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On The Solution to the "Fundamental Equation" of Inventory Theory [1/2/]

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The mathematical theory of the inventory problem has attained a certain completeness in recent years so that further results must depend on specialization. From the economic point of view one of the first questions would undoubtedly be what the solutions look like in cases typical of economic applications. There has been little study, however, of the inventory problem as formulated for an indefinite number of time periods, assuming "stationarity and independence". Thus it has even been an open question under what conditions the solution is of the so called (s,S) type.[3/] By this one means an ordering behavior reminiscent of a thermostat: whenever stocks drop below a critical level, they are filled up to another fixed level. In this paper we shall show

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3/ The question of when an (s,S) policy is optimal for a one period problem has been studied in [4]. For problems somewhat similar to ours see [1], [2], and [5].

that such an (s,S) policy is indeed optimal, whenever 1) quantity demanded (at given prices) in different periods is a random variable subject to independent and identical frequency distributions which have a differentiable density everywhere; 2) that the ordering cost consists of a fixed cost plus a cost proportional to the amount ordered; and 3) that the loss due to unfilled demand is proportional to the unfilled demand. With the solution known to be of this type, it may still be a difficult problem to determine the values of the two limiting stock levels s and S. We shall show that the inventory equation may be solved explicitly for simple types of distributions such as the negative exponential and Gamma distributions, though practical considerations may limit the usefulness of this approach to the former alone. Finally we shall give a numerical example based upon a case study of the warehousing problem of a container manufacturer that one of us (Muth) has made. We shall not restate here the elements of the existing theory on the inventory problem, rather we will start out with an exposition in economic terms of the well known inventory equation. [cf. [3], p. 205, eq. 4.6].

1. Formulation of the Problem. Assumptions and Notation

Suppose a business firm has to decide at the beginning of each time period whether to order a certain commodity or not, and if so in what quantities.

1.1. The quantities disposed of in different time periods are random variables, independently and identically distributed. We shall require that the distribution $F(x)$ have a density to be denoted by $f(x)$. Thus $dF(x) = f(x)\,dx$. Particular distributions considered will be the negative exponential distribution

$$(1) \qquad f(x) = \frac{1}{m} e^{-\frac{x}{m}}$$

where $m > 0$ denotes the mean, and the amma distribution

(2) $$f(x) = \frac{j^k}{\Gamma(k)} x^{k-1} e^{-jx},$$ for k a positive integer.

If we take $k = 1$ and $j = \frac{1}{m}$, (2) reduces to (1).

1.2. For reasons which appear later we shall denote the amount ordered at the beginning of a period by $y - x, y$ denoting the stock after receipt of the commodity shipment and x the stock before. As we assume that the time between placement of the order and receipt of the shipment is short relative to the length of the period (at the beginning of which the order is contemplated) we may say also that x is the initial stock and y the starting stock.

The reason that an inventory is carried at all even though shipments may be had at fairly short notice might be of course that there is a certain cost of ordering in excess of a proportional amount which may be directly imputed to the commodity cost. It may be, however, that physical or other limitations prevent ordering except at certain points in time in some cases. We shall assume that the indivisible cost element is a fixed cost, so that total ordering cost is of the form

(3) $$k(x,y) = B\,\delta(y - x) + b(y - x)$$

where $\delta(u)$ the Kronecker function, is defined to be

(4) $$\delta(y) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ if } u \begin{Bmatrix} > \\ \leq \end{Bmatrix} 0.$$

1.3. During each period there will arise a certain cost of carrying the stock and an average loss from unfilled orders. (Both losses and costs will be termed "the loss" henceforth). The cost of carrying stock per unit of time will be assumed proportional to the quantity of stock. The expected value of the quantity of stock at any moment of time during the period is in turn proportional to the starting stock (though with a proportionality factor that varies for different time points). The total cost of carrying stock over the period is therefore itself proportional to the starting stock = ry, say. On the assumption that during the period itself demand is distributed independently from time to time r can be calculated from the carrying cost per unit of time and stock and the average rate of stock decrease = the mean of the demand distribution / the length of the period.

The probability that a shortage of $z - y$ units of stock arises during this period is $f(z)$. Thus the expected value of shortage is $\int_y^\infty (z-y)\, dF(z)$

If we assume a constant penalty per unit of short supply, p, say, the total loss <u>during</u> the period under consideration is

$$(5) \qquad h(y) = ry + p \int_y^\infty (z-y)\, dF(z)$$

In order that it should pay at all to hold stock we must have that

$$(6) \qquad p > r + b$$

for otherwise it would be cheaper to just incur the penalty.

