COWLES COMMISSION DISCUSSION PAPER: ECONOMICS: 240

Third Thoughts on Liquidity and Uncertainty.*

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A. Degrees of information, redefined.

Let [u] be a known set of alternative events, and [p] the corresponding set of probabilities. Distinguish the following cases:

incomplete information on [u]

1. [p] not known (ignorance).
2. as in 1., but the following is known: a) data d; b) some characters of the joint distribution of [u] and d.

complete information on [u]

3. [p] is known.
4. as in 3; and one element of [p] is 1 (certainty).

A.1 Intermediate cases possible. For example, the case when some characters of the distribution [p] are known, such as: $p_1 \geq p_2$, $p_1 + p_2 = 1$.

A.1.1 This particular case can be called: "ordinal probabilities characterizing future conditions." This is not the same thing as "ordinal probabilities characterizing preferences." The latter would necessitate the revision of the theorem on maximization of expected utility.

B. Profit function reformulated.

Profit in year 0, $z_0 = \rho(x) - Px$, where ρ is the revenue function and P the buying price of input. Use input x, at buying price, as numéraire, i.e., put $P \equiv 1$.

B.1 We have two "commodities"; a) "input" (proportional to "investment," the factor of proportionality being the maximum rate of use of the asset in question); b) "revenue."

*.Partly used in amending the introductory part of Mrs. Bronfenbrenner's minutes of the Staff Meeting of December 22, 1948.

C. Insistence on the physical or institutional character of the liquidity coefficient ℓ. ℓ is the ratio between two contemporary prices (price per unit of increase and the price of unit of decrease of input). It is independent of any change of prices occurring between the dates of actual buying and selling of a given asset. In this respect, ℓ is comparable to the ratio of raw material outlay to the value of a finished product, a physically or institutionally determined quantity.

C.1 The change in market prices, as between two successive year-ends, would reflect itself in the revenue function ρ; thus: $\rho_1 = \rho_o \cdot u$, where u is the (multiplicative)"shift" (see D).

D. In addition to the case of additive shift, $\rho_1(x) = \rho_o(x - u)$, studied in Cowles Commission Discussion Paper, Economics No. 235, the multiplicative case has been now studied. Nothing new arises. In particular, three intervals for u occur again, analogous to (α), (β), (γ) in Section 4.4 of the earlier paper.

E. Case of Ignorance, revised.

In the earlier paper, regret r was minimaxed with respect to x, u under the condition that y takes for each u, its profit-maximizing value $\hat{y}(u)$. What we actually need, however, is to find values of x and y (say, $\bar{x}$, $\bar{y}$) that yield $\mathrm{Min}_{x,y}\mathrm{Max}_u r$. We want to study the effect upon the best investment $\bar{x}$, thus defined, of liquidity ℓ. Obviously, if $\ell = 1$, or if u cannot take positive values the case is identical with 4.4, Case 1: the best investment in this case will be called $\hat{x}_\alpha$. Consider now the case $\ell = 0$ and show that then the regret will be minimax at a certain value of x, $\bar{x} \leq x_\alpha$: investment in illiquid asset is equal to or smaller than that in a liquid one, all other things being equal.

Suppose first that $[u]$ consists of two values only: u_α in the interval (α), and u_β in the interval (β), as defined in 4.4: $u_\alpha \leq 0$, $u_\beta > 2c$. Then, using the respective optimal inputs (call them $\hat{x}_\alpha$, $\hat{y}_\alpha$; $\hat{x}_\beta$, $\hat{y}_\beta$), given in 4.4, we can compute the corresponding maximum profits, $\hat{z}_\alpha$, $\hat{z}_\beta$, and hence obtain the regrets, $r_\alpha = \hat{z}_\alpha - z_\alpha$, $r_\beta = \hat{z}_\beta - z_\beta$. The difference

$$r_\alpha - r_\beta = u_\alpha - c + \rho(x + y + u_\beta) - \rho(x + y + u_\alpha),$$

is an increasing function of $x + y$. It vanishes when $x + y$ has a certain value, w (say), depending on u_α, u_β. Therefore $r_\alpha >, <, = r_\beta$ according as $x + y >, <, = w$. Minimax regret is therefore found as follows: minimize r_α for the cases $x + y > w$; minimize r_β for the cases $x + y < w$; then choose the smaller of the two minima. Note that r_α can itself have two minima, one at some $y \geq 0$ and one at some $y < 0$; and similarly for r_β. Inspection shows that all minima considered are either at $x = \hat{x}_\alpha = c(b - 1)$ or at $x = \hat{x}_\beta = c(b - 2)$. And, depending on the size of the possible shifts u_α, u_β, either $\hat{x}_\alpha$ or $\hat{x}_\beta$ yields the smallest of the minima. Therefore the minimax regret is obtained at an investment value which is $\leq \hat{x}_\alpha$, and thus best investment under illiquidity is smaller than or equal to the best investment under full liquidity.

This result is easily shown to be true also in the more general case: when $[u]$ contains any number of elements, belonging to all three intervals defined in 4.4: (α), (β), and the intermediate interval (γ). Again, minimax regret will be obtained when investment is either $\hat{x}_\alpha$ or $\hat{x}_\beta$: the possibility of events of intermediate kind (γ) has no effect. Hence, Laplace's rule of equiprobability of unknown events is no guide to action.

F. Further lines of investigation:

F.1: The case of many inputs and outputs.

F.2: A longer horizon. This will also include the case (insurance, gambling) when uncertainty prevails already with respect to the revenue of the year 0.

F.3: General variation in the revenue function, instead of a one-parametric shift.

F.4: Making ℓ a random variable (not an important complication, I believe).

F.5: Ordinal probabilities on the utility side (see A.1.1), implying reformulation of the behavior axioms.

F.6: Fitting the "minimax regret" principle into the other behavior axioms.

G. In the earlier paper, q, the buying or selling price of input, is a discontinuous function of the input increment, y: $q = 1$ or ℓ according as $y \geq 0$ or $y < 0$. It may be mathematically more convenient to consider q as the limit of a continuous function, e.g.,

$$q(y) = \lim_{k \to \infty} \frac{\ell + e^{kx}}{1 + e^{kx}}.$$

By using the continuous function until the maximum (or minimum) solution is obtained, and then passing to the limit, one will perhaps avoid the bothersome inspection of alternative intervals.