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# Liquidity of Assets and Contracts under Complete Information

by Jacob Marschak

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1. It is proposed to study how the demand of rationally acting men for an asset or contract depends on its liquidity.

1.1 Man's behavior is called rational (consistent) if, faced with the same alternatives, he will make the same choices. While the actual behavior of men is not consistent, the implications of consistent behavior, or the so-called pure economic theory, deserve study for two reasons: (1) as a possible first approximation to the description of actual behavior; (2) as a set of practical norms which, while derivable from the general norm of consistency, may be easier to apply in special cases.

## 2. LIQUIDITY

Let $x_0$, $x_1$ = rates of input of a certain service (in man-hours, machine-hours, etc.) in years 0 and 1 respectively. If $x_1 > x_0$, and the asset or contract that yields the service is expanded, a price, $p_0$, is to be paid, per unit of input added. If $x_1 < x_0$, and the asset or contract that yields the service is reduced, a certain amount, $p_0\ell$ is released, per unit of input subtracted. The ratio $\ell (0 \leq \ell \leq 1)$ will be called liquidity. Of the cases drawn on GRAPH I, we shall consider line bb as sufficiently realistic, though cc is somewhat more general.

### 2.1 Special forms

2.1.1 Liquidity as marketability of an asset. Here $p_0\ell$ = second-hand or scrap value of a machine, the selling price of a (non-standardized) real estate after advertising or agent costs, etc. In perfect market (single shares, bonds, grains) $\ell = 1$. Note (on GRAPH I) that continuous line dd ordinarily used for monopoly does not meet our case as well as do cases bb or cc.

2.1.2 Liquidity as physical convertibility of an asset. Here $\ell = 1$ implies costless change of physical firm or location. For raw materials, $\ell$ is larger than for finished ("specific") goods.

2.1.3 Liquidity of contracts. Here $\ell < 1$ if the dissolution of contract entails legal penalty or some other cost.

2.2 The following extensions will not be attempted here:

2.2.1 If one admits (a) preference for present vs. future; (b) change in market prices, then the ratio between two money costs must be replaced by a ratio between two marginal utilities (a "marginal rate of substitution").

2.2.2 Horizon might be extended beyond 2 time-units.

## 3. DEGREES OF INFORMATION

3.1 A model (N = size of sample).

3.1.1 $N \to \infty$: complete information approached.

3.1.1.1 (Special case: $N = \infty$ and all probabilities = 0 or 1: certainty).

3.1.2 $N = 0$: ignorance.

3.1.3 $0 < N < \infty$ : incomplete information.

3.1.3.1 (Special case: N grows in time: sequential information).

3.2 All probabilities involved here are subjective.

3.3 At least parts of the model can be extended to the case when probabilities are degrees of belief not depending on observed frequencies; provided the firm can make choices between bets.

3.4 Extension to "ordinal probabilities" will not be discussed here.

3.5 Granted certain plausible postulates, the consistent man (1.1) maximizes the expected (= mean) value of utility.*

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*See Measurable Utility and the Theory of Assets, Cowles Commission Discussion Paper, Economics 226, 226A presented at Madison meeting of Econometric Society, 1948; abstract in Econometrica, 1949.

## 4. EFFECT OF LIQUIDITY ON DEMAND IN CASE OF CERTAINTY

4.1 The firm <u>knows</u> (as in 3.1.1.1) that x input units will produce in years 0 and 1, respectively, $\pi_0(x)$ and $\pi_1(x)$ output, measured in money units. Write $x_1 - x_0 \equiv y$. Firm chooses, at beginning of year 0, those values of x, y that maximise two-years' profit $z = z_0 + z_1$ where

$$z_0 = \pi_0(x_0) - p_0 x_0; \quad z_1 = \pi_1(x_0 + y) - p_0 x_0 - q p_0 y,$$

where $q = 1$ if $y \geq 0$; $q = \ell$ if $y < 0$.

<u>Problem</u>: Investigate the effect of $\ell$ upon x and y for a given change in the production function.

4.2.1 Choose units so that $p_0 \equiv 1$.

4.2.2 Assume a one-parametric "shift" in production function, thus:

$$\pi_0(x) = \pi_1(x + u)$$ (i.e., total cost needed to produce a given output is changed by a constant)

4.2.3 Assume, as on GRAPH II, that marginal product function is linear,

$$\pi_0'(x) = b - x/c; \quad \pi_1'(x) = b - (x + u)/c; \quad c > 0.$$

On all CHARTS of this paper, $b = 4$, $c = 1$.

4.2.4 From now on we write x for $x_0$; and $x + y$ for $x_1$.

4.3 The problem becomes: "$\hat{x}$, $\hat{y}$ are values of x, y that maximize

$$s(x, y) = a + bx - x^2/2c + a + b(x + y + u) - (x + y + u)^2/2c - 2x - qy,$$

where $q = 1$ if $y \geq 0$; $q = \ell$ if $y < 0$; a = constant.

Express $\hat{x}$, $\hat{y}$ as functions of both u and $\ell$."

4.3.1 Especially, show that, given the shift $u > 0$ (increase in total cost for given output), $\hat{x}$ is smaller when $\ell = 0$ than when $\ell = 1$: GRAPH II.

4.4 Solution of 4.3. Putting $\partial s/\partial x = \partial s/\partial y = 0$, solve for x, y:

$$y = y(u, q) = 2c(1 - q) - u$$

$$x = x(u, q) = c(b - 2 + q)$$

Case 1. Perfect liquidity: $\ell = 1$, hence $q = 1$. $\hat{x} = x(u, 1) = c(b - 1)$, $\hat{y} = y(u, 1) = -u$ } See GRAPH III A.

Case 2. Imperfect liquidity (See GRAPH III B): $0 \leq \ell < 1$.

