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# A Theorem on Characteristic Roots

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The purpose of this note is to prove, in a simple fashion, a theorem which is in the same vein as that proved by Slater in C.C.D.P. Mathematics 404. One might add here that both Slater's (or more properly, Chipman's) result, as well as the result of this paper, are very special cases of theorems by Frobenius in his paper "Über Matrizen aus nicht negativen Elementen" in the Sitzungsberichten der Preussichen Akademie der Wissenschaften, in 1912. The proof given now is quite simple and uses a result due to Debreu in his paper C.C.D.P. Mathematics 405 where it is proved in a very simple and elementary way.

Let $A=(a_{ij})$ denote henceforth a square non-negative matrix $(a_{ij}\geq 0)$. Let $S_i = \sum_j a_{ij}$. We write $\operatorname{Max}_i S_i = \|A\|$.

The following can be readily verified:

1. $\|A^m\| \leq \|A\|^m$ $\{$in fact $\|AB\| < \|A\| \; \|B\|\}$ for all positive integers $m$.
2. If $\lambda$ is a characteristic root of $A$, $|\lambda| \leq \|A\|$.
3. If $\lambda$ is a characteristic root of $A$, then $\lambda^m$ is a characteristic root of $A^m$.

Debreu's theorem runs as follows:

Let $A=(a_{ij})$, $a_{ij} \geq 0$, $a_{ii} > 0$ for all i,j. Suppose that $\lambda = \|A\|$ is a characteristic root of A. Then if $\alpha$ is any characteristic root of A such that $|\alpha| = \|A\|$ then $\alpha = \lambda$.

<u>Definition</u>: $z(A)$ = number of $a_{ii}$ which are zero.

<u>Definition</u>: If $A=(a_{ij})$, $A^t = (a_{ij}^{(t)})$

<u>Lemma 1</u>: If $A=(a_{ij})$, $a_{ij} \geq 0$, then for all positive integers t, $z(A^t) \leq z(A)$. Moreover, $a_{ii}^{(t)} = 0$ only if $a_{ii} = 0$.

<u>Proof</u>: $a_{ij}^{(t)} = \sum_{v_1 v_2 \ldots, v_{t-1}} a_{iv_1} a_{v_1 v_2} a_{v_2 v_3} \ldots a_{v_{t-1} j}$

Thus $a_{ii}^{(t)} = a_{ii}^t$ + non-negative terms.

So $a_{ii}^{(t)} = 0$ only if $a_{ii}^t = 0$, that is, if $a_{ii} = 0$.

This proves the lemma.

<u>Lemma 2</u>: If A is a non-singular, non-negative matrix, and $z(A) > 0$, then for some $t > 0$, $z(A^t) < z(A)$.

<u>Proof</u>: Suppose for all integers $t > 0$, $z(A^t) = z(A)$. Since A is a non-singular, it satisfies a polynomial

(1) $A^m + \lambda_1 A^{m-1} + \ldots + \lambda_m = 0$, $\lambda_i$ are real, $\lambda_m \neq 0$.

Since $z(A^t) = z(A)$ for all t, $a_{ii} = 0$ implies $a_{ii}^{(t)} = 0$ for all t.

Thus (1) gives us

$a_{ii}^{(m)} + \lambda_1 a_{ii}^{(m-1)} + \lambda_{m-1} a_{ii} + \lambda_m = 0$ and since each $a_{ii}^{(t)} = 0$, $\lambda_m = 0$, a contradiction. Whence the theorem has been proved.

We are now in a position to prove the theorem on characteristic roots for which we set out.

<u>Theorem</u>: Let $A$ be a non-singular, non-negative matrix with $\|A\| = 1$. Suppose that $\lambda$ is a characteristic root of $A$ with $|\lambda| = 1$. Then $\lambda$ must be a root of unity.

<u>Proof</u>: Since $\lambda$ is a characteristic root of $A$, $\lambda^s$ is a characteristic root of $A^s$. Thus $1 = |\lambda|^s = |\lambda^s| \leq \|A^s\| \leq \|A\|^s = 1$. So $\|A^s\| = 1$ for all positive integers $S$. $A^s$ is certainly non-negative and non-singular and has a characteristic root, $\lambda^s$, with $|\lambda^s| = 1$, that is $A^s$ has the same conditions on it as $A$ does. Pick a $t$ for which $z(A^t) < z(A)$ (lemma 2). Reapply the same argument to $A^t$. We eventually get an $A^r$, non-singular, non-negative, $\|A^r\| = 1$, $A^r$ having $\lambda^r$ as a characteristic root where $|\lambda|^r = 1$ and where $z(A^r) = 0$. Thus all the conditions of Debreu's theorem are fulfilled, so the only characteristic root, $\alpha$, of $A^r$ with $|\alpha| = 1$ must be $\alpha = 1$. Since $\lambda^r$ is such an $\alpha$, $\lambda^r = 1$ and $\lambda$ is an $r^{\underline{th}}$ root of unity.