Quadratic Forms Definite Under Linear Constraints

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This paper offers a proof of a theorem whose applications in the classical theories of economic equilibrium are numerous.

$x, A, B$ are matrices of orders n.1, n.n, n.m. $x$ being a matrix,

$M_{p,q}$ is obtained from $M$ by keeping only the elements in the first $p$ rows and the first $q$ columns; $M_p$ stands for $M_{p,p}$. Primed letters denote transposes.

Th. 1 $x'Ax > 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if there exists a number $\lambda$ such that $x'Ax + \lambda x'B'x$ is a positive definite quadratic form.

It is sufficient.

Now, the function $y(x) = \frac{x'Ax}{x'B'B'x}$ is continuous on the set $\{x: x'x = 1$ and $B'x \neq 0\}$, and tends to $-\infty$ whenever $x$ tends to a boundary point; it has therefore a finite maximum $\lambda^*$. Any $\lambda > \lambda^*$ has the desired property.

Th. 2 $|A + \lambda BB'|$ is a polynomial in $\lambda$ whose term of highest order (possibly null) is $(-1)^m \begin{vmatrix} A & B \\ B' & 0_{m,m} \end{vmatrix} \lambda^m$.

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From \[
\begin{bmatrix}
A & \lambda B \\
B' & -I_m
\end{bmatrix}
\begin{bmatrix}
I_n & 0_{n\times m} \\
B' & I_m
\end{bmatrix}
= \begin{bmatrix}
A + \lambda BB' & \lambda B \\
0_{m \times n} & -I_m
\end{bmatrix}
\] follows
\[
A, \lambda B = (-1)^m A + \lambda BB'
\]

In the development of the left-hand determinant a term contains the highest possible power of $\lambda$ if in every one of the last $m$ columns one takes an element of $\lambda B$. Such terms are unaffected if $-I_m$ is replaced by any other $m \times m$ matrix: take $0_{m \times m}$.

Th. 3 Let $A$ be symmetric and $|B_{m\times m}|$ be different from zero. $x'Ax > 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if

\[
(-1)^m \begin{vmatrix}
A & B_{r \times m} \\
B' & 0
\end{vmatrix} > 0 \quad \text{for } r = m+1, \ldots, n.
\]

1) Necessity: Consider the equations

\[
\begin{cases}
Ax + B\xi = 0 \\
B'x = 0
\end{cases}
\]

where $\xi$ is an $m \times 1$ matrix. A solution $\begin{bmatrix}x \\ \xi \end{bmatrix}$ is such that $x'Ax + x'B\xi = 0$, i.e., $x'Ax = 0$. This must imply $x = 0$, therefore $B\xi = 0$, and, since $|B_{m \times m}| \neq 0$, $\xi = 0$. The system must have no other solution than 0, i.e.,

\[
\begin{vmatrix}
A & B \\
B' & 0
\end{vmatrix} \neq 0.
\]

From th. 1, for every $\lambda > \lambda^*$ one must have $|A + \lambda BB'| > 0$.

From th. 2 one must have $(-1)^m \begin{vmatrix}A & B \\
B' & 0
\end{vmatrix} > 0$.

This argument can be made for any $r$, $m \leq r \leq n$.

2) Sufficiency: I shall prove that the coefficient of the term of highest order in $\lambda$ of $|A_{r \times r} + \lambda B_{r \times m} B'_{r \times m}|$ is positive whatever be $r \leq n$.

It will therefore be possible to choose $\lambda$ large enough to make these $n$ leading minors positive and consequently $[A + \lambda BB']$ positive definite.

a) If $r > m$, it is true by assumption.

b) If $r \leq m$, the development technique used in the proof of th. 2 shows that every term of $|A_{r \times r} + \lambda B_{r \times m} B'_{r \times m} + I_m|$ of order higher than $r$ vanishes.
The $r$th order term is $\sum (-1)^{m-r} \left| \begin{array}{c} A_r \\ \tilde{B}_r \\ -B_r \end{array} \right| \lambda^r$ where $\tilde{B}_r$ is any $r\times r$ submatrix of $B_{rm}$ whose columns are in the natural order.

This term is equal to $(-1)^m \lambda^r \sum |\tilde{B}_r|^2$, and finally the coefficient of $\lambda^r$ in $|A_r + \lambda B_{rm} \tilde{B}_r|_2$ is $\sum |\tilde{B}_r|^2$ which cannot vanish since $|B_{mn}| \neq 0$.

A similar argument proves

**Th. 4** Let $A$ be symmetric and $|B_{mn}|$ be different from zero. $x'Ax < 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if

$$(-1)^r \left| \begin{array}{c} A_r \\ B_{rm} \\ \tilde{B}_r \end{array} \right| > 0$$

for $r = m+1, \ldots, n$. 