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Saddle Point Existence Theorems[1/] [*]

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General properties of saddle points and of the MinMax operator are first proved. A historical note then shows how saddle point existence theorems (and corresponding fixed point theorems) of an increasing generality have originated and revealed themselves as powerful tools in economics. After having introduced a few basic topological concepts this paper gives very general conditions under which a real valued function of two variables has a saddle point.[2-3/]

1. Saddle Points and MinMax Operator

Only finite Euclidean spaces R^{ℓ}, R^m, R^n will be considered here.

In this section $x \in X \subset R^{\ell}$, $y \in Y \subset R^m$ and $f(x\ y)$ is a real-valued function on XxY. A saddle point of f is a point $(x_o y_o)$ such that

(1) $$\underset{y}{\text{Min}}\ f(x_o y) = f(x_o y_o) = \underset{x}{\text{Max}}\ f(x\ y_o)$$

If X and Y are compact[4/] and if f is continuous

a) $\underset{y}{\text{Min}}\ f(x\ y)$ [resp. $\underset{x}{\text{Max}}\ f(x\ y)$] is a continuous function of x (resp. y).

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The continuity of f on the compact set XxY implies uniform continuity [4, Chap. II, § 4, Th. 2]: thus to every $\mathcal{E} > 0$ corresponds $\eta > 0$ such that " $|x'-x| < \eta$ and $|y'-y| < \eta$ " implies " $|f(x'y')-f(x\ y)| < \mathcal{E}$ ".

We then prove that $|x_2-x_1| < \eta$ implies $|\underset{y}{\text{Min}}\ f(x_2y) - \underset{y}{\text{Min}}\ f(x_1y)| < \mathcal{E}$.

Take y_1 such that $f(x_1y_1) = \underset{y}{\text{Min}}\ f(x_1y)$.

$$|f(x_2y_1) - f(x_1y_1)| < \mathcal{E} \quad \text{i.e.,} \qquad f(x_2y_1) < f(x_1y_1) + \mathcal{E} \text{ i.e.,}$$

$\underset{y}{\text{Min}}\ f(x_2y) < \underset{y}{\text{Min}}\ f(x_1y) + \mathcal{E}$. One proves similarly $\underset{y}{\text{Min}}\ f(x_1y) < \underset{y}{\text{Min}}\ f(x_2y) + \mathcal{E}$.

b)[5] $\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ f(x\ y) \leq \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f(x\ y)$.

Let $A = \left\{x' : \underset{y}{\text{Min}}\ f(x'y) = \underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ f(x\ y)\right\}$ [6], $B = \left\{y' : \underset{x}{\text{Max}}\ f(x\ y') = \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f(x\ y)\right\}$.

If $x' \in A$ and $y' \in B$,

(2) $\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ f(x\ y) = \underset{y}{\text{Min}}\ f(x'y) \leq f(x'y') \leq \underset{x}{\text{Max}}\ f(x\ y') = \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f(x\ y)$.

The result follows from a comparison of the first and last terms.

c) <u>The existence of a saddle point</u> (x_oy_o) <u>implies the equality</u>

$$\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ f = \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f\ [= f(x_oy_o)].$$

From the definition (1) follows

(3) $\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ f(x\ y) \geq \underset{y}{\text{Min}}\ f(x_oy) = f(x_oy_o) = \underset{x}{\text{Max}}\ f(x\ y_o) \geq \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f(x\ y)$

which together with b) gives the result. It also gives

$\underset{x}{\text{Max}}\ f(x\ y_o) = \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f(x\ y)$ i.e., $y_o \in B$, and similarly $x_o \in A$.

d) <u>The equality</u> $\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ f = \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f$ <u>implies the existence of a saddlepoint.</u>

Assume that the equality holds and take $x_o \in A$, $y_o \in B$,

(2) gives $\quad \underset{y}{\text{Min}}\, f(x_o y) = f(x_o y_o) = \underset{x}{\text{Max}}\, f(x\, y_o)$,

which is the definition (1) of a saddle point $(x_o y_o)$. We have incidentally proved

e) the set of saddle points is either empty or equal to AxB.

2. Historical Note[7/]

Let $\quad S_n = \left\{ z \in R^n:\ z_k \geq 0 \ \text{ for } \ k = 1, \ldots, n \text{ and } \sum_{k=1}^{n} z_k = 1 \right\}$.

