CHARACTERISTIC ROOTS OF NON-NEGATIVE MATRICES

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The purpose of this note is to give an elementary proof of theorems closely related with the Theorem (3) stated by J. Chipman (Econ. 2016) and proved by M. Slater (Math. 404).

A matrix $M = \left[a_{ij}\right]$ is said to be non-negative if all its elements are non-negative. We shall denote by s_i the sum of the elements of the i^{th} row $s_i = \sum_j a_{ij}$.

Theorem 1 Given a non-negative square matrix, for every characteristic root λ, $|\lambda| \leqq \underset{i}{\text{Max}}\ s_i$

Proof: If λ is a characteristic root, $\exists$ a non-zero vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_h \\ \vdots \\ x_n \end{pmatrix}$ such that $Mx = \lambda x$. Let x_h be a component x with the largest modulus: $\sum_j a_{hj} x_j = \lambda x_h$

$$\left|\lambda x_h\right| \leqq \sum_j a_{hj} \left| x_j \right| \leqq s_h \left|x_h\right|$$

i.e. $|\lambda| \leqq s_h \leqq \underset{i}{\text{Max}}\ s_i$

Theorem 2 Given a non-negative square matrix with positive diagonal elements, if a characteristic root λ has a modulus equal to $\underset{i}{\text{Max}}\ s_i$, there is one and only one characteristic root having this property namely $\underset{i}{\text{Max}}\, s_i$.

Proof: $|\lambda| = \underset{i}{\text{Max}}\, s_i$, let x be a characteristic vector associated with λ, and x_h a component of x with the largest modulus,

$$(1) \qquad \sum_j a_{hj}\, x_j = \lambda x_h$$

and, as seen in the proof of theorem 1, $|\lambda| \leqq s_h$, therefore $s_h = \underset{i}{\text{Max}}\, s_i$.

From (1), $$(2) \qquad \sum_j \frac{a_{hj}}{s_h} x_j = \frac{\lambda}{s_h} x_h. \quad \text{and } a_{hh} \neq 0$$

In the complex plane, the point x_h is on the circle of center 0 and radius $|x_h|$, every other point x_j is inside or on this circle. The center of gravity of x_h $\left(\text{with a } \underline{\text{positive}} \text{ mass } \frac{a_{hh}}{s_h}\right)$ and of these other points x_j $\left(\text{with masses } \frac{a_{hj}}{s_h}\right)$ will therefore be <u>inside</u> the circle $\left(\text{and this would contradict (2) since } \frac{|\lambda|}{s_h} = 1\right)$ unless all the x_j which have a non-zero mass are equal to x_h. But then it is possible to divide both members of (2) by x_h and we are left with $\frac{\lambda}{s_h} = 1$.

This theorem 2 is different from the theorem proved by M. Slater in the following respect: I assume that $a_{ii} > 0$ for every i where he assumes only that $a_{ii} \geqq 0$; as a consequence of this more restrictive assumption I can prove that if for a given characteristic root λ, $|\lambda| = \underset{i}{\text{Max}}\, s_i$, $\underset{i}{\text{Max}}\, s_i$ is the <u>only</u> characteristic root whose modulus is equal to $\underset{i}{\text{Max}}\, s_i$.