($\alpha$) for values of u such that $y(u, 1) \geq 0$:

$$\hat{y} = y(u, 1) = -u, \text{ as in Case 1}$$

$$\hat{x} = x(u, 1) = c(b - 1), \text{ as in Case 1}$$

$$u \leq 0;$$

($\beta$) for values of u such that $y(u,1) < 0$ and that even $y(u, \ell) < 0$:

$$\hat{y} = y(u, \ell) = 2c(1 - \ell) - u$$

$$\hat{x} = x(u, 1) = c(b - 2 + \ell)$$

$$u > 2c(1 - \ell) > 0$$

($\gamma$) for values of u such that $y(u, 1) < 0 \leq y(u, \ell)$:

$$\hat{y} = 0,$$

$$\frac{ds(\hat{x}, 0)}{dx} = 0, \ \hat{x} = c(b - 1) - u/2$$

$$0 < u \leq 2c(1 - \ell)$$

4.4.1 Meaning of the intervals ($\alpha$), ($\beta$), ($\gamma$) in terms of capacity.

In ($\alpha$) the shift of production function calls for an increase in input after year 0. This means also an increase in capacity as the asset acquired or contract made at the beginning of year 0 was such as to suit the initial input $\hat{x}$.

In ($\beta$) the shift calls for decrease of input. In this case, if the second-hand price of the asset = 0, part of capacity created in year 0 may remain unused during year 1; that is, it was not absurd to plan in year 0 for unused capacity in year 1.

In ($\gamma$) the shift in production function would call for decrease of input if the second-hand price equaled the price of new equipment; but with a lower second-hand price it is preferable to start with smaller capacity in year 0 and to maintain it unchanged, and fully used, in year 1.

4.5 Problem of several (n) kinds of inputs (and, correspondingly, several kinds of assets or contracts), each having different liquidity $\ell^i$ ($i = 1, \ldots, n$). Find the best initial inputs (proportional to investments) $\hat{x}^i$ and the best increments $\hat{y}_1^i$, $\hat{y}_2^i$ for successive years 1, 2, ... as functions of $\ell^i$ and of a known shift u in the production function. It is conjectured that (as on GRAPH III B), $\partial \hat{x}^i / \partial \ell^i \geq 0$.

4.6 We conclude that differences in the liquidity of various assets affect the relative demand for them even under conditions of certainty. Examples: till money, and pipe-line stocks held to provide for predicted changes (seasonal or otherwise) in production and market conditions.

## 5. EFFECT OF LIQUIDITY ON DEMAND IN THE GENERAL CASE OF COMPLETE INFORMATION (3.1.1)

5.1 Notation: Small letters: non-random variables (quantities and functions)

Capital letters: variables that are in general random

Roman letters: quantities (mostly)

Greek letters: functions (mostly)

5.2 At the beginning of year 0, the production function $\pi_0(x)$ for that year is known, but the production function for the next year is random. Specifically, replace equation in 4.2.2 by the following:

$$\pi_1(x) = \pi_0(x + U)$$

where the "random shift" U has probability density function $\phi(U)$.

5.3 We shall assume $\mathcal{E} U = 0$ and denote $\mathcal{E} U^2$ by $\sigma^2$. We shall discuss how the best investment (best initial input) $\hat{x}_0$ depends upon liquidity $\ell$ and upon riskiness $\sigma^2$ of the contract or asset in question.

5.4 The two-years' profit z in 4.2 becomes (putting $p_0 = 1$ and writing x for $x_0$; x + Y for $x_1$: 4.2.4)

$$Z = Z(x, Y) = \pi_0(x) - x + \pi_0(x + Y + U) - x - QY,$$

where $Q = 1$ if $Y \geq 0$; $Q = \ell$ if $Y < 0$.

Here Y, and therefore Q, is random, because the choice of best value for Y, to be made at beginning of year 1, will depend upon the value which the shift U will have taken by then. At the beginning of year 0, the firm determines:

(a) the ("best") value $\hat{x}$ of x, and

(b) the ("best") function $\hat{\eta}$ ), —

such that if $x = \hat{x}$ and $Y = \hat{\eta}(U)$, then the expectation $\mathcal{E}Z$ has a maximum, i.e.

$$\mathcal{E}Z[\hat{x},\ \hat{\eta}(U)] \geq \mathcal{E}Z[x,\ \eta(U)]$$

for any x, $\eta$ (cf. 3.3), or

$$\mathcal{E}Z[\hat{x},\ \hat{\eta}(U)] = \mathrm{Max}_x\ \mathrm{Max}_\eta\ \mathcal{E}Z$$

5.4.1 $Z = Z_1 + Z_2$ is used here as a sample case of utility function of money profits instead of a more general one, say $\omega(Z_1, Z_2)$; cf. 2.2.1. Expanding $\omega$ into Taylor series (as in Marshall, Principles, Math. Appendix IX) we approximate $\mathcal{E}\omega$ by a linear combination of $\mathcal{E}Z$, the variances of $Z_1$ and $Z_2$ (sometimes called "risks"), their correlation etc. The firm is thus concerned with higher moments and not only with the means of profits.

5.4.1.1 In the expansion just mentioned, the coefficient of the variance of $Z_i$ $(i = 1, 2)$ is $\partial^2\omega/\partial Z_i^2$. Only if this is negative, i.e. if the marginal utility of profit decreases with profit, is there "risk aversion" (cf. Friedman and Savage, JPE, 1948).

5.4.1.2 Further generalization of the utility function might make it dependent on other commodities besides money—e.g. power (for a firm's manager), consumers' goods (for the householder).

5.4.1.3 However, the simple utility function $\omega(Z_1, Z_2) = Z_1 + Z_2 = Z$, and hence the maximizing of $\mathcal{E}Z$, will suffice to illustrate the main propositions of this paper.

5.4.1.4 The statement of 5.4 is equivalent to the Neumann-Morgenstern definition of strategy. See also A. Hart's contribution to the Schultz memorial volume; and my paper on Measurable Utility and Theory of assets, p. 20.

5.4.2 We also omit here the extension to horizons of more than two time-units.

5.5 We shall show that

(5.5.1) $\text{Max}_x \text{ Max}_\eta \; \mathcal{E} Z[x, \eta(U)] = \text{Max}_x \; \mathcal{E} \text{Max}_\eta \; Z[x, \eta(U)]$ ;

for this, it suffices to show that

(5.5.2) $\text{Max}_\eta \; \mathcal{E} Z[x, \eta(U)] = \mathcal{E} \text{ Max}_\eta \; Z[x, \eta(U)]$ .

Suppose that

(5.5.3) $\text{Max}_\eta Z[x, \eta(U)] = Z[x, \hat{\eta}(U)]$ ;

that is, for any U, x, $\eta$

$$Z[x, \hat{\eta}(U)] \geq Z[x, \eta(U)] \;;$$

multiplying by $\phi(U)$ (non-negative) and summing over U,

$$\mathcal{E} Z[x, \hat{\eta}(U)] \geq \mathcal{E} Z[x, \eta(U)] \,,$$

$$\mathcal{E} Z[x, \hat{\eta}(U)] = \text{Max}_\eta \; \mathcal{E} Z[x, \eta(U)] \;;$$

This, by (5.5.3) is equivalent to (5.5.2); hence (5.5.1) is proved.