In his first study on the theory of games [12], J. von Neumann proved

(I) Let $f(x\, y)$ be a continuous real-valued function defined for $x \in S_\ell$ and $y \in S_m$.
If for every $x_o \in S_\ell$ and for every real number α $\left\{ y \in S_m:\ f(x_o y) \leq \alpha \right\}$ is convex, and if for every $y_o \in S_m$ and for every real number β $\left\{ x \in S_\ell :\ f(x\, y_o) \geq \beta \right\}$ is convex, then f has a saddle point.

He applied this result to the case where $f(x\, y)$ is a bilinear form $\sum_{ij} a_{ij} x_i y_j$ and thus obtained the central theorem of the theory of games. (I) was clearly a more powerful tool than this particular case required and proofs taking full advantage of the bilinearity were given by J. Ville [16], J. von Neumann and O. Morgenstern [14], Chap. III, Section 17, D. Gale [9, p. 287-297], H. Weyl [10, p. 19-25], D. Gale, H. W. Kuhn and A. W. Tucker [10, p. 81-87].

In another paper on economics [13], J. von Neumann later proved

(II) Let $X \subset R^\ell$ and $Y \subset R^m$ be two compact convex sets. If U and V are two closed subsets of XxY such that
for every $x_o \in X$, $U_{x_o} = \left\{ y \in Y:\ (x_o y) \in U \right\}$ is nonempty, closed, and convex,

<u>and for every $y_o \in Y$, $V_{y_o} = \{x \in X: (x\ y_o) \in V\}$ is non-empty, closed, and convex then U and V have a common point.</u>

He used this result to show that the ratio of two bilinear forms $\varphi_{(x\ y)} = \frac{\Sigma\ a_{ij}\ x_i\ y_j}{\Sigma\ b_{ij}\ x_i\ y_j}$ (where $x \in S_\ell$, $y \in S_m$, and for all (ij) $a_{ij} \geq o$, $b_{ij} \geq o$, $a_{ij} + b_{ij} > o$ 8/) has a saddle point. 9/ He thus established that a certain economic system actually has an equilibrium position (giving one of the few instances where an economist did not stop at a mere counting of equations and unknowns).

L. H. Loomis [11], L. L. Dines [6] later gave simpler proofs of the existence of a saddle point for $\varphi(x\ y)$ by taking full advantage of the bilinearity of numerator and demoninator.

In 1941 S. Kakutani [8] showed that (II) immediately implied

(III) <u>In the statement of (I), S_ℓ and S_m can be replaced by two arbitrary compact, convex sets $X \subset R^\ell$ and $Y \subset R^m$ respectively.</u>

He also gave a more simple proof of (II) which he made appear as a direct consequence of a fixed point theorem generalizing Brouwer's. 10/

(III) Has been used recently by M. Slater in a study [15] whose origin is the economic problem of efficient organization of production where the usual linearity assumptions [9] are replaced by certain mild convexity assumptions.

Saddle point existence theorems thus revealed themselves as powerful tools in economics and one wishes naturally to have the most general conditions under which a real-valued function of two variables has a saddle point. It is along this line that our attempt lies. It will be shown that (III) can be generalized in several directions and that, in particular, convexity assumptions are completely irrelevant.

3. Basic Topological Concepts

Two sets in R^n are said to be homeomorphic when it is possible to set up between them a one-to-one bicontinuous (h and h^{-1} continuous) correspondence h (called a homeomorphism). Two homeomorphic sets are topologically equivalent.

A convex cell C in R^n is determined by r points z^k ($k = 1, \ldots, r$); it is the set

$$C = \left\{ z : z = \sum_{k=1}^{r} \zeta_k z^k \;,\; \zeta_k \geqq 0 \text{ for } k = 1, \ldots, r, \; \sum_{k=1}^{r} \zeta_k = 1 \right\}.$$

Such a set is closed.

The product of two convex cells $A \subset R^{\ell}$ and $B \subset R^m$ is a convex cell $C \subset R^{\ell+m}$. Let A be generated by the p points x^i ($i = 1, \ldots, p$) and B by the q points y^j ($j = 1, \ldots, q$). Denote by C the convex cell in $R^{\ell+m}$ generated by the pq points (x^i, y^j).

$A \times B$, the product of two convex sets, is convex; as it contains each one of the points (x^i, y^j), $A \times B \supset C$.

