffices to show that $\mathfrak{y}$ and $\mathfrak{z}$ are proportional. We determine the number c so that $\mathfrak{y} - c\mathfrak{z} = \mathfrak{h} \geq 0$ and so that one component of $\mathfrak{h}$ vanishes.

By this last property and (d), $\mathfrak{h}$ can not be an extreme-vector; on the other hand, $A\mathfrak{h} = r\,\mathfrak{h}$, so $\mathfrak{h}$ would be an extreme-vector if $\mathfrak{h} \neq 0$. So $\mathfrak{y} = c\mathfrak{z}$.

(h) r is a simple root of $\Phi(x)$.

The assertion states that $\Phi'(r) \neq 0$. By the well known rule of differentiating a determinant, $\Phi'(r)$ is the trace of P, the adjoint of $rE - A$. From (g) it follows that

$$\text{rank } (rE - A) = n-1, \quad \text{so} \quad P \neq 0.$$

Furthermore, P satisfies $(rE - A)\,P = 0$. From this, every nonvanishing column of P must be a characteristic vector of A belonging to r, and so, from (g), contains only elements of the same sign. The same result holds for the rows of P, as is seen by going to transposes. So all the elements of P must have the same sign, and $\Phi'(r) = \text{trace } P \neq 0$. (Since r is the largest real zero of $\Phi(x)$, $\Phi'(r)$, and so every element of P, is positive.)

With this the proof of Theorem I is completed. The proof of Theorem II rests on a partially new Theorem III (by Frobenius[4], p. 516, the case $|\beta| = r$, essential for the argument, is missing).

III. Let $A = (a_{\mu\nu})$ be an indecomposable matrix with nonnegative elements, $B = (b_{\mu\nu})$ a matrix with complex elements $(\mu, \nu = 1, 2, \ldots, n)$. Suppose $|b_{\mu\nu}| \leq a_{\mu\nu}$ for all μ, ν. If r is the maximal root of A and

β is an arbitrary characteristic root of B , then $|\beta| \leq r$. In case of equality, that is $\beta = re^{i\varphi}$, then B has the form

(5) $$B = e^{i\varphi} D A D^{-1}$$

where D is a diagonal matrix whose diagonal elements all have absolute value 1; in particular we then always have $|b_{\mu\nu}| = a_{\mu\nu}$.

(The converse that every matrix B of the form (5) has a characteristic root $re^{i\varphi}$ is clear.)

Proof of III[7/]. From

(6) $$\beta \mathfrak{z} = B \mathfrak{z}$$

it follows, as in Ie, that

(7) $$|\beta| \mathfrak{z}^* \leq B^* \mathfrak{z}^* \leq A \mathfrak{z}^*$$

and from this $|\beta| \leq r_{\mathfrak{z}^*} \leq r$. If the limiting case, $|\beta| = r$, occurs, then by (7) $\mathfrak{z}^*$ is an extreme-vector, so we have, by Id

(8) $$|\beta| \mathfrak{z}^* = A \mathfrak{z}^*, \qquad \mathfrak{z}^* > 0,$$

and so by (7)

(9) $$B^* = A, \qquad |b_{\mu\nu}| = a_{\mu\nu}$$

By definition of $\mathfrak{z}^*$ it follows that $\mathfrak{z} = D \mathfrak{z}^*$ where D is a diagonal matrix whose diagonal elements have absolute value 1. If we introduce this, together with $\beta = re^{i\varphi}$ into (6) we obtain

(10) $$|\beta| \mathfrak{z}^* = C \mathfrak{z}^* \qquad \text{where}$$

(11) $$C = e^{-i\varphi} D^{-1} B D, \quad C^* = B^* = A.$$

From (8), (10) and (11) it follows that $C \mathfrak{z}^* = C^* \mathfrak{z}^*$, thus since $\mathfrak{z}^* > 0$,

$$C = C^*, \quad C = A, \quad B = e^{i\varphi} D A D^{-1}.$$

Proof of II.

(a) The nature of the characteristic roots of largest absolute value. Suppose there are exactly k roots of (1) which have the largest possible absolute value r, say

$$(12) \qquad \alpha_x = re^{i\varphi_x} \qquad (0 = \varphi_1 < \varphi_2 \leq \varphi_3 \leq \dots \leq \varphi_k < 2\pi) .$$

The conditions of III are satisfied by B = A, together with $\beta = \alpha_x$; thus there exists a diagonal matrix D_x with

$$(13) \qquad A = e^{i\varphi_x} D_x A D_x^{-1} .$$

By I, r is a simple characteristic root of A, thus $re^{i\varphi_x}$ is a simple characteristic root of the matrix on the right hand side; that is, of A. We show that, modulo 2π, they form an additive group, by transforming (13) with $D^{\pm 1}_\lambda$, we get

$$A = e^{i(\varphi_x \pm \varphi_\lambda)} T A T^{-1} \qquad (T = D_x D_\lambda^{\pm 1}).$$

So $re^{i(\varphi_x \pm \varphi_\lambda)}$ is a characteristic root of A, so it must be one of the numbers (12). From this demonstrated group property we obtain

$$(14) \qquad \varphi_x = (x - 1)\frac{2\pi}{k} \qquad (x = 1, 2, \dots, k) .$$

(b) The rotational invariance of the spectrum follows immediately from (13) and (14). The totality of roots of (1) admits exactly the rotation group of the regular k-gon.

