Characteristic Roots of Non-Negative Matrices

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The purpose of this note is to give an elementary proof of theorems closely related with the Theorem (3) stated by J. Chipman (Econ. 1916) and proved by M. Slater (Math. 404).

A matrix $M = \begin{bmatrix} a_{i,j} \end{bmatrix}$ is said to be non-negative if all its elements are non-negative. We shall denote by $s_i$ the sum of the elements of the $i$th row $s_i = \sum_j a_{i,j}$.

Theorem 1. Given a non-negative square matrix, for every characteristic root $\lambda$, $|\lambda| \leq \max_i s_i$.

Proof: If $\lambda$ is a characteristic root, $\exists$ a non-zero vector $x = \begin{bmatrix} x_1 \\ x_h \\ x_n \end{bmatrix}$ such that $Mx = \lambda x$. Let $x_h$ be a component $x$ with the largest modulus: $\sum_j a_{h,j} x_j \leq s_h x_h$.

$$|\lambda x_h| \leq \sum_j a_{h,j} |x_j| \leq s_h |x_h|$$

i.e. $|\lambda| \leq s_h \leq \max_i s_i$.

Theorem 2. Given a non-negative square matrix with positive diagonal elements, if a characteristic root $\lambda$ has a modulus equal to $\max_i s_i$, there is one and only one characteristic root having
this property namely \( \text{Max} s_i \).

Proof: \( |\lambda| = \text{Max}_i s_i \), let \( x \) be a characteristic vector
associated with \( \lambda \), and \( x_h \) a component of \( x \) with the largest
modulus,

\[
\sum_j a_{hj} x_j = \lambda x_h
\]

and, as seen in the proof of theorem 1, \( |\lambda| \leq s_h \)
therefore \( s_h = \text{Max}_i s_i \).

From (1), (2)

\[
\sum_j \frac{a_{hj}}{s_h} x_j = \frac{\lambda}{s_h} x_h \quad \text{and} \quad a_{hh} \neq 0
\]

In the complex plane, the point \( x_h \) is on the circle of center
0 and radius \( |x_h| \), every other point \( x_j \) is inside or on this
circle. The center of gravity of \( x_h \) (with a positive mass \( \frac{s_{hh}}{s_h} \)) and
of these other points \( x_j \) (with masses \( \frac{s_{hj}}{s_h} \)) will therefore be inside
the circle (and this would contradict (2) since \( \frac{|\lambda|}{s_h} = 1 \)) unless all
the \( x_j \) which have a non-zero mass are equal to \( x_h \). But then it is
possible to divide both members of (2) by \( x_h \) and we are left with

\[
\frac{\lambda}{s_h} = 1.
\]

This theorem 2 is different from the theorem proved by \( M \).

Slater in the following respect: I assume that \( a_{ii} > 0 \) for every
\( i \) where he assumes only that \( a_{ii} \geq 0 \); as a consequence of this
more restrictive assumption I can prove that if for a given character-
istic root \( \lambda \), \( |\lambda| = \text{Max}_i s_i \), \( \text{Max}_i s_i \) is the only
characteristic root whose modulus is equal to \( \text{Max}_i s_i \).