On the other hand if $\xi_i \geqq 0$ for $i = 1, \ldots, p$, $\sum_{i=1}^{p} \xi_i = 1$, and $\eta_j \geqq 0$ for $j = 1, \ldots, q$, $\sum_{j=1}^{q} \eta_j = 1$, then $\xi_i \eta_j \geqq 0$ for all (ij), $\sum_{ij} \xi_i \eta_j = 1$, and

$$\sum_{ij} \xi_i \eta_j (x^i, y^j) = \left(\sum_{ij} \xi_i \eta_j x^i, \sum_{ij} \xi_i \eta_j y^j \right) = \left(\sum_i \xi_i x^i, \sum_j \eta_j y^j \right).$$

The last member is a general point of $A \times B$, the first one is a point of C, therefore $A \times B \subset C$.

Summing up, $A \times B = C$.

A geometric polyhedron is the union of a finite number of convex cells in R^n. It is clearly closed.

The product of two geometric polyhedra P, Q is a geometric polyhedron. Let $P = \bigcup_{i=1}^{p} A_i$ ($Q = \bigcup_{j=1}^{q} B_j$) where the A_i (the B_j) are convex cells in R^{ℓ} (in R^m). The relation $P \times Q = (\bigcup_i A_i) \times (\bigcup_j B_j) = \bigcup_{ij} (A_i \times B_j)$

proves the result.

A <u>polyhedron</u> is a set in R^n homeomorphic to a geometric polyhedron (called geometric antecedent of the first one). Examples of polyhedra in R^3 are a hollow torus, a solid (closed) torus.

The product of two polyhedra is a polyhedron. Let $\mathbb{P}$, $\mathbb{Q}$ be two polyhedra in R^{ℓ}, R^m, P, Q their geometric antecedents, f, g the corresponding homeomorphisms; $P \times Q$ is a geometric polyhedron and $(x\ y) \longrightarrow (f(x)\ \ g(y))$ is a homeomorphism from $P \times Q$ to $\mathbb{P} \times \mathbb{Q}$.

Let $I = \{t: \ o \leq t \leq 1\}$ denote the closed interval $[o,1]$ on the real line. A non-empty set Z of R^n is said to be <u>contractible</u>, or more precisely deformable into a point $z_o \in Z$, if there exists a continuous function $H(t,z)$ (called a deformation) taking $I \times Z$ into Z such that for all $z \in Z$ $H(o,z) = z$ and $H(1,z) = z_o$. This rigorous definition corresponds to the intuitive notion of a set which can be shrunk <u>over itself</u> to a point by a continuous deformation.

The product of two contractible sets $X \subset R^{\ell}$, $Y \subset R^m$ is contractible. Take two deformations F and G of X, Y into $x_o \in X$, $y_o \in Y$ respectively. $H[t, (x\ y)] = [F(t,x)\ G(t,y)]$ is clearly a deformation of $X \times Y$ into $(x_o\ y_o)$.

4. The Existence Theorem

The real function $f = \dfrac{e^{\varphi} - 1}{e^{\varphi} + 1}$ of the real variable φ is monotonically increasing from -1 to +1 when φ increases from $-\infty$ to $+\infty$. It establishes a one-to-one correspondence between the closed interval $[-1, +1]$ and the set $\overline{R}$ of all real numbers to which are added two elements

$-\infty$ and $+\infty$. Open sets in $\bar{R}$ are defined as images of the usual open sets in $[-1, +1]$, an order is defined in $\bar{R}$ as an image of the usual order in $[-1, +1]$. $\bar{R}$ endowed with this topology and this order is called the completed real line. From this point of view the elements $-\infty$ and $+\infty$ behave exactly like two ordinary numbers.[11] We can now state:

Th. Let $X \subset R^{\ell}$ and $Y \subset R^m$ be two contractible polyhedra and $\varphi(x\ y)$ a continuous function from $X \times Y$ to the completed real line.

If for every $x_o \in X$ the set $U_{x_o} = \left\{y \in Y:\ \varphi(x_o\ y) = \underset{y \in Y}{\mathrm{Min}}\ \varphi(x_o\ y)\right\}$ is contractible and if for every $y_o \in Y$ the set $V_{y_o} = \left\{x \in X:\ \varphi(x\ y_o) = \underset{x \in X}{\mathrm{Max}}\ \varphi(x\ y_o)\right\}$ is contractible,

then φ has a saddle point.[12]

The proof uses as a lemma a particular case of a fixed point theorem due to E. G. Begle [2] and then proceeds along lines similar to S. Kakutani's.