5.6 Therefore, to find $\hat{x}, \hat{\eta}$, we proceed as follows: solve

(5.6.1) $\partial Z(x, Y)/\partial Y = 0$

for Y, obtaining $Y = \hat{\eta}(U)$, which involves x. Substituting,

(5.6.2) $\mathcal{E} Z(x, Y) = \mathcal{E} Z[x, \hat{\eta}(U)] = \mathcal{E} Z$,

now a function of x only. Then solve for x the equation

(5.6.3) $d\mathcal{E}Z/dx = 0$;

it will be satisfied by $x = \hat{x}$.

5.7 Apply this to the case of linear marginal product function (4.2.3). The problem in 4.3 becomes:

"Write $Y = \eta(U; \ell)$; $V = x + Y + U$; then $\hat{x}$ and $\hat{\eta}(U; \ell)$ are a constant and a function that maximise

$$\mathcal{E} Z(x, Y) = a + bx - x^2/2c + a + bV - V^2/2c - 2x - QY,$$

where $Q = 1$ if $\eta(U, 1) \geq 0$; $Q = \ell$ if $\eta(U, 1) < 0$; and $\mathcal{E} U = 0$.

Express $\hat{x}$ as a function of liquidity $\ell$ and of the parameters of the probability function $\phi(U)$, such as the variance, $\sigma^2 = \mathcal{E} U^2$, of random shifts of the production function. In particular, find the signs of $\partial\hat{x}/\partial\ell$ and $\partial\hat{x}/\partial\sigma$."

5.8 Proceeding as in (5.6.1), we obtain, for the three intervals analogous to those in 4.4:

($\alpha$) for $U \leq c(b-1) - x$: $\hat{\eta}(U) = c(b-1) - x - U$; $V = c(b-1)$; $Q = 1$.

($\beta$) for $U \geq c(b-\ell) - x$: $\hat{\eta}(V) = c(b-\ell) - x - U$; $V = c(b-\ell)$; $Q = \ell$.

($\gamma$) for $c(b-1) - x < U < c(b-\ell) - x$:

$$\hat{\eta}(U) = 0;\ V = x + U;$$

Proceeding further, as in (5.6.2), (5.6.3):

$$dEZ/dx = b - 2 - \hat{x}/c + \int_{m_1}^{m_\ell} [b - (\hat{x} + U)/c]\ \phi(U)dU$$

$$+ \int_{-\infty}^{m_1} \phi(U)dU + \int_{m_\ell}^{\infty} \ell\phi(U)dU = 0,$$

where $m_\ell = (b-\ell)c - \hat{x}$, $m_1 = (b-1)c - \hat{x}$.

5.9.1 Note: At perfect liquidity $\ell = 1$, the equation in 5.9 yields

$$\hat{x} = (b-1)c,$$

independent of the random shifts, and identical with the solution in 4.4, Case 1.

5.9.2 On GRAPH IV, the equation in 5.9 is plotted for $\phi(U)$ normal and for $\ell = 0$ and $\ell = 1$: $\hat{x}$ is expressed as a function of $\sigma$.

5.10 To find the sign of $\partial\hat{x}/\partial\ell$, differentiate the equation in 5.9 with respect to $\ell$, and write for the cumulative probability function $\int_{-\infty}^{m} \phi(U)dU = \psi(m) \geq 0$. We obtain

$$(\partial\hat{x}/\partial\ell) \cdot [1 + \psi(m_\ell) - \psi(m_1) + c\,\phi(m_1)] = c\,\psi(m_\ell);$$

and since $c > 0$, and $m_1 < m_\ell$, $\psi(m_1) < \psi(m_\ell)$,

$$\partial\hat{x}/\partial\ell \geq 0$$

That is, initial investment increases with liquidity.

5.11 It can be further shown that investment $\hat{x}$ decreases with increasing variance of the random shift. By introducing $U^* = U/\sigma$, a random variable with zero mean, unit variance and a probability density function $\phi^*(U^*)$, Herman Chernoff has shown that for continuous distribution functions,

$$\frac{d\hat{x}}{d\sigma} = -\int_{n_1}^{n_\ell} U^* \ \phi^*(U^*)\ dU^* / [1 + \int_{n_1}^{n_\ell} \phi^*(U^*)dU^*] \leq 0,$$

where $n_i = m_i/\sigma$ $(i = 1, \ell)$. (The equality sign applies when $\ell = 1$: see 5.9.1).

5.12 Chernoff has shown, moreover, that investment $\hat{x}$ is bounded as follows:

(5.12.1) as $\sigma \rightarrow 0$, $\hat{x} \rightarrow c(b - 1)$;

(5.12.2) as $\sigma \rightarrow \infty$, $x \rightarrow c[b - 2 + \ell + (1 - \ell)\ \psi^*(0)]$,

where $\psi^*(m) = \int_{-\infty}^{m} \phi^*(U^*)dU^*$, so that, for symmetrical functions, $\psi^*(0) = 1/2$.

While (5.12.1) simply repeats the result obtained in the case of certainty (4.4: Case 2, for $u = 0$), the result (5.12.2) is interesting: as the variance of the random shift increases indefinitely, investment does not fall to zero; for example, if $\ell = 0$ and the distribution is symmetrical, $\hat{x}$ approaches $(b - 3/2)c$. (See GRAPH IV for the more special case of normal distribution and for $b = 4$, $c = 1$).

## 6. NOTE ON THE REMAINDER OF THE PAPER

The possible role of liquidity under conditions of incomplete information (called "uncertainty" by F. Knight as I understand him), with ignorance as the limiting case, can be only sketched out here as a program. The work to be done would essentially apply the principle of maximized utility to actions based on limited data. This idea, in its application to statistical inference, was developed by Abraham Wald as early as 1939. (One of the more recent papers is "Foundations of a General Theory of Sequential Decision Functions," Econometrica, 1948). The following remarks aim mainly at acquainting general economists with this approach of statisticians.

