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The Classical Tax-Subsidy Problem

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1. Introduction

The present paper attempts to give firm foundations for a mathematical formula which has appeared recurringly in the literature since J. Dupuit's first study [2]. The formula referred to is a particular case of (17) given as an evaluation of the economic loss attributable to a system of indirect taxes and subsidies.

The actuality of this problem hardly needs to be emphasized. The importance of indirect taxes (state sales taxes, tax on transportation, gasoline, tobacco, luxury goods, entertainment, etc.) is already considerable; a general federal sales tax has been considered and there is little doubt that this question will come up again in the near future with increased insistence. Subsidies, on the other hand, allegedly justified on various grounds are more and more massively used in modern fiscal systems.

An approximation formula is sought in this paper and differentials appear to be the appropriate mathematical tool. Moreover their power allows one to go easily through calculations which would otherwise be forbidding.

2. The Economic System

With every one of the m consumption-units is associated a consumption vector x_i $(i = 1, \ldots, m)$ and a numerical satisfaction function $s_i(x_i)$ (defined but for an arbitrary monotonically increasing function) representing completely its tastes.

With every one of the n production-units is associated an input-vector y_j $(j = 1, \ldots, n)$ and a production surface whose equation is $e_j(y_j) = 0$. (We are never concerned here with technological inefficiency and we are therefore not interested in the whole set of production possibilities but only in the set of efficient points.)

Finally one must have $\sum_i x_i + \sum_j y_j \leqq z^o$, where z^o is the available resources vector.

The economic efficiency of a situation defined by s^o $(s_1^o, \ldots, s_m^o)$ is ρ, the coefficient of resource utilization of the economic system, which is the smallest fraction of the actually available resources which allows one to achieve the standard of living s^o [1]. According to the results of [1], the economic situation described by s^o and ρz^o is optimal and therefore there exists a price-vector p (and vectors x_i and y_j) such that

$$
(1) \quad \begin{cases} s_i(x_i) = s_i^o & e_j(y_j) = 0 \\ \operatorname{grad} s_i = \sigma_i\, p & \operatorname{grad} e_j = \xi_j\, p \\ \sum_i x_i + \sum_j y_j = \rho z^o & \end{cases}
$$

3. Variation of the Coefficient of Resource Utilization with the Standard of Living

For our purpose we are interested in the variation in ρ resulting from a variation in s^o away from an optimal situation (for which $\rho = 1$). The functions $s_i(x_i)$, $e_j(y_j)$ and the vector z^o are data.

We will first calculate the first and second order differentials of ρ under these conditions. In this section differentials will always be denoted by the symbol δ. Let $S_i = \left[\frac{\partial^2 s_i}{\partial x_{hi} \ \partial x_{ki}}\right]$ be the hessian matrix of $s_i\ (x_i)$, and similarly E_j be the hessian matrix of $e_j\ (y_j)$. Differentiate all equations of (1):

$$(2) \quad \begin{cases} \operatorname{grad} s_i \cdot \delta x_i = \delta s_i^0 \quad \operatorname{grad} e_j \cdot \delta y_j = 0 \\ S_i\, \delta x_i = \sigma_i\, \delta p + p\, \delta \sigma_i \quad E_j\, \delta y_j = \varepsilon_j\, \delta p + p\, \delta \varepsilon_j \\ \sum_i \delta x_i + \sum_j \delta y_j = z^0 \delta \rho \end{cases}$$

The two first equations can be rewritten

$$(3) \qquad \sigma_i\, p'\, \delta x_i = \delta s_i^0\ ,\ \varepsilon_j\, p'\, \delta y_j = 0$$

and the last one yields

$$p' \sum_i \delta x_i + p' \sum_j \delta y_j = p'\ z^0 \delta \rho$$

and therefore from (3)

$$(4) \qquad \sum_i \frac{\delta s_i^0}{\sigma_i} = p'\ z^0 \delta \rho$$

The first differential of ρ is thus obtained. The differential of the loss itself is $\delta\,[p'\ z^0\ (1 - \rho)] = -\,p'\ z^0\, \delta \rho$. When an approximate evaluation of this loss is wanted, (4) can be used[1] unless, however, $\delta \rho = 0$. In this case (which happens to be the one in which we are interested) the second order differential is needed. We proceed to calculate it. As the only case where it is interesting to obtain it (when one is concerned only with an approximation formula of the type used here) is the one where the first order differential is null

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1. For the purpose of practical evaluation one must note that

$$\sum_i \frac{ds_i^0}{\sigma_i} = p'\ dx$$

we will calculate the second order differential under this specific assumption.