(c) In order to bring A to the announced form (2) (which is only necessary for k > 1) we rearrange the rows, and in the same way, the columns of A so that the components of a characteristic vector which belongs to $\alpha_2 = re^{\frac{2\pi i}{k}}$ appear as ordered according to their arguments. We assume, without loss of generality, D_2 to be of the form

$$D_2 = \begin{pmatrix} E_1 e^{i\delta_1} & & & \\ & E_2 e^{i\delta_2} & & \\ & & \ddots & \\ & & & E_g e^{i\delta_g} \end{pmatrix} \qquad (g \geq 1)$$

where the δ_γ modulo 2π are all different and the E_γ are unit matrices (not necessarily of the same degree). Introducing the corresponding splitting of A into g^2 submatrices $A_{\gamma\rho}$ into equation (13) for $x = 2$ we obtain

$$A_{\gamma\rho} = e^{i(\frac{2\pi}{k} + \delta_\gamma - \delta_\rho)} A_{\gamma\rho} \qquad (1 \leq \gamma, \rho \leq g)$$

and so

(15) $$A_{\gamma\rho} = 0 \quad \text{when } \frac{2\pi}{k} + \delta_\gamma \not\equiv \delta_\rho \mod 2\pi.$$

Hence it follows that for each γ there is at most one ρ (and to each ρ at most one γ) for which $A_{\gamma\rho} \neq 0$; on the other hand, since A is indecomposable, there must exist at least one; so there is exactly one, and $\delta_\rho \equiv \frac{2\pi}{k} + \delta_\gamma$. In particular, occuring amonst the δ_γ must be the k values $x \frac{2\pi}{k} + \delta_1$. We suppose the ordering so chosen that these occur in the first k places. Then

$$\delta_x = \delta_1 + (x-1)\frac{2\pi}{k} \qquad (x = 1, 2, \ldots, k)$$

and A has the form

$$\begin{pmatrix} 0 & A_{12} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & A_{23} & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & A_{k-1,k} & 0 \\ A_{k1} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & B \end{pmatrix}$$

Here, only the first k rows and columns can occur, since A was supposed to be indecomposable. Thus Theorem II is proved.

We close with two supplementary remarks. The first concerns the important case $k = 1$, in which r is the only characteristic root of maximal size (following Frobenius, the nonnegative indecomposable matrix A is, in this case, called primitive). It certainly occurs, as Perron[3] noted, in the case that a power $A^m > 0$; namely one recognizes from the representation (2) that in the case $k > 1$ each power of A contains zeros. Frobenius[2], p. 463, proved the following converse to this theorem:

if A is primitive, then for some fixed least number p, $A^m > 0$ for $m \geqq p$. Frobenius gave no particular statements about this p, although using his discussion one can obtain the estimate $p \leqq 2n^2 - 2$ where n is the number of rows of A. A more thorough analysis yields, (which we communicate here without proof), the best possible bound, depending only on n, as $p \leqq 2n^2 - 2n+2$. Equality occurs, for instance, for

$$A = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ 1 & 1 & 0 & . & . & . & 0 \end{pmatrix} .$$

The second remark is in the following direction; the maximal root, r, has, besides the Maximum-Minimum property (4) also a Minimum-Maximum property, namely

$$(16) \qquad r = \min_{\mathfrak{x} > 0} \max_{\mu} \frac{\Sigma a_{\mu\nu} x_\nu}{x_\mu} .$$

It amounts to the right side of Collatz's inequality

$$\min \frac{\sum a_{\mu\nu} x_\nu}{x_\mu} \leqq r \leqq \max \frac{\sum a_{\mu\nu} x_\nu}{x_\mu} \qquad (17)$$

as (4) does with respect to the left. (Collatz[6] proves (17) only for $A > 0$, although the simple proof, similar to If, yields it for $A \geqq 0$ and indecomposable). The symmetry of (4) and (16) points out that (16) could have possibly been made the basis of the whole theory, as was done here with (4). In fact, one could go part of the way in this direction. However, in the proof that r is a largest characteristic root, the simple argument of Ie is rendered inapplicable, because the estimate, yielded by the triangle inequality, is in the wrong direction.

(From the Mathematische Zeitschrift, 52, p. 642-648.)

FOOTNOTES

1. Translation undertaken under contract between the Cowles Commission for Research in Economics and The RAND Corporation.

2. G. Frobenius, "Über Matrizen aus nicht negativen Elementen," Sitziungsber Preus. Akad. Wiss., Berlin, 1912, p. 456-477.

3. O. Perron, "Zur Theorie der Matrices," Math. Ann. 64, p. 248-263 (1907).

4. G. Frobenius, "Über Matrizen aus positiven elementen," Sitzungsber. Preus. Akad. Wiss., Berlin, 1908, p. 471-476; 1909, p. 514-518.

5. Frobenius has this statement in another form: if the powers, x^n, x^m, x^ℓ,.... actually occur in $\Phi(x)$, then k is the greatest common divisor of $n-m$, $m-\ell$, ...

6. L. Collatz, "Einschliesungssatz für die characteristischen Zahlen von Matrizen," Math. Zeitschr., 48, p. 221-226 (1942).

7. A similar argument occurs in a special case by V. Romanovsky, "Sur les zéros des matrices stocastiques," comptes Rend. Acad. Sci. Paris, 192, p. 266-269 (1931).