## 7. THE CASE OF IGNORANCE

7.1 Examples of ignorance as defined in 3.1.2:

7.1.1 The firm knows the production function $\pi_0(\ )$, the liquidity $\ell$ of the asset or contract in question, andtwo values, $u_1$ and $u_2$, that the shift u of the production function can take; but it has no information to estimate probabilities attached to these two values. It has to choose x, y.

7.1.2 Same as before but, instead of two constants $u_1$ and $u_2$, the firm knows that $\phi(U)$, the probability distribution of the random shift U, has zero mean and is normal, but the variance $\sigma^2$ is not known.

7.2 If the unknown conditions $(u, \sigma)$ were determined by a rational opponent in a game, a rational choice by our firm , in case 7.1.1, would be as was shown by von Neumann and Morgenstern: it would "minimax the loss," i.e. make a choice of x, y such as to obtain:

$$\text{Min}_{x,y}\, \text{Max}_u \left[- z(x, y; u)\right] ;$$

and, in case 6.1.2, it would "minimax the expected value of the loss,"

$$\text{Min}_{x,\eta}\, \text{Max}_\sigma \left[- \mathcal{E}\, Z(x, \eta(U); \sigma\right]$$

7.2.1 In words: In a game, choose a policy (x and y, or x and $\eta$) such that, even if the unknown conditions (u or $\sigma$) should turn out to be the ones most adverse to this policy, the expected gain (z or $\mathcal{E}$ Z) would be at least as large as the expected gain that would result from any other policy, and from conditions most adverse to that other policy.

7.3 If conditions are not determined by an inimical rational opponent, there is no reason for this maxim of behavior. L. J. Savage has suggested (orally) that not the loss but the "regret" be minimaxed: the "regret" being the difference between the gain actually obtained and the gain that would be obtained if choice had been made in full knowledge of conditions. Thus, the firm chooses the policy that would yield, in case 7.1.1, (7.3.1) $\text{Min}_{x,y}\ \text{Max}_i \left[\text{Max}_{x,y}\ z(x,\ y;\ u_i) - z(x,\ y;\ u_i)\right]$ ,

where i = 1, 2. And in the case 7.1.2 the policy should yield

(7.3.2) $\text{Min}_{x,\eta}\ \text{Max}_s \left[\text{Max}_{s,\eta}\ \mathcal{E} Z(x,\ y;\ s) - \mathcal{E} Z(x,\ y;\ s)\right]$ ,

where s are the values that can be taken by $\sigma$.

(CHARTS VI - VII, to be supplied later)

7.4 It is felt, and has been confirmed by tentative numerical examples/ that the optimal policies thus determined would involve liquidity $\ell$. But I have not worked out the problem in detail.

## 8. THE GENERAL CASE OF INCOMPLETE INFORMATION

8.1 We now modify the example 7.1.2 as follows:

The firm knows the production function $\pi_0$, the liquidity $\ell$ of the asset or contract in question, and knows that $\phi(U)$, the distribution of random shifts, is normal with zero-mean. In addition, it can obtain, in the year 0, certain data, to be denoted by $D_o$, that are related to the future random production shifts U. For example, the numbers D may be sample previews of crops. Denote the (unknown) joint distribution of $D_o$, U by $\delta(D_o,\ U)$. In particular, $\delta$ may be characterized by variances and regression coefficients. To choose a policy means to decide how to react to information $D_o$, i.e. what functions $\xi$, $\epsilon$ will yield optimal initial investment $\hat{X} = \xi(D_o)$ and optimal increment in year 1, $Y = \epsilon(D_o)$, determined on the basis of information in year 0. Applying the principles of 7.3.2, we have to choose $\xi$, $\epsilon$ so as to obtain

$$\text{Min}_{\xi,\epsilon}\ \text{Max}_{\delta}\ (\text{Max}_{\xi,\epsilon}\ \mathcal{E} Z - \mathcal{E} Z),$$

where $Z = Z\left[\xi(D_o),\ \ \epsilon(D_o)\right]$ and $\delta(U,\ D)$ is an unknown joint distribution function.

8.2 Again, it is conjectured that if the optimal decision functions—say, $\hat{\xi}$, $\hat{\epsilon}$ — are actually evaluated for our economic examples, they will turn out to depend on liquidity.

8.3 In 8.1, the only data considered were those available at beginning of year 0, $D_o$. Actually, further data, say $D_1$, will become available in the course of that year. Then $D_1$ must be included as argument in the joint distribution function $\delta$ and in the decision function $\epsilon$. This will affect the optimal values of $\xi$, $\epsilon$.

8.3.1 In particular, it will probably turn out that if the liquidity of the asset is low, it is advantageous (i.e. it will "minimax the regret") to have less of it during the year 0, in order to wait for additional information. Thus, "sequential information" listed in 3.1.3.1 as a special case, is likely to be of importance for the theory of liquidity.

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(Correction: Wherever the word GRAPH appears, it should be replaced by CHART).

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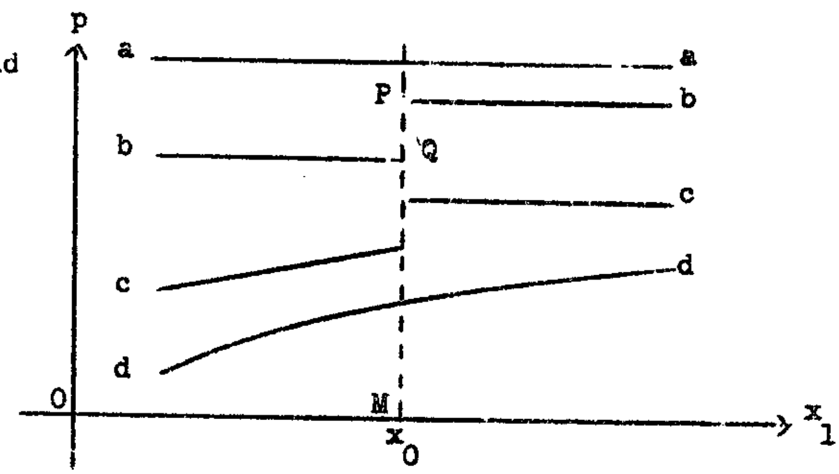
GRAPH I