1.4. At the end of the period a certain stock will remain which is either positive $= y-z$, if demand fell short of starting stock, $z < y$, or zero otherwise. The probability that a stock $y - z$ remains is $dF(z)$, and that a stock 0 remains is $1 - F(y)$. Since we assume "stationarity and independence", that is, that demands in different periods are independently and identically distributed random variables and that all loss parameters are unchanging over time, the problem will repeat itself exactly one period hence, except that we will have a different initial stock.

We then introduce the function $\ell(x)$ which denotes the loss over <u>all</u> future periods that results from an <u>initial</u> stock x if the optimal policy is applied in all periods. From [3], pp. 205-208, we know this exists and is unique. Starting from the next period this loss is $\ell(0)$ with probability $1 - F(y)$ and $\ell(y - z)$ with probability $dF(z)$ according to the previous distribution. Adding up the expected loss during the period under consideration and for all following periods and discounting the latter with a discount factor a, $0 \leq a < 1$, we have

$$(7) \qquad h(y) + a\,\ell(0)\,(1 - F(y)) + a \int_0^y \ell(y - z)\, dF(z)$$

for the expected loss resulting from a <u>starting</u> stock y. The loss associated with an <u>initial</u> stock x is obviously the minimum with respect to y of this expression plus ordering cost $k(x,y)$, with the provision that $y \geq x$.

$$(8) \qquad \ell(x) = \inf_{y \geq x} \left\{ k(x,y) + h(y) + \ell(0)\,(1 - F(y)) + a \int_0^y \ell(y - z)\, dF(z) \right\}$$

1.5. This is the fundamental inventory equation for an unlimited number of periods. In the present case it reads explicitly

$$(9) \qquad \ell(x) = \inf_{y \geq x} \left\{ B\,\delta(y - x) + b(y - x) + ry + p \int_y^\infty (z - y)\, dF(z) \right.$$
$$\left. + a\,\ell(0)\,(1 - F(y)) + a \int_0^y \ell\,(y - z)\, dF(z) \right\}$$

If the loss function $\ell(x)$ has been found this equation may be used in turn to determine y as a function of x, that is the ordering policy.

One simple type of an ordering policy is the following: If initial stock is less than or equal to an amount s, order up to S, where $0 \leq s \leq S$. Otherwise don't order. Thus

$$(10) \qquad y = \begin{Bmatrix} S \\ x \end{Bmatrix} \text{ if } \quad x \begin{Bmatrix} \leq \\ > \end{Bmatrix} s.$$

In terms of the loss function $\ell(x)$ this so-called (s,S) policy means that

$$(11) \qquad \ell(x) \begin{Bmatrix} = \\ < \end{Bmatrix} B + b(S - x) + \ell(S) \text{ if } \quad x \begin{Bmatrix} \leq \\ > \end{Bmatrix} s, \quad \text{and} \quad x \leq S.$$

Equation (11) arises in the following way: If $x \leq s$, the (s,S) policy says order up to S, that is, an amount $S - x$. But then, the sum of discounted expected losses in all future periods is $\ell(S)$, since this depends only upon the starting stock, and the cost of ordering, $B + b(S - x)$. Hence $\ell(x) = B + b(S - x) + \ell(S)$. Now, for $s < x \leq S$, the policy says don't order. But if we do not order the optimal positive amount $S - x$ it must be that it is "cheaper" to order nothing, that is, that $\ell(x) < B + b(S - x) + \ell(S)$.

In the following section we shall find sufficient conditions for an $\ell(x)$ with this property to be a solution to equation (9).

2. Proof of optimality of an (s,S) policy.

Let a solution $\chi(x)$ be known and substituted in the right hand side of (9). Then this becomes a function of x and y alone and the equation may be written for shortness

$$\chi(x) = -bx + \inf_{y \geq x} (B\,\delta(y-x) + g(y)) \tag{12}$$

where

$$g(y) = by + h(y) + a\,\chi(0)\,(1 - F(y)) + a\int_0^y \chi(y-z)\,d\,F(z). \tag{13}$$