Let Z be a set in R^n and ϕ a function associating with each $z \in Z$ a subset $\phi_{(z)}$ of Z, the graph of ϕ is the subset of $Z \times Z$, $\left\{(z z'):\ z' \in \phi_{(z)}\right\}$. The multi-valued function ϕ is said to be continuous if its graph is closed.[13]

A fixed point of ϕ is a point z_o such that $z_o \in \phi_{(z_o)}$.

Lemma. Let Z be a contractible polyhedron in R^n and $\phi: Z \to Z$ a continuous multi-valued function such that for every $z \in Z$ the set $\phi_{(z)}$ is contractible. Then ϕ has a fixed point.[14-15]

Consider $Z = X \times Y$; this set, the product of two contractible polyhedra in R^{ℓ} and R^m, is a contractible polyhedron in $R^{\ell+m}$ (see Section 3). Define on Z the multi-valued function ϕ as follows: if $z = (x\ y)$, $\phi_{(z)} = V_y \times U_x$. Since U_x (resp. V_y) is contractible for all $x \in X$ (resp. for all $y \in Y$), $\phi_{(z)}$ is contractible for all $z \in Z$ (Section 3). To be able to apply the lemma it remains only to show that ϕ is continuous.

For this first define in $X \times Y$ two sets U and V

$$U = \left\{(x\ y):\ \ y \in U_x\right\} \quad , \qquad V = \left\{(x\ y):\ \ x \in V_y\right\} .$$

It has been shown in Section 1.a that $\underset{y \in Y}{\text{Min}}\ \varphi_{(x\ y)}$ is a continuous function of x on X; the function $\psi_{(x\ y)} = \underset{y \in Y}{\text{Min}}\ \varphi_{(x\ y)}$ is therefore continuous on XxY. The equivalent definition of $U = \left\{(x\ y):\ \ \varphi_{(x\ y)} = \psi_{(x\ y)}\right\}$ makes it appear as the set of (x y) for which two continuous functions are equal. U is thus closed. A similar argument applies to V.

The graph of ϕ is the subset of ZxZ

$\left\{(zz'):\ \ z' \in \phi_{(z)}\right\}$ i.e., the subset of (XxY) x (XxY)

$$\left\{(x\ y\ x'y'):\ \ x' \in V_y \text{ and } y' \in U_x\right\} = \left\{(x\ y\ x'y'):\ \ (x'y) \in V \text{ and } (x\ y') \in U\right\} .$$

But for the order of the terms this is UxV which is <u>closed</u>.

The conclusion of the lemma is then that there exist $z_o \in Z$ such that $z_o \in \phi_{(z_o)}$, i.e., $x_o \in X$ and $y_o \in Y$ such that $(x_o y_o) \in V_{y_o} x U_{x_o}$, i.e., $x_o \in V_{y_o}$ and $y_o \in U_{x_o}$, i.e., $\varphi_{(x_o y_o)} = \underset{x \in X}{\text{Max}}\ \varphi_{(x\ y_o)}$ and

$\varphi_{(x_o y_o)} = \underset{y \in Y}{\text{Min}}\ \varphi_{(x_o y)}$ which is the definition (1) of a saddle point $(x_o y_o)$ for φ.

A final remark can be made. Let X, Y be subsets of R^{ℓ}, R^m and $\varphi_{(x\ y)}$ a continuous function from XxY to the completed real line. Denote by $\bar{X}$, $\bar{Y}$ the closures of X, Y; the closure of XxY is easily seen to be $\bar{X}x\bar{Y}$. If for any $z \in \bar{X}x\bar{Y}$ $\varphi_{(x\ y)}$ has a limit when $(x\ y) \in XxY$ tends to z there exists a unique continuous function $\bar{\varphi}$ on $\bar{X}x\bar{Y}$ which coincides with φ on XxY [4, Chap. I, § 6, Th. 1]. If moreover $\bar{X}$,$\bar{Y}$ are contractible polyhedra and if the conditions imposed by the existence theorem on $\bar{\varphi}$ are satisfied

$$\underset{x \in \bar{X}}{\text{Max}}\ \underset{y \in \bar{Y}}{\text{Min}}\ \bar{\varphi} = \underset{y \in \bar{Y}}{\text{Min}}\ \underset{x \in \bar{X}}{\text{Max}}\ \bar{\varphi}, \quad \text{which implies}$$

$$\underset{x \in X}{\text{Sup}}\ \underset{y \in Y}{\text{Inf}}\ \varphi = \underset{y \in Y}{\text{Inf}}\ \underset{x \in X}{\text{Sup}}\ \varphi .$$ [16]