p = price (per input unit) paid
or retained by firm

$x_1$ = input rate in year 1

$x_0$ = initial input rate



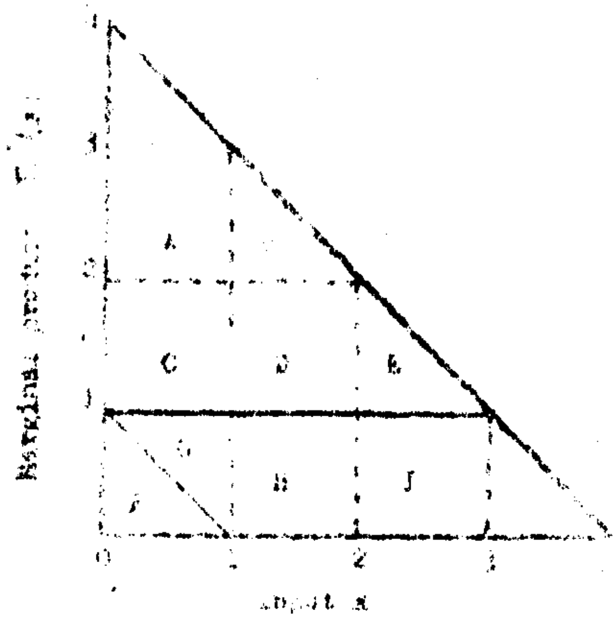

aa: perfect liquidity: $\ell = 1$

bb: imperfect liquidity (admitting discontinuity): $\ell$ = MQ/MP = constant $< 1$

cc: ditto, $\ell$ variable $< 1$

dd: imperfect liquidity (continuous case).

## GRAPH II




[illegible] inputs and [illegible] perfect liquidity and illiquidity.

It is known that [illegible] function represented by [illegible] in year 0 will shift by 3 input units [illegible] in year 1. Input price [illegible] throughout.

Case $\ell = 1$ (perfect liquidity)

Best inputs: $x_0 = 3$, $x_1 = 0$;

Profits: $z_0 = A + B +$ [illegible] $+ E$

$z_1 =$ [illegible]

Case $\ell = 0$ (perfect illiquidity)

If again $x_0 = 3$, $x_1 = 0$, then

Profits: $z_0 = A + B + C + D + E$

$z_1 = -(F + G + H + I)$

A better (in fact, the best*) choice is:

$x_0 = 2$, $x_1 = 1$; then

Profits: $z_0 = A + B + C + D$

$z_1 = F - (G + H)$. Then total profit $(z_0 + z_1)$ exceeds that of previous choice by $(F + I - E)$.

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* See GRAPH III B, with $u = 3$.

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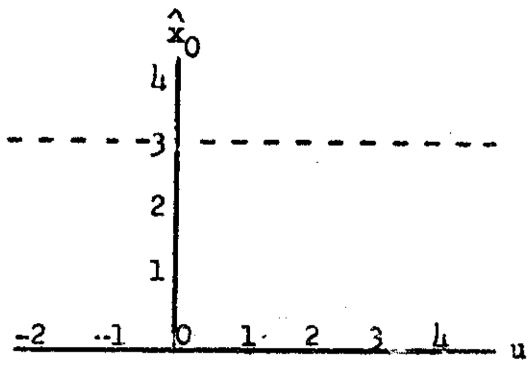
## GRAPH III

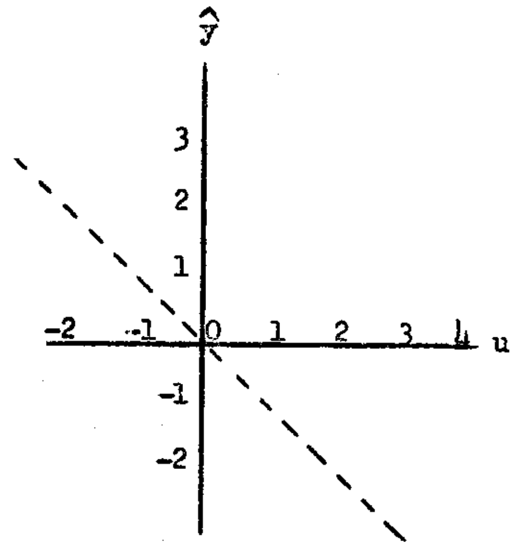
Optimal initial input ($\hat{x}_0$) and optimal change in input ($\hat{y}$) as functions of a known shift (u) in production function.

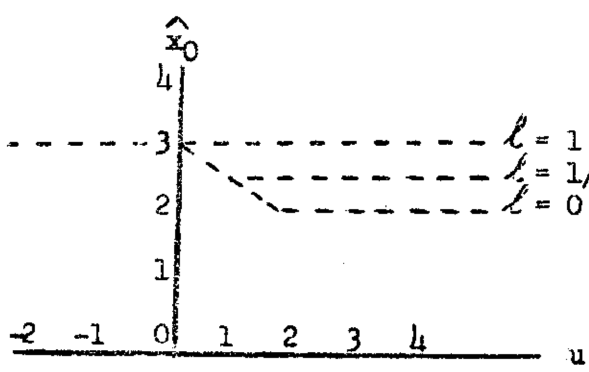
(A): Perfect liquidity ( $\ell$ = 1)

(B): Varying degrees of liquidity ($\ell$ = 0, 1/2, 1).

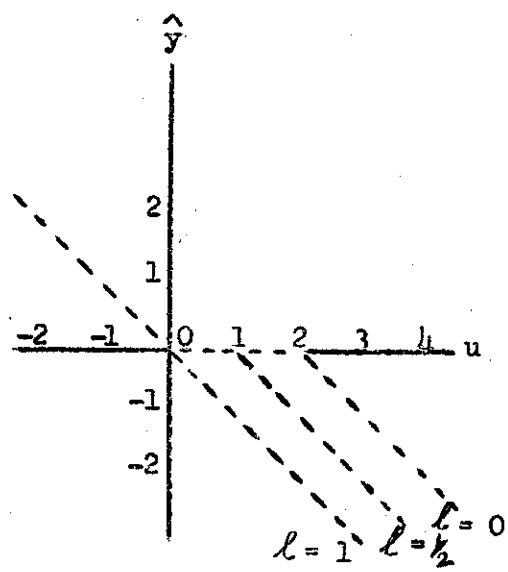



(A)

