COWLES COMMISSION DISCUSSION PAPERS, ECONOMICS, No. 215

Systems of linear production functions

Tjalling C. Koopmans

February 10, 1948

1. Introduction

In a sequence of articles by Schlesinger (1), Wald (2, 3), von Neumann (4), the technology of production is described as a set of possible processes P_k, $k = 1, \ldots, K$, each conducted with non-negative intensity

(1) $$x_k \geq 0,$$

and characterized by the fact that the process produces or consumes the goods G_n, $n = 1, \ldots, N$, at the respective rates (per unit of time)

(2) $$\gamma_{kn}$$

where a positive value of γ_{kn} indicates production, a negative value indicates "being used up" in the production of something else. With Schlesinger and Wald, the purpose is to let the analysis, from given demand functions, determine not only the quantities produced of various goods, but also which goods are free. Von Neumann generalizes this to answer simultaneously the question which processes are used.

Wald and Schlesinger consider a static case and assume a market for each factor and each product. Theydo not explicitly require optimum allocation of resources, but impose zero profits instead.

Von Neumann considers an economy expanding at a constant rate and maximizes the rate of expansion by proper choice of the intensities x_i of the various processes. This requires an assumption regarding consumption, which he treats as consisting of the minimum necessities of life required to carry on production. Consumption is thus one of the processes P_k, which produces labor. All output not required for consumption is invested immediately. Von Neumann does not elaborate on prices, except the interest rate, which equals the rate of expansion as the result of a minimax theorem. This analysis is based on a fundamental topological theorem, Brouwer's "fixed point theorem."

It is the purpose of this note:

1. To initiate an exploration as to how far one can get with more elementary mathematical tools,

2. To demonstrate the concept of optimal prices without the concept of a market,

3. To prepare for a justification of the concept of a "general transformation function" expressing the convertibility of all goods through production, as used by Lange (5),

4. To provide the basis for the concept of an aggregate production function connecting "value of product," "value of capital" and quantities of primary factors of production.

This note is in the nature of a progress report, incomplete in many respects. We shall explore the production side only, leaving a blank for supplementary assumptions regarding the formation of incomes and of consumers' demand. For possible simplification, or in any case because of limited time of preparation, we shall require the analysis only to answer which processes are used, assuming that we know already which primary factors and which goods are not free.

The great generality of our model of productive processes should be stressed. The only substantial limitation is the disregard of indivisibilities. The concept of a process carried out with a given intensity does not coincide with the concept of the productive activities of one firm. Therefore, <u>within one process</u>, there are no diseconomies of scale, because what one firm can do can be multiplied by any integer by having more firms of the same kind operating. Decreasing returns for the economy as a whole are seen to arise from basic limitations on the amounts of primary factors available to the economy.

Cases where factors of production are continuously substitutable for each other, according to some family of isoquant surfaces in factor space, can be accounted for in our analysis as follows: Cover the surface to any required density with discrete points, and let each point define a process P_k.

By dating the intensities and outputs, dynamic problems can be analyzed with the same mathematical theory. We limit ourselves mainly to the static interpretation.

2. Notations and definitions

Let $a = \{a_i\}$, $b = \{b_i\}$ ... be vectors. We introduce the following notations

Symbol	Meaning	If $b = 0$, a is called
$a > b$	$a_i > b_i$ for all i	positive
$a \geq b$	$a_i \geqq b_i$ for all i $a_i > b_i$ for some i	semi-positive
$a \geqq b$	$a_i \geqq b_i$ for all i	non-negative
$a \not\geq b$	either $a_i = b_i$ for all i, or, if $a_i > b_i$ for some i, then $a_i < b_i$ for some other i.	non-semipositive
$a \equiv b$	$a = b$ by definition	

We shall call the point set $a \geqq 0$ the non-negative a-space.