By virtue of the δ function (12) simplifies to

$$\chi(x) = -bx + \mathrm{Min}\,(g(x),\ B + \inf_{y \geq x} g(y)) \tag{14}$$

2.1. Suppose first that

$$g(0) \leq B + \inf_{y \geq 0} g(y) \tag{15}$$

Then, according to (12), (13), and (5)

$$\chi(0) = g(0) = h(0) + a\,\chi(0) = p\int_0^\infty z\,dF(z) + a\,\chi(0)$$

or

$$\chi(0) = \frac{1}{1-a}\,p\int_0^\infty z\,d\,F(z)$$

The assumption (15) therefore implies

$$\frac{1}{1-a}\,p\int_0^\infty z\,d\,F(z) \leq B + \inf_y \chi(y) \tag{16}$$

This means that the penalty for not meeting any demand is less than the smallest loss that can be achieved with an inventory. The optimal policy is therefore not to hold any stocks which may also be described by

$$s = S = 0.$$

2.2. We turn to the more interesting case in which

$$(17) \qquad g(0) > B + \inf_{y \geq 0} g(y) = B + g(S)$$

say, where $S > 0$ (and possibly infinite). By its definition (13) $g(x)$ is continuous. Therefore $g(x) \geq B + g(S)$ for all x of some interval $0 \leq x \leq s$. In general there will then be a sequence of intervals with endpoints $s_0, s_1, \ldots s_m$ such that

$$(18) \qquad \begin{aligned} g(x) &\geq B + g(s) \qquad s_{2n-1} \leq x \leq s_{2n} \in I_1 \quad \text{, say,} \\ g(x) &\leq B + g(s) \qquad s_{2n} \leq x \leq s_{2n+1} \in I_2, \quad \text{say,} \end{aligned}$$

where $n = 1, 2, \ldots.$ and $s_0 = 0$. However we need not consider intervals beyond any point $s_m > S$. For starting from $x = 0$ one shall never have a inventory $x > S$.

From (14) one sees that

$$\begin{aligned} &(19a) \\ &(19b) \end{aligned} \qquad \not{l}(x) = \begin{cases} -bx + B + g(S) & x \in I_1 \\ h(x) + a\,\not{l}(0)\,(1 - F(x)) + a \int_0^x \not{l}(x - z)\, dF(z) & x \in I_2 \end{cases}$$

<u>Theorem</u> Suppose that $0 \leq a < 1$, $p \geq r$, and that $F(x)$ is a twice differentiable distribution function. Then for any $\not{l}(x)$ defined by (19) and any $h(x)$ defined by (5), the function $g(x)$ defined by (13) is convex.

Proof: Notice that (19b) may be rewritten

$$(20) \qquad \ell(x) = h(x) + a\,\ell(0)\,(1 - F(x)\,) + a \int_0^x f(x - z)\,\ell(z)\,dz \quad x \in I_2$$

The right hand side is differentiable, hence $\ell(x)$ is in the interior of I_2. Differentiation of (19b) is therefore permissible and yields

$$(21) \qquad \ell'(x) = h'(x) + a \int_0^x f(x - z)\,\ell'(z)\,dz \qquad x \in \overset{\circ}{I}_2$$

Since h' and f are differentiable, so is ℓ' again and we obtain

$$22 \qquad \ell''(x) = h''(x) + a\,\ell(0)\,f(x) + \int_0^x \ell''(x - z)\,f(z)\,dz$$

Both (20) and (21) are valid in the interior of I_2. At endpoints s_m right derivatives and left derivatives exist toward the interior of I_2. By (5) $h''(x) + a\,\ell(0)\,f(x) = [p + a\,\ell'(0)]\,f(x)$, but by (17), (19a), and (5) $\ell'(0) = -b$ so that

$$(23) \qquad h''(x) + a\,\ell'(0)\,f(x) = [p - ab]\,f(x) \geq 0$$

From (19a) it follows that $\ell'' = 0$ for $x \in I_1$. In the equation (22) only those $l''(x)$ occur therefore under the integral for which $x \in I_2$. Now $l''(0) = 0$, the terms preceding the integral in (22) are non-negative by (23), and $f(x)$ is non-negative by definition. It follows recursively that $l''(x) \geq 0$ for all x in the interior of I_2. (This argument can be made rigorous by constructing the convergent and non-negative Neumann series to this Volterra integral equation). In the interior of I_1 $\ell'' = 0$ as we have seen. From (13) it now follows by differentiation that $g''(x)$ exists and is non-negative in the interior of I_1 and I_2. To prove that $g(x)$ is convex it suffices to show that in all endpoints s_m the left hand and right hand derivatives of $g(x)$ agree. Now

$$(24) \qquad g'_+ - g'_- = [b + h' - a\, \ell(0)\, f(x) + a \int_0^x f'(x - z)\, \ell(z)\, dz]_{s_m - 0}^{s_m + 0} = 0$$

since the expression in brackets is continuous. This concludes the proof of the theorem.