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Footnotes

1. Acknowledgment is due to Cowles Commission staff members and their guests, in particular to I.N. Herstein and J. Milnor. I am indebted to J.L. Koszul and D. Montgomery for references [7] and [2]. I also had the privilege of consulting with S. MacLane and A. Weil on the contents of [7].

2. But for some parts of footnote (14) the exposition is to a large extent elementary and self-contained.

3. Section 2 contains a more developed introduction to the subject.

4. In R^n a compact set is a closed, bounded set.

5. Results b–e have already been proved in [14], Chap. III, Section 13.

6. $\{z: \ P\}$ denotes the set of z which have the property P.

7. The reading of this rather detailed historical note is not necessary for the understanding of the following sections.

8. All the products $x_i y_j$ are nonnegative, they add to unity, one of them $x_{i'} y_{j'}$ is therefore positive. If $a_{i'j'} > 0$ (resp. $b_{i'j'} > 0$) the numerator (resp. the denominator) is positive. As $a_{i'j'} > 0$ and/or $b_{i'j'} > 0$ an indeterminacy of the form $\frac{0}{0}$ cannot occur for φ. However φ may tend to $+\infty$.

9. This result looks more general than the existence of a saddle point for a bilinear form; it is actually equivalent to it.

a) If it is known that $\varphi_{(x\ y)}$ has a saddle point, to prove that $\Sigma c_{ij} x_i y_j$ also has one, take for all (ij) $b_{ij} = 1$

and $a_{ij} = \alpha + c_{ij}$ (this makes φ equal to $\alpha + \Sigma c_{ij} x_i y_j$), where α is chosen such that for all (ij) $a_{ij} \geqq 0$.

b) Conversely assume that $f_{(x\ y)} = \Sigma c_{ij} x_i y_j$ has a saddle point.

$$\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ \varphi \leqq \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ \varphi \qquad \text{(Section 1.b)}$$

and we wish to prove that equality holds. Suppose that it does not and choose λ such that (1) $\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ \varphi < \lambda < \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ \varphi$. (1) is equivalent to: for any x there exists a y such that $\varphi_{(x\ y)} < \lambda$ i.e., such that $\Sigma c_{ij} x_i y_j < 0$ (where $c_{ij} = a_{ij} - \lambda b_{ij}$) and similarly for any y there exists an x such that $\Sigma c_{ij} x_i y_j > 0$. This means in turn that

$$\underset{x}{\text{Max}}\ \underset{y}{\text{Min}}\ f < 0 < \underset{y}{\text{Min}}\ \underset{x}{\text{Max}}\ f \text{, a contradiction.}$$

10. The fixed point theorem:

(II') Let Z be a compact convex set in R^n and $z \rightarrow \phi_{(z)}$ a continuous function associating with each z in Z a nonempty closed convex subset $\phi_{(z)}$ of Z, then there exists a $z_o \in Z$ such that $z_o \in \phi_{(z_o)}$.

The sense in which ϕ is continuous is defined later in this footnote.

The equivalence of (II) and II') is easily seen.

a) (II') $\longrightarrow$ (II) is proved very simply in [8].

b) Let us therefore prove that (II) $\longrightarrow$ (II'). Consider in ZxZ two sets U, V defined as follows: $U = \{(z\ z'): \ z' \in \phi_{(z)}\}$; U, the graph of ϕ, is closed (this is the very definition of the continuity of ϕ). $V = \{(z\ z'): \ z' = z\}$; V is clearly closed. All the assumptions of (II) are satisfied: U and V intersect. Let $(z_o z_o') \in U \cap V$: obviously $z_o \in \phi_{(z_o)}$.

