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Homogeneous Systems

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1. Introduction

The properties of homogeneous functions have recently been put to a number of important uses. In particular, they have played a key role in the arguments over the nature of the (static) "Keynesian" system.*

*Cf. especially F. Modigliani. Liquidity Preference and the Theory of Interest and Money. Econometrica, January, 1944, p. 45 ff.

Policy recommendations have, directly or indirectly, been based on the presence or absence of homogeneity and this fact in itself would justify a thoroughgoing investigation of the general properties of the homogeneous function.

I know of only one systematic discussion of the homogeneous system, viz, that by Oscar Lange.*

*Price Flexibility and Employment. Bloomington, Indiana, 1944, p. 99 ff.

The present note is an attempt to supplement and generalize some of Lange's results.

2. Notation

To simplify writing, we use some rudimentary vector notation:

$$y = (y_1, \ldots, y_n)$$

$$y' = (y_1, \ldots, y_m) \qquad 1 \leqslant m \leqslant n$$

$$y'' = (y_{m+1}, \ldots, y_n)$$

so that

$$y = (y', y'').$$

$$\pi = (\pi_1, \ldots, \pi_{m-1}) \qquad \pi_i = \frac{y_{i+1}}{y_1}$$

t is a scalar. k is zero or a positive integer.

3. Definition

A function $f(y)$ is said to be homogeneous in y' of degree k if

(1) $f(ty', y'') = t^k f(y)$

4. Corollary I

If $f(y)$ is homogeneous in y' of degree k then*

*Instead of y_1 any other component of y' could have been chosen in equations (2) and (3).

(2) $f(y) = y_1^k f(\frac{1}{y}, y', y'')$

or

(3) $f(y) = y_1^k g(\pi, y'')$

5. Corollary 2

Given an "underdetermined" system of equations

(4) $f_i(y) = 0 \qquad i = 1, 2, \ldots, n-1$

where the f_i are homogeneous in y' of degree k_i respectively, we may replace the system (4) by

(5) $y_1^{k_1} g_i (\pi, y'') = 0 \qquad i = 1, 2, \ldots, n-1$

or, provided $y_1 \neq 0$, by

(6) $g_i (\pi, y'') = 0 \qquad i = 1, 2, \ldots, n-1$

which is no longer "underdetermined" with regard to the n-1 components of π and y''. However, the actual magnitudes of the components of y' do remain indeterminate.

6. Corollary 3

Given a determinate system of equations

(7) $f_i(y) = 0, \qquad i = 1, 2, \ldots, n$

the assumption that the f_i are homogeneous in y' of degree k_i respectively leads to "overdeterminacy." For by a procedure identical with that used in the previous section we obtain a system of n equations

(8) $\quad g_i\,(\pi, y'') = 0 \qquad\qquad i = 1, 2, \dots, n$

involving only n-1 unknowns.

7. Corollary 4

Given a system of equations

(9) $\quad f_i(y) \qquad\qquad\qquad i = 1, 2, \dots, n$

where $f_j(j = 1, 2, \dots, n-1)$ are homogeneous in y' of degree k_i respectively, it is possible to determine the n-1 components of π and y'' from the first n-1 equations of the system.

Thus π and y'' are independent of the functional form of f_n and of the parameters which enter f_n.

The n-th equation may then be written as

(10) $\quad f_n(y_1, \pi, y'') = 0$

and considered as determining y_1.

8. Generalisation

Homogeneity of functions and its implications are of particular interest in economic applications, but it should be noted that it is but a very special case of a more general type of situation.

Thus let there be given a system of equations

(11) $\quad f_i(y) = 0 \qquad\qquad i = 1, 2, \dots, r$

and let there exist a transformation carrying the vector y into a vector $\sigma = (\sigma_1, \dots, \sigma_n)$ such that

(12) $\quad f_i(y) = g_i(\sigma) \qquad i = 1, 2, \dots, r$

where the g_i have the following factorization property:

$$(13)\qquad g_i(\sigma) = g_{1i}(\sigma')g_{2i}(\sigma'') \qquad i = 1, 2, \dots, r$$

$$\sigma' = (\sigma_1, \dots, \sigma_m)$$

$$\sigma'' = (\sigma_{m+1}, \dots, \sigma_n)$$

Then, provided

$$(14)\qquad g_{1i}(\sigma') \neq 0 \qquad i = 1, 2, \dots, r$$

we obtain a system of equations

$$(15)\qquad g_{2i}(\sigma'') = 0 \qquad i = 1, 2, \dots, r$$

which is overdetermined, determined, or underdetermined as $n - m \lesseqgtr r$.

Clearly, the above corollaries 2, 3, and 4 are special cases of this.

PART 6

Economic Applications

The properties of homogeneous systems have recently been used with success for a number of purposes

I

Independence of "subsystems"

A. A number of controversies have centered around the lack of dependence of a variable (or group of variables) on certain "sectors" of the economy. A classical example is the question: does the rate of interest depend on the quantity of money and the shape of the liquidity preference function?