From the convexity of $g(y)$ and from equation (14) it now follows that only one interval exists in I_2, say $s \leq x \leq \bar{s}$. Since $S \in I_2$, as seen e.g. from (14), the solution $\ell(x)$ is of the form

$$(25) \qquad \ell(x) = \begin{cases} -bx + B + g(S) & 0 \leq x \leq s \\ h(x) + a\, \ell(0)\, (1 - F(x)\,) + a \int_0^x \ell\,(x-z)\, dF(z) & \\ \qquad < -bs + B + g(s) & s < x \leq S \end{cases}$$

where $g(S) = \underset{s \leq x \leq \bar{s}}{\operatorname{Min}} g(x)$. This proves that $\ell(x)$ determines an (s,S) policy.

3. Determination of the Parameters s and S.

3.1. In order to find s and S it is necessary to solve the equation (19b). The initial conditions are

$$(26) \qquad \ell(s) = B + g(S) - bs = \ell(0) - bs$$

Instead of (19b) consider the simpler equation (21) obtained from it by differentiation

$$\text{(27)} \qquad \begin{aligned} \chi'(x) &= h'(x) + a \int_0^x \chi'(x-z)\, d\, F(z) = \\ & h'(x) - a\, b\, (F(x) - F(x-s)) + a \int_0^{x-s} \chi'(x-z)\, dF(z) = \\ & r - p \,.\, (1 - F(x)) - a\, b\, (F(x) - F(x-s)) \\ & \quad + a \int_0^{x-s} \chi'(x-z)\, d\, F(z) \end{aligned}$$

$$\text{(28)} \qquad \chi'(x) = r - p(1 - F(x)) - ab\, (F(x) - F(x-s)) + a \int_s^x \chi'(z)\, f\,(x-z)\, dz$$

Solutions for particular distribution functions $F(x)$ will be obtained in sections 3.2 and 3.3. Assume now that equation (28) has been solved and let the solution, which still depends on a parameter s, be called $\chi'(x,s)$. With the initial conditions (26) for $\chi(x)$ we obtain

$$\chi(x,s) \;=\; B + g(S) - bs + \int_s^x \chi'(z,s)\, dz$$

According to (14) $g(S) = \chi(S) + bS$, so that

$$\chi(x,s) \;=\; B + \chi(S) + b(S - s) + \int_s^x \chi'(z,s)\, dz$$

Substituting $x = S$ we obtain

$$\text{(29)} \qquad \int_s^S \chi'(x,s)\, dx \;=\; b(s - S) - B$$

a first equation to determine s and S. The second equation is simply the minimum condition $g'(S) = 0$ or

$$\text{(30)} \qquad \chi'\,(S,s) \;=\; -b$$

This follows from the fact that S is the minimizer of $g(x)$; but using equations (13) and (19b), the latter since $s \leq S$, we have $g(x) = bx + \chi(x)$, and hence $g'(S) = b + \chi'(S) = 0$. We see that to obtain S and s it is enough to have the solution $\chi'(x,s)$ of (28) instead of that of (19b).

3.2. Let now the exponential distribution $F(x) = 1 - e^{-\frac{x}{m}}$, $f(x) = \frac{1}{m} e^{-\frac{x}{m}}$ be substituted in (28).

$$\chi'(x) = r - pe^{-\frac{x}{m}}$$
$$+ ab\left(e^{\frac{-x-s}{m}} - e^{-\frac{x}{m}}\right) + \frac{a}{m}\int_0^{x-s} \chi'(x-z)\, e^{-\frac{z}{m}}\, dz$$

Multiplying both sides with $e^{\frac{x}{m}}$ and denoting

(31) $\qquad \chi'(x)\, e^{\frac{x}{m}} = u(x) \qquad$ we obtain

(32) $\qquad u(x) = re^{\frac{x}{m}} - p - ab\left(e^{\frac{s}{m}} - 1\right) + \frac{a}{m}\int_0^{x-s} u(x-z)\, dz$

The last term may also be written $\frac{a}{m}\int_s^x u(z)\, dz$.