11. The closed finite interval and the continuous monotonic function chosen to define the completed real line are obviously irrelevant and a direct definition can be given [5]. However it may be helpful to keep in mind the ismorphism

$$f = \frac{e^{\varphi} - 1}{e^{\varphi} + 1} \text{ between } [-1, +1] \text{ and } \bar{R}.$$

The substitution of the completed real line $\bar{R}$ for the real line R is very natural in a study of this problem; it is necessary if one wants to account for cases where $\varphi_{(x\ y)}$ tends to $-\infty$ or $+\infty$ (see for example the ratio of two bilinear forms of Section 2).

All the results of Sections 1-2 hold if the set R in which the function takes its values is replaced by $\bar{R}$.

12. Note that the compactness of X and Y and the continuity of φ imply that U_{x_o} and V_{y_o} are nonempty and compact.

13. This continuity implies that $\phi_{(z)}$ is closed for every z in Z.

14. The statement [2, p. 546] of E. G. Begle is much more general.

Definitions. The coefficient group is an arbitrary field. An acyclic set has the same homology groups as a set consisting of one point. A compact space is defined as in [4, Chap. I, § 10, Definition 1], an $\mathcal{L}c$ (homology locally connected) space as in [2, p. 544].

Th. Let Z be an acyclic compact $\mathcal{L}c$ space and ϕ: Z $\longrightarrow$ Z a continuous multi-valued function such that for every $z \in Z$ the set $\phi_{(z)}$ is acyclic. Then ϕ has a fixed point.

(1) An absolute neighborhood retract (the definition of an ANR is that of [3, p. 222]) is a compact $\mathcal{L}c$ space [2, p. 544],

one obtains, therefore, as a particular case the fixed point theorem of S. Eilenberg and D. Montgomery [7, p. 215].

The lemma is a further particular case since
(2) a contractible set (in the sense of Section 3) is acyclic,
(3) a polyhedron is an ANR (a simple characterization (7) of an ANR is given below for finite dimensions).

Only in order to have a simple statement of the lemma did we accept this loss of generality.

One naturally wonders if the best compromise between simplicity and generality has been picked here. One might think for example of limiting oneself to the case where X and Y (and therefore Z) are closed cells (a closed cell is homeomorphic to the unit cube $\{z: |z_k| \leq 1 \text{ for } k = 1, \ldots, n\}$ of some Euclidean space R^n: it is a very simple case of contractible polyhedron) but this would rule out the possibility for X and/or Y to be made of several parts of different dimensions and it does not seem unlikely that such cases might arise in economics.

On the other hand a more general saddle point theorem can be obtained. From (1), (2),
(4) "absolute retract" (as defined in [3, p. 222]) is equivalent to "contractible A.N.R." [3, p. 229],
and the fixed point theorem, follows the new
<u>Lemma* Let Z be an absolute retract and ϕ: Z ⟶ Z a continuous multi-valued function such that for every $z \in Z$ the set $\phi_{(z)}$ is contractible. Then ϕ has a fixed point.</u>

From there a proof entirely analogous to that of Section 4 can be built to establish the new

<u>Th.* Let X and Y be two absolute retracts and $\varphi_{(x\ y)} \cdots$ then φ has a saddle point.</u>

It is sufficient to remark that
(5) the product of two absolute retracts is an absolute retract [1, p. 197],
(6) all definitions, proofs and results of Section 1 hold if X and Y are two arbitrary compact metric spaces.

Finally it may be useful to recall that
(7) for finite dimensions "ANR" is equivalent to "locally contractible (as defined in [3, p. 235-236]) compact metric space" [3, p. 240].

15. We showed in Section 2 how two problems in economics gave rise to J. von Neumann's generalization (II) of Brouwer's fixed point theorem. This led to S. Kakutani's article which in turn led to S. Eilenberg and D. Montgomery's, and then to E.G. Begle's papers. Their powerful fixed point theorems are striking examples of valuable contributions to one of the most abstract kinds of mathematics whose origin can be traced directly to economics.

16. The existence theorem* of footnote (14) generalizes, in several directions, the central theorem of the theory of games (existence of a saddle point for a bilinear form defined on the fundamental simplexes S_ℓ, S_m of two finite Euclidean spaces R^ℓ, R^m). This result has been extended in a different direction in the note by J. Ville [16], in the work of A. Wald (see [17] and his papers quoted in his bibliography [17, p. 171-172]), and in Part II of [10].