Differentiate (4)

$$\sum_i \left[\frac{\delta^2 s_i^o}{\sigma_i} - \frac{\delta s_i^o \delta \sigma_i}{\sigma_i^2} \right] = p' z^o \delta^2 \quad (5)$$

The only term unknown to us so far is $\delta \sigma_i$. We obtain it in the following way. From (2) and (3)

$$\frac{1}{\sigma_i} S_i \, \delta x_i - p \frac{\delta \sigma_i}{\sigma_i} = \delta p$$

$$p' \, \delta x_i = \frac{\delta s_i^o}{\sigma_i}$$

Define $\Sigma_i = \begin{bmatrix} \frac{1}{\sigma_i} S_i & p \\ p' & 0 \end{bmatrix}$ one has

$$\Sigma_i \begin{bmatrix} \delta x_i \\ -\frac{\delta \sigma_i}{\sigma_i} \end{bmatrix} = \begin{bmatrix} \delta p \\ \frac{\delta s_i^o}{\sigma_i} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \delta x_i \\ -\frac{\delta \sigma_i}{\sigma_i} \end{bmatrix} = \Sigma_i^{-1} \begin{bmatrix} \delta p \\ \frac{\delta s_i^o}{\sigma_i} \end{bmatrix} \quad (6)$$

Let us define $\Sigma_i^{-1} = \begin{bmatrix} X_i & \delta_i \\ \delta'_i & c_i \end{bmatrix}$ where δ_i is a column vector and c_i a number.[2] From (6)

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2. The elements of X_i are the classical substitution coefficients studied in detail for example in [6] p. 103-117. It is clear here that the matrices S_i, Σ_i, Σ_i^{-1} and therefore X_i are symmetrical. The convexity of the set $\{ x_i : s_i(x_i) \geq s_i^o \}$ implies that X_i is negative semi-definite. So are X, Y_j, Y, Z.

$$\delta x_i = X_i \delta p + \gamma_i \frac{\delta s_i^o}{\sigma_i} - \frac{\delta \sigma_i}{\sigma_i} = \gamma_i' \delta p + c_i \frac{\delta s_i^o}{\sigma_i} \tag{7}$$

Define $X = \sum_i X_i$ and $\Gamma = [\gamma_1, \ldots \gamma_m]$, from (7)

$$\sum_i \delta x_i = X \delta p + \Gamma \left[\frac{\delta s_i^o}{\sigma_i}\right] \tag{8}$$

It must be remarked that x_i and y_j play perfectly symmetrical roles in (1), if one replaces s_i^o by 0. One has therefore

$$\sum_i \delta y_j = Y \delta p \tag{9}$$

where Y is constructed from the E_j in the same way that X is constructed from the S_i. Adding (8) and (9) one finds $(X + Y) \delta p = - \Gamma \left[\frac{\delta s_i^o}{\sigma_i}\right]$.

Denote $X + Y = Z$ and assume that the price vector p is normalized in some way: $\delta p = - Z^* \Gamma \left[\frac{\delta s_i^o}{\sigma_i}\right]$. From (7),

$$- \frac{\delta \sigma_i}{\sigma_i} = - \gamma_i' Z^* \Gamma \left[\frac{\delta s_i^o}{\sigma_i}\right] + c_i \frac{\delta s_i^o}{\sigma_i}$$

$$- \sum_i \frac{\delta s_i^o}{\sigma_i} \frac{\delta \sigma_i}{\sigma_i} = - \left[\frac{\delta s_i^o}{\sigma_i}\right]' \Gamma' Z^* \Gamma \left[\frac{\delta s_i^o}{\sigma_i}\right] + \sum_i c_i \left(\frac{\delta s_i^o}{\sigma_i}\right)^2 \tag{10}$$

Finally, defining $M = - \Gamma' Z^* \Gamma$, and using (5) and (10)

$$p' z^o \delta^2 \rho = \sum_i \frac{\delta^2 s_i^o}{\sigma_i} + \left[\frac{\delta s_i^o}{\sigma_i}\right]' M \left[\frac{\delta s_i^o}{\sigma_i}\right] + \sum_i c_i \left(\frac{\delta s_i^o}{\sigma_i}\right)^2 \tag{11}$$

The second order differential of the loss itself is

$$\delta^2 [p' z^o (1 - \rho) = - p' z^o \delta^2 \rho$$

and the approximation formula is

$$- \frac{1}{2} p' z^o \delta^2 \rho \tag{12}$$

4. Evaluation of the Loss Associated with a Tax-subsidy System

In this section differentials will be denoted by the symbol d.

Let us start now on what is actually our problem. Constraints imposed on the economic system are

$e_j\,(y_j) = 0$, $j = 1, \ldots, m$, i.e., defining $e = [e_j]$, $e = 0$.

$\sum_i x_i + \sum_j y_j = z^0$, i.e., defining $\sum_i x_i + \sum_j y_j = z$, $z = z^0$;

we assume that there is no technological inefficiency, and no unused physical resources. Under those constraints we want to have a maximal s. One recognizes a typical Lagrange problem.