Differentiating equation (32)

$$u'(x) = \frac{r}{m} e^{\frac{x}{m}} + \frac{a}{m} u(x)$$

The differential equation

(33) $\qquad u'(x) - \frac{a}{m} u(x) = \frac{r}{m} e^{\frac{x}{m}}$

is linear of the first order with constant coefficients and an inhomogeneous term which is an exponential function. The solution of the homogeneous equation is simply

$$v = c\, e^{\frac{a}{m} x}$$

A particular solution of the inhomogeneous equation obtained by "variation of coefficients"

$$v = e^{\frac{ax}{m}} \int_0^x e^{-\frac{ay}{m}} \frac{r}{m} e^{\frac{y}{m}} dy = e^{\frac{ax}{m}} \frac{r}{m} \int_0^x e^{\frac{y}{m}(1-a)} dy$$
$$= \frac{r}{1-a} \left[e^{\frac{x}{m}} - e^{\frac{ax}{m}} \right]$$

The initial condition which the solution must satisfy, is obtained from (32) for $x = s$

(34) $$u(s) = (r - ab)\, e^{\frac{s}{m}} - p + ab$$

By some easy calculation

(35) $$u(x) = - A(s)\, e^{\frac{ax}{m}} + \frac{r}{1-a}\, e^{\frac{x}{m}}$$ where

(36) $$A(s) = a\left(b + \frac{r}{1-a}\right) e^{\frac{1-a}{m} s} + (p - ab)\, e^{-\frac{as}{m}}$$

or observing (31)

(37) $$\chi'(x,s) = - A(s)\, e^{\frac{a-1}{m} x} + \frac{r}{1-a}$$

Thus (30) assumes the form

$$A(s)\, e^{\frac{a-1}{m} S} = \frac{r}{1-a} + b$$

(38) $$S = \frac{m}{a-1} \log\left(\frac{\frac{r}{1-a} + b}{A(s)} \right)$$

$$= \frac{m}{1-a} \log \frac{A(s)}{b + \frac{r}{1-a}}$$

(29) becomes

$$A(s)\,\frac{m}{1-a}\left(e^{\frac{a-1}{m}S} - e^{\frac{a-1}{m}s}\right) = (s - S)\,(b + \frac{r}{1-a}) - B$$

Substituting S from (38) we obtain after some light calculation

(39) $$v = -\beta\, e^{-v} + \log\left\{ae^{v} + \beta\right\} + c,$$

where $c = (1-a)\,[1 + \frac{\alpha B}{m}]$ and

$$v = \frac{s}{m},\ \alpha = \frac{1}{b + \frac{r}{1-a}},\ \beta = \frac{p - ab}{b + \frac{r}{1-a}}$$

This equation can be solved by iteration, for instance, if $\frac{p-b}{r}$ is significantly smaller than e. This follows easily from the iterativity condition[4] and the observation that S is of the order of magnitude of m. For computation we may rewrite (38) as:

(38') $$S = \frac{m}{1-a}\,[-av^{*} + \log\left\{ae^{v^{*}} + \beta\right\}]$$

where v^{*} satisfies (39)

3.3. We shall sketch briefly how the solution may be obtained in the more general case of a Gamma distribution. We shall assume $b = 0$, i.e., that constant cost factors have been absorbed in the profits, that is to say, the penalty p. Substituting (2) in (28) we have

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[4] The iteration $x_{n+1} = f(x_n)$ converges to a solution $x = f(x)$ provided $|f'(x)| < 1$ in the neighborhood of the solution and the initial value was in this neighborhood.

$$\chi'(x) = r - \frac{p\, j^k}{\Gamma(k)} \int_x^\infty z^{k-1}\, e^{-jy}\, dy$$

$$+ \frac{aj^k}{\Gamma(k)} \int_s^x \chi'(y)\,(x - y)^{k-1}\, e^{\,j(y - x)}\, dy,$$

for k a positive integer.

Multiplying the equation with e^{jx} and putting again $u(x) = \chi'(x)\, e^{jx}$ we obtain an integral equation of the simple type

$$(40) \qquad u(x) = g(x) + c \int_s^x (x - z)^{k-1}\, u(z)\, dz$$

where (41) $c = \dfrac{a\, j^k}{\Gamma(k)}$ is a constant and

$$g(x) = r\, e^{jx} - \frac{p\, j^k}{\Gamma(k)} \int_x^\infty y^{k-1}\, e^{j(x-y)}\, dy$$

$$(42) \qquad = r\, e^{jx} - pj^{k-1} \sum_{n=0}^{k-1} \frac{x^n}{\Gamma(n)}$$

The integral equation (40) is converted into a differential equation by differentiating it k times

$$u^{(k)}(x) = c\, \Gamma(k - 1)\, u(x) + g^{(k)}(x) \qquad \text{or}$$

$$(43) \qquad u^{(k)}(x) - \frac{a\, j^k}{k}\, u(x) = j^k\, re^{jx}$$

with k initial conditions

$$(44) \qquad u^{(m)}(s) = g^{(m)}(s) = rj^m e^{js} - pj^{k-1} \sum_{n=0}^{k-m-1} \frac{s^n}{\Gamma(n)} \quad m = 0,\ldots,k-1$$