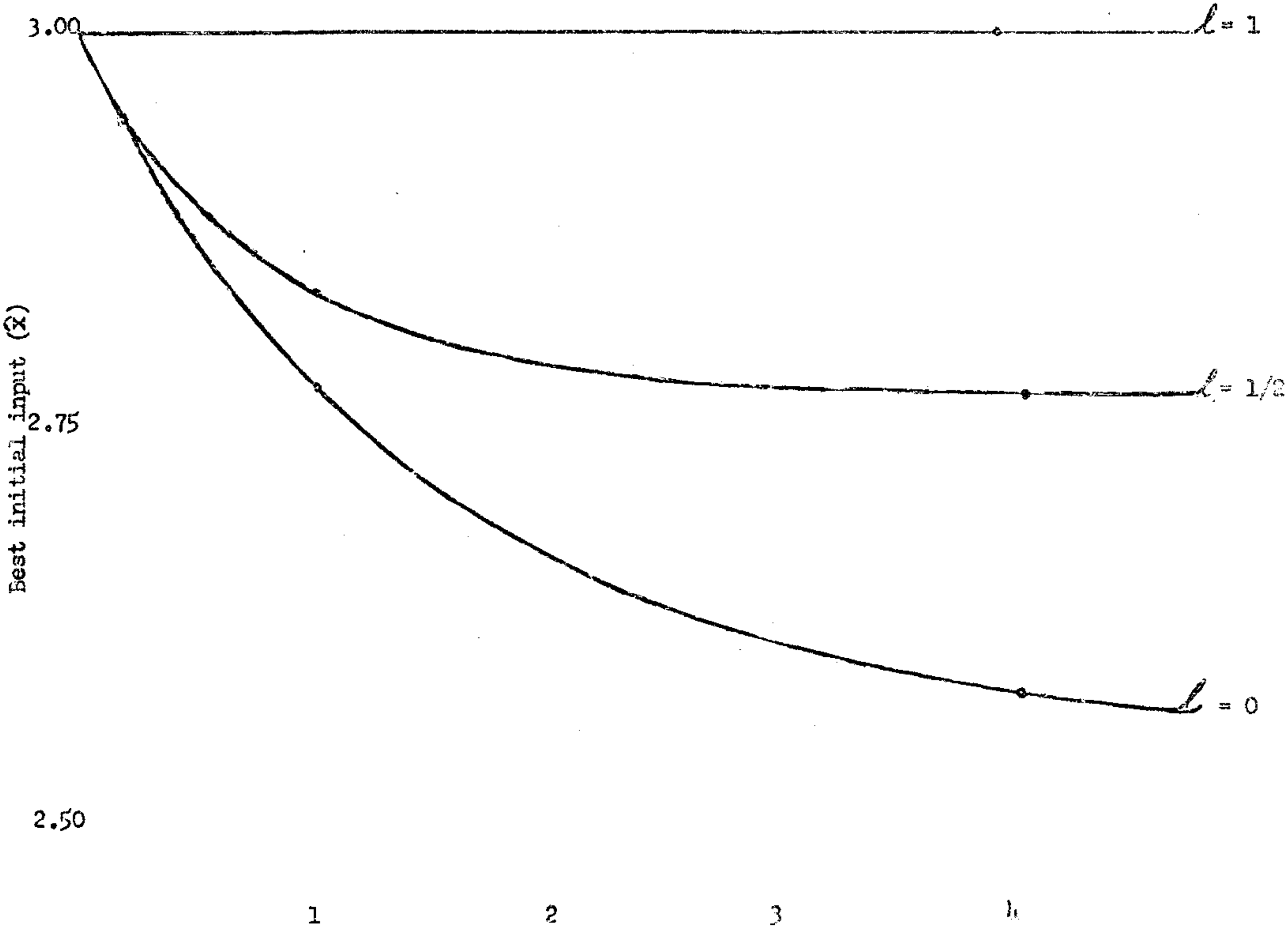



(B)

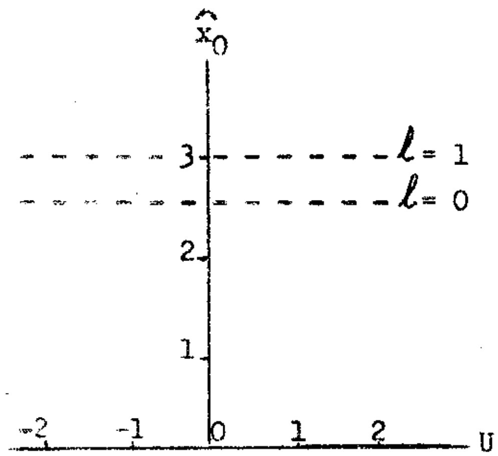


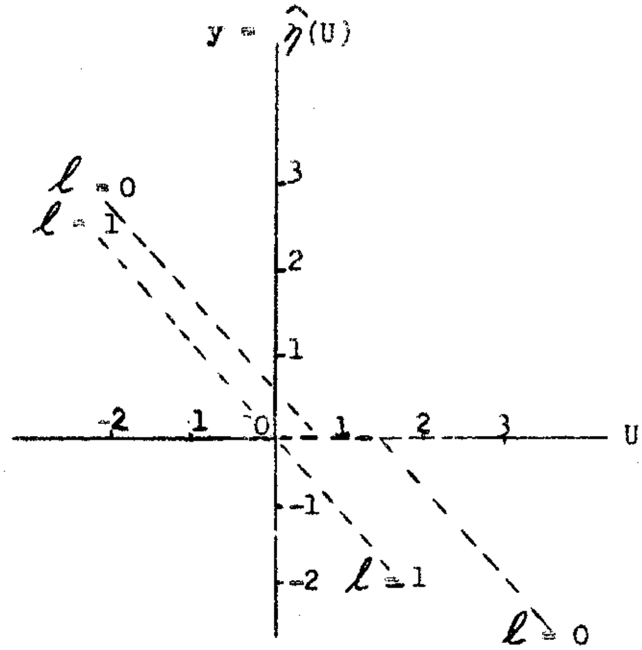
GRAPH IV




Dependence of best initial input ($\hat{x}$) upon liquidity ($\ell$) of the asset or contract, and upon the known standard deviation ($\sigma$) of the probability distribution (known to be normal, with zero mean) of a future shift in the production function.

Limiting values of best initial input ($\hat{x}_0$) and best change in input ($y = \hat{\eta}(U)$) as functions of a random shift U, symmetrically distributed with zero mean,* and with standard deviation increasing indefinitely ( $\sigma \to \infty$ ); two alternative degrees of liquidity ( $\ell$ = 0; 1) of the asset or contract in question are assumed.