$\exists$ 3 vectors $[1/\sigma_i]$, $[1/\varepsilon_j]$, p such that

$$\left[\frac{1}{\sigma_i}\right]' ds + \left[\frac{1}{\varepsilon_j}\right]' de - p'\, dz = 0, \tag{13}$$

where the x_i and y_j are now free variables. This is indeed the path to a concise proof of the basic theorem of the new welfare economics. From (13) it is clear that if we impose the constraints $e = 0$, $z = z^0$ and therefore $de = 0$, $dz = 0$: $\sum_i \frac{ds_i}{\sigma_i} = 0$, and if we set $\delta s_i = ds_i$ in (4) we find $d\rho = 0$ as it was indicated earlier.

Thus we start from an optimal situation and we set up a system of indirect taxes and subsidies: t_h^+ on any unit of the h^{th} commodity when it is an input and t_h^- when it is an output (t is positive for a tax, negative for a subsidy). Let us write the production function under the form $e_j\,(y_j^+, y_j^-) = 0$, where y_j^+ is made of the positive components of y and y_j^- of the negative components. The equilibrium position of any firm is characterized by

$$\partial e_j/\partial y_j^+ = \varepsilon_j\,(p + t^+) \qquad \partial e_j/\partial y_j^- = \varepsilon_j\,(p - t^-)$$

and the differential relation

$$\frac{\partial e_j}{\partial y_j^+}\, dy_j^+ + \frac{\partial e_j}{\partial y_j^-}\, dy_j^- = 0$$

can be written

$$(14) \qquad (p + t^+) \cdot dy_j^+ + (p - t^-) \cdot dy_j^- = 0$$

Differentiate (14) taking into account the fact that in the initial situation $t^+ = 0 = t^-$:

$$(dp + dt^+) \cdot dy_j^+ + (dp - dt^-) \cdot dy_j^- + p \cdot (d^2y_j^+ + d^2y_j^-) = 0$$

Making a summation over j, taking account of

$$\sum_j (y_j^+ + y_j^-) + x = z^0 ,$$

and defining

$y^+ = \sum_j y_j^+$, $y_j^- = \sum_j y_j^-$, one finds

$$(15) \qquad dt^+ \cdot dy^+ - dt^- \cdot dy^- = dp \cdot dx + p \cdot d^2x$$

The problem is now to show how the right-hand side of this relation can be linked to the ds_i and d^2s_i and from there to $d^2\xi$.

From grad $s_i = \sigma_i p$, follows $ds_i/\sigma_i = p' dx_i$.

Differentiate:

$$\frac{d^2s_i}{\sigma_i} - \frac{ds_i d\sigma_i}{\sigma_i^2} = dp' \, dx_i + p' \, d^2x_i$$

Make a summation over i:

$$\sum_i \frac{d^2s_i}{\sigma_i} - \frac{ds_i \, d\sigma_i}{\sigma_i^2} = dp' \, dx + p' \, d^2x$$

and from (15)

$$(16) \qquad \sum_i \frac{d^2s_i}{\sigma_i} - \frac{ds_i \, d\sigma_i}{\sigma_i^2} = dt^+ \cdot dy^+ - dt^- \cdot dy^-$$

Finally $\dfrac{d\sigma_i}{\sigma_i} = -\gamma_i' \, dp - c_i \dfrac{ds_i}{\sigma_i}$ so that

from (16) and (11)

$$p' z^{o} d^{2} p = dt^{+} \cdot dy^{+} - dt^{-} \cdot dy^{-} - \sum_{i} \frac{ds_i}{\sigma_i} \gamma_i' \, dp + \left[\frac{ds_i}{\sigma_i}\right]' M \left[\frac{ds_i}{\sigma_i}\right] \qquad (17)$$

The loss itself is given by the approximation formula

$$- \frac{1}{2} p' z^{o} d^{2} p$$

In this formula ds_i/σ_i can be replaced everywhere by $p' \, dx_i$ for the purpose of evaluation.

In most of the studies of this subject (see for example [5]) it was explicitly assumed (or it had to be implicitly assumed in order to make the result correct) that $ds_i = 0$ ($i = 1, \ldots, m$). In this case the two last terms of (17) vanish and it becomes very simple indeed, but there is little reason to expect that in the transformation of the economic system caused by the introduction of taxes and subsidies, such a condition is satisfied.

In more elaborate attempts (see f. ex. [3], [4]) the new economic situation (after the setting up of taxes and subsidies) was compared with the (fixed) initial optimal situation instead of being compared, as it is here, with the most appropriate (variable with the tax-subsidy system) optimal situation. As a consequence an error of the same order of magnitude as the term evaluated was made.

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