B. Modigliani* has constructed a system which makes use of

*loc. cit.

of Corollary 4, by making all equations of his sytem, except the liquidity preference equation, homogeneous in prices and wages. Using the symbols

W = wages

P = prices

N = employment

X = real output

r = rate of interest

M_0 = quantity of money (a parameter),

his system may be written as follows

(1a)	$W - PX'(N) = 0$	(demand for labor)
(1b)	$W - PG(N) = 0$	(supply of labor)
(2)	$E(r, X, P) = 0$	(equality of *ex ante* savings and investment)

(3) $\quad M_0 - L(r, PX) = 0$ $\quad$ ("liquidity preference" equation)

where X, G, E, and L are known functions of their arguments.

E (= the excess of ex ante investment over ex ante savings) is homogeneous in P of the first degree* and equation (2) can therefore be written as

(2') $\quad E(r, X, 1) = 0 \quad (P \neq 0)$

or

(2'') $\quad F(r, X) = 0$

*Actually Modigliani uses an equation which is a special case of (2), viz.

(2a) $\quad E_1(r, XP) = 0$

and assumes E_1 to be homogeneous in XP. This, however, leads to overdeterminacy as can be seen from the fact that r could then be obtained either from equations (1) and (2a) or (2a) and (3).

Modigliani then proceeds to show that equations (1) and (2'') determine r, which, therefore, is independent of the shape of L and of the value of M_0. On the other hand P is determined from equation (3) since X is known from equation (1) and r from equations (1) and (2'').

C. Modigliani's point is that the only difference between his system and the Keynesian system is the homogeneity of (1b), while the conclusions are strictly "orthodox"*. This gives perhaps too much emphasis to the "rigid" or "irrational"* behavior of the workers, assumed in the Keynesian version of equation (1b) which may be written as

*The rate of interest is a "real" (non-monetary) phenomenon while the price level depends on the quantity of money.

*Both of these terms are used as synonymous with lack of homogeneity.

(1b') $\quad W - H(P, N) = 0$

where the left member is assumed to be non-homogeneous in W and P.

For suppose that we should have retained the assumption of "rational" behavior of workers, while discarding the homogeneity assumption of either the demand for labor equation (i.e. equation 1a) or the savings-investment equation (i.e. equation 2).

It can be seen that here as in the usual Keynesian approach (based on equation 1b') we should find it impossible to solve for r without using the liquidity preference equations. Thus Keynes conclusions are based on the assumption that at least one* (no matter

*The non-homogeneous nature of the supply of labor function in the Keynesian system is not an accident, however.
If we are to acknowledge the existence of "involuntary employment" without abandoning the assumption of equilibrium in the labor market, we must have a horizontal supply of labor curve.

which) among the equations (1a), (1b), and (2) is non-homogeneous!

D. While the "irrationality" of workers' behavior is stressed, the importance of the non-homogeneous nature of the liquidity preference equations tend to be overlooked.

If one should drop the assumption that the quantity of money is a fixed parameter and make it a genuine variable of the system in such a way that

$$K(r, X, P) = M(r, X, P) - L(r, XP)$$

should become homogeneous in P while (1b') were non-homogeneous, the resulting system would be

(1a)	$W - PX'(N) = 0$	(demand for labor)
(1b')	$W - H(P, N) = 0$	(supply of labor)
(2)	$F(r, X) = 0$	(savings-investment)
(3)	$K(r, X, 1) = R(r, X) = 0$	(liquidity preference)

In this system r and X are determined by equations (2, 3), real wages then follow from (1a) while money wages and prices are obtained from (1b')

II

"Totally homogeneous" systems

A. Professor Lange* considers a system where all the equations

*Op. cit., p. 102, equations (4.11)

are homogeneous, some of zero, some of first degree.

In order to avoid overdeterminacy (see Corollary 3) he should have considered the original (non-homogeneous) system as "underdetermined." This may, for instance, be accomplished by regarding one of the n equations as an identity.

In any case, once the overdeterminacy is removed, Lange's point is an important application of our Corollary 1 viz. that while the "relative price" vector π is determinate, the actual prices y' are not, so that various price and wage levels are compatible (at equilibrium) with the same values of π relative prices (e.g. real wages) and other variables (e.g. the level of employment).

B. It should be noted that Lange's reference to Keynes at this point brings out the existence of two distinct "Keynesian" systems: 1. quantity of money given, rate of interest variable, horizonal labor supply curve; 2. a "totally homogeneous" system with the rate of interest as given.

While Modigliani treats the first variety and is thus able to determine all variables, including money wages and prices, Lange considers the second and, therefore, obtains a system with money wages and price level indeterminate, although real wages are determinate.

C. A word might not be out of place about the interpretation of the fact that in Lange's version of the Keynesian system one equation should be considered as an identity.

It would seem proper to select for this role the excess supply of securities (or money) equations.

For, as Lange points out,* "total homogeneity" of the system

*op. cit., p. 103, footnote 21.

implies a "neutral monetary system." Assuming, as argued by Lange,* that individual excess supply of securities is homogeneous in commodity prices, we might say that a "neutral monetary policy" consists in equating (identically) the central bank's excess demand for securities to the public's excess supply.

*op. cit., p. 102

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