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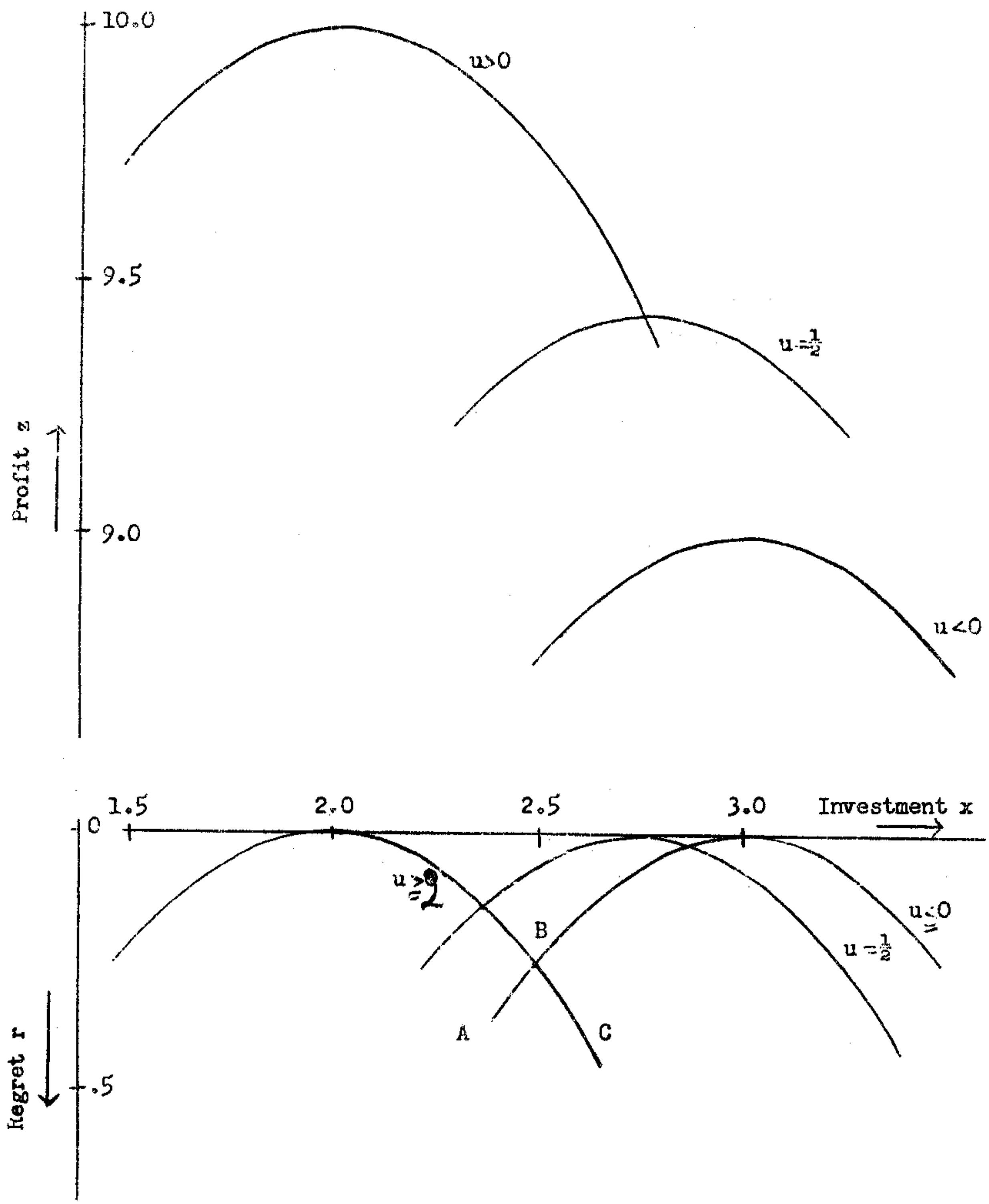
*Note: Since this graph is contructed for mean U = 0, it can be compared with the case of certainty ( $\sigma$ = 0) plotted on GRAPH III, for u = 0 only. The comparison is as follows:

| | | | |
|---|---|---|---|
| $\sigma = 0$ ; $0 \leq \ell \leq 1$ | : | $\hat{x}_0 = 3$ , | $\hat{y} = 0$ |
| $\sigma \to \infty$; $\ell = 1$ | | $\hat{x}_0 = 3$ , | $y = \hat{\eta}(0) = 0$ |
| $\sigma \to \infty$; $\ell = 0$ | | $\hat{x}_0 = 2\ 1/2$ , | $y = \hat{\eta}(0) = 1/2$ |




Profits and regrets as functions of investment (initial input) x, assuming that optimal change (y) in input will be chosen. Future shift (u) unknown. Liquidity ($\ell$) = 0.

Locus of maximal regrets: ABC. <u>Point of minimax regret</u>: B.

Hence best investment at $\ell = 0$: x = 2.5

(Compare best investment at $\ell = 1$: x = 3.0)

Liquidity of Assets and Contracts under Complete Information

Amended version of Section 7:

7.1 Examples of ignorance as defined in 3.1.2:

7.1.1 The firm knows the production function $\pi_0(\ )$, the liquidity $\ell$ of the asset or contract in question, and a set (u) of values, that the shift u of the production function can take; but it has no information to estimate probabilities attached to each of these values. It has to choose x, y.

7.1.2 Same as before but, instead of a set (u) of constants, the firm knows that $\phi(U)$, the probability distribution of the random shift U, has zero mean and is normal, and that standard deviation $\sigma$ is an element of a set $(\sigma)$ of constants.

7.1.3 The two cases 7.1.1 and 7.1.2 can be considered as special cases of the following: The form and all but one parameters of $\phi(U)$ are known. In case 7.1.1, $\sigma = 0$, and $\mathcal{E}U$ is unknown; in 7.1.2, $\mathcal{E}U = 0$, and $\sigma$ is unknown.