A particular solution of the inhomogeneous equation (43) is

$$v(x) = \frac{r}{1 - \frac{a}{k}} e^{jx}$$

The general solution of the homogeneous equation equals

$$(45) \qquad v(x) = \sum_{n=0}^{k-1} c_n \exp\left(j\left|\left(\frac{a}{k}\right)^{\frac{1}{k}}\right| e^{\frac{i2\pi n}{k}}\right)$$

Determination of the c_n from the initial conditions leads to the linear equation system

$$(46) \qquad \sum_{k=0}^{k-1} c_n e^{i\frac{2\pi mn}{k}} = \left[pj^{k-m-1} \sum_{n=0}^{k-m-1} \frac{s^n}{\Gamma(n)} + \frac{are^{js}}{k-a}\right] \left|\left(\frac{a}{k}\right)^{-\frac{m}{k}}\right| \exp\left(-js\left|\left(\frac{a}{k}\right)^{\frac{1}{k}}\right| e^{\frac{i2\pi m}{k}}\right)$$

$$= q(s,m), \quad \text{say.}$$

Since $\sum_{n=0}^{k-1} e^{i\frac{2\pi m_1 n}{k}} \overline{e^{i\frac{2\pi m_2 n}{k}}} = \sum_{n=0}^{k-1} e^{i2\pi\frac{(m_1-m_2)n}{k}} = \delta_{m_1 m_2} \cdot k$

the matrix $\left(e^{i\frac{2\pi mn}{k}}\right)$ is k times a unitary matrix and its inverse is the conjugate transposed with $\frac{1}{k} e^{-i\frac{2\pi mn}{k}}$ as the element in the m^{th} row and n^{th} column. Thus

$$(47) \qquad c_n = \frac{1}{k} \sum_{m=0}^{k-1} e^{-i\frac{2\pi nm}{k}} q(s,m)$$

And so

$$(48) \qquad u(x) = \frac{r}{1 - \frac{a}{k}} \; e^{jx} + \frac{1}{k} \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} q\,(s,m)\, e^{-i\frac{2\pi mn}{k}} \exp\left(j \left| \left(\frac{a}{k}\right)^{\frac{1}{k}} \right| e^{i\frac{2\pi n}{k}} \right)$$

Since $\mathcal{L}'(x) = u(x)\, e^{-jx}$ the further steps are as under 3.

4. Numerical Example.

The loss function which we have studied is a "realistic" formulation of that of the container manufacturer mentioned above and, we believe, is a fairly representative type in economic applications. It is quite similar to those upon which the so-called "economic lot size" formulas, which are in rather general use (see [6]), are based. It differs from the latter primarily in taking explicit account of randomness in quantity demanded and the "penalty" for shortage. It is not directly applicable to the container makers problem, however, because we have failed to take explicit account of the time lag in the delivery of orders. For purposes of illustration, however, we shall use the parameters relevant to the stocking of a machine part and give the numerical solution to (36) and (37), assuming this lag to be negligible.

We shall assume that the distribution of demand is negative exponential with mean $m = 100$ units/period, where the period is three months in length. Also, $B = \$20$, $b = \$0.30$, $r = \$0.15$, and $a = 0.975$. It is difficult to assign a value to p, the unit cost of shortage. However, for this case it seem that its smallest conceivable value would be \$1.575, while it might be as great as \$15.075. The parameters for the optimal order policy are then:

p	s	S
\$1.575	113	271
\$15.075	358	516

For $p = \$1.575$, the solution was also approximated iteratively using a procedure similar to that suggested in [3]; the values so found correspond closely to those given above.

We also note that if one wishes to order so as to minimize the expected loss for one period only, then it is easily shown that for our loss function the optimal order policy is also an (s,S) policy, provided that the distribution of demand has a density $f(x)$. Using the above parameter values (with $p = \$1.575$) one easily finds $s = 45$, $S = 120$, approximately. It is thus apparent that the parameters of the optimal order policy are quite different for the infinitely many time period problem than for the one period problem.

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