7.2 If the unknown conditions $(u, \sigma)$ were determined by a rational opponent in a game, a rational choice by our firm, in case 7.1.1, would be as was shown by von Neumann and Morgenstern: it would "minimax the loss," i.e. make a choice of x, y such as to obtain:

$$\text{Min}_{x,y}\ \text{Max}_u \left[ - z(x, y; u) \right] ;$$

and, in case 6.1.2, it would "minimax the expected value of the loss,"

$$\text{Min}_{x,\eta}\ \text{Max}_{\sigma} \left[ - \mathcal{E}\, Z(x, \eta\, (U); \sigma) \right]$$

7.2.1 In words: In a game, choose a policy (x and y, or x and $\eta$ ) such that, even if the unknown conditions (u or $\sigma$) should turn out to be the ones most adverse to this policy, the expected gain (z or $\mathcal{E}$ Z) would be at least as large as the expected gain that would result from any other policy, and from conditions most adverse to that other policy.

7.3 If conditions are not determined by an inimical rational opponent, there is no reason for this maxim of behavior. L. J. Savage has suggested (orally) that not the loss but the "regret" (or "miss") be minimaxed. In the case 7.1.1, the "regret," r, is the difference between the gain actually obtained and the gain that would be obtained if choice had been made in full knowledge of conditions. We shall work out this case first.

7.3.1 We define regret:

$$r = \operatorname{Max}_{x,y} z(x,y;\, u) - z(x,y;\, u) = r(x,\, y,\, u).$$

The firm chooses x,y so as to achieve

$$\operatorname{Min}_{x,y} \operatorname{Max}_{u \text{ in } (u)} r(x,y,\, u) = r(\bar{x},\, \bar{y},\, \bar{u}), \text{ say.}$$

We are interested in the effect of liquidity $\ell$ upon best investment, $\bar{x}$.

7.3.1.1 The profit-maximizing values $\hat{x}$, $\hat{y}$ derived in 4.4 depended, in general, on u: $x = \hat{x}(u)$; $y = \hat{y}(u)$, say. For a given u, the regret

$$r(x,\, y,\, u) = z(\hat{x},\, \hat{y};\, u) - z(x,\, y;\, u)$$

is a quadratic in x, y, because the profit z is: see 4.3. When $(x,\, y) = (\hat{x},\, \hat{y})$, r reaches its maximum value, zero. It suffices to consider, for each u, the relation between r and x when $y = \hat{y}(u)$, viz.,

$$r(x,\, \hat{y},\, u) = z(\hat{x},\, \hat{y};\, u) - z(x,\, \hat{y},\, u).$$

Again, $r(x,\, \hat{y},\, u)$ is a quadratic in x, with maximum value zero reached when $x = \hat{x}(u) = \hat{x}$. Therefore

$$r(x,\, \hat{y},\, u) = -\, (x - \hat{x})^2/c.$$

This is represented by each of the three parabolas r = r(x) in the <u>lower</u> part of CHART VI. Their vertices have abscissas $\hat{x}$, derived in 4.4. We shall denote them, for each of the three intervals ($\alpha$), ($\beta$), ($\gamma$), by $\hat{x}_\alpha$, $\hat{x}_\beta$ and $\hat{x}_\gamma$ (u), respectively.

7.3.1.2 Suppose, as on CHART VI, that the set $(u)$ consists of the following values of u: One or more values in $(\alpha)$; one or more values in $(\beta)$; one value in $(\gamma)$--say 1/2. It is seen that the minimax regret $r = \bar{r}$ is given by the intersection of the two extreme parabolas; the abscissa of this point is, by symmetry (since the two parabolas differ with respect to their abscissas only)

$$\bar{x} = (\hat{x}_\alpha + \hat{x}_\beta)/2 = c(2b - 3 + \ell)/2.$$

In this case $\bar{x}$ increases with $\ell$.

7.3.1.3 Note that this solution applies whenever the two known bounds of u lie in $(\alpha)$ and $(\beta)$ respectively.

7.3.1.4 As a limiting case, this includes one in which u can take any real value. Note also that the same solution is obtained in the case of complete information about the symmetrical (e.g. a uniform) distribution $\phi(U)$ with $\sigma \rightarrow \infty$: see (5.12.2)!

7.3.1.5 If the two bounds of u do not lie in $(\alpha)$ and $(\beta)$ respectively, it is seen, by inspecting the values of $\hat{x}$ in 4.4, that $\bar{x}$ is either an increasing function of $\ell$ or does not depend on $\ell$.

7.4 Proceed now to the case 7.1.2. Here the minimax regret, $\bar{r}$, is replaced by the "minimax mean value of regret"

$$\text{Min}_{x,\eta}\ \text{Max}_{\sigma \text{ in } (\sigma)}\ \mathcal{E}\, R(x, \eta(U); \sigma),$$

where $R(x, \eta(U); \sigma) = \text{Max}_{x,\eta} Z(x, \eta(U); \sigma) - Z(x, \eta(U); \sigma)$

$$= Z(\hat{x}, \hat{\eta}(U); \sigma) - Z(x, \eta(U); \sigma)$$

Since $\mathcal{E}Z$ in 5.7 is a quadratic in x, $\mathcal{E}R$ is a quadratic that, for a given $\sigma$, has its maximum value, zero at $x = \hat{x}$, $\eta = \hat{\eta}$. Reasoning as in 7.3.1.1 and using (5.12.1), (5.12.2), we obtain

(7.4.1) $\bar{x} = \left[\hat{x}(0) + \hat{x}(\infty)\right]/2 =$

$$c\left[2b - 3 + \ell + (1 - \ell)\psi^*(0)\right]/2;$$

7.4.2 Since $\phi^*(0) < 1$, we note that for $\ell < 1$, the value of $\bar{x}$ obtained in 7.4 is larger than that obtained in 7.3.1.4, 7.3.1.2; to know that $\mathcal{E}U = 0$ and be ignorant about $\sigma$ justifies smaller curtailment of investment than to know that $\sigma = 0$ and be ignorant of $\mathcal{E}U$.

7.4.3 We have by (7.4.1) $d\bar{x}/d\ell > 0$: the size of commitment (asset or contract) increases with its liquidity, if the firm is ignorant about the variance of the future random shift in production cost.

7.5 In a more general case, $\sigma$ is known to be contained between two bounds, $\sigma_1 > \sigma_2$. Then, for a given $\ell$,

$$2\bar{x} = \hat{x}(\sigma_1, \ell) + \hat{x}_2(\sigma_2, \ell).$$

Therefore, by 5.10, $d\bar{x}/d\ell \geq 0$. (Use CHART IV as an example: mark $\bar{x}$ for $\ell = 0$ and for $\ell = 1/2$).