3. Optimum allocation in the static case with capital saturation

Let the quantity "produced" of the good G_n be denoted by y_n. This is negative in the case of a primary factor used up in production, and positive in the case of a commodity wanted for consumption. It may have either sign for a good simultaneously in both categories. Let x and y denote column vectors containing all process intensities and all (positive and negative) outputs. Then production consists of a mapping of the non-negative x-space X on a set Y in the y-space by the linear transformation

$$y = \Gamma x. \tag{3}$$

If $\rho(M)$, $r(M)$, $c(M)$ denote the rank, number of rows and number of columns of a matrix M, we shall assume that

$$\rho(\Gamma) = r(\Gamma) = N \tag{4}$$

For this it is necessary but not sufficient that the number K of possible processes is at least equal to the number N of goods. The assumption implies further that, if $x > 0$, $y = \Gamma x$ can be varied in the small without restriction through variation of x.

Definition: A point y in Y (i.e. a point y such that there exists an x in X with $y = \Gamma x$) will be called **efficient**, if, for every

$$x^* \equiv x + \xi \tag{5}$$

in X,

$$y^* - y \equiv \eta = \Gamma \xi \cancel{\geqq} 0. \tag{6}$$

?. $\ngeq$

In an efficient point, therefore, no output of a good can be increased without simultaneously decreasing the output of another good (or increasing the input of a primary factor). We shall investigate the set Y_e of efficient points in Y.

The image $y = \Gamma x$ of an internal point $x > 0$ of X cannot be efficient. For then ξ is unrestricted in the small, and because of (4), a ξ can be found to produce a sufficiently small η in any direction, including $\eta > 0$. Therefore, a certain positive number of elements of x must vanish before y can be efficient. For any vector x we introduce, after possible rearrangement of elements (denoted by $\sim$), a partitioning

(7) $$x \sim \begin{pmatrix} x_0 \\ x_+ \end{pmatrix}, \quad x_0 = 0, \quad x_+ > 0, \quad r(x_0) \equiv K_0, \quad r(x_+) \equiv K_+,$$

and a corresponding partitioning

(8) $$\Gamma \sim (\Gamma_0 \quad \Gamma_+)$$

of Γ, which, of course, depends on x, although we do not express that in the notation.

In order that the image $y = \Gamma x$ of x be efficient, it is necessary that for every ξ such that

(9) $$\xi_0 \geqq 0, \qquad \xi_+ \quad \text{unrestricted},$$

we have

(10) $$\eta = \Gamma \xi = \Gamma_0 \xi_0 + \Gamma_+ \xi_+ \ngtr 0$$

Let us first consider all ξ's such that

(11) $$\xi_0 = 0.$$

Because of their presence in (9), efficiency requires that

(12) $$\eta = \Gamma_+ \xi_+ \ngtr 0 \qquad \text{for all } \xi_+$$

Again, (12) requires that

(13) $$\rho(\Gamma_+) < N$$

because if we had $\rho(\Gamma_+) = r(\Gamma_+) \equiv N$, a ξ_+ could be found to produce any sufficiently small η, including positive ones.

Let us now explore the case where

(14) $$\boxed{\rho(\Gamma_+) = N - 1}$$

This is not necessary for efficiency of y, but we conjecture that it occurs almost everywhere in Y_e. If (14) holds there exists a vector p of order N, unique but for normalization, such that

(15) $$\boxed{p'\Gamma_+ = 0.}$$

Furthermore, the efficiency of y requires that p can be so normalized that

(16) $$p > 0.$$

For, because of (14), every sufficiently small η such that

(17) $$p'\eta = 0$$

can be produced by a

$$\xi = \begin{pmatrix} 0 \\ \xi_+ \end{pmatrix}$$

with x* in X. If any component of p were zero, or if any two components were of opposite sign, therefore, (12) could be contradicted. The elements of p will be recognized as "accounting" prices, the equation (17) as a local "facet" of the transformation function. If productive units charge each other with these (marginal cost) prices, and if the management of each unit produces whatever it does produce at a minimum cost, then the output of the whole economy will correspond to an efficient point.

We continue the exploration of (14) and now follow up the requirement (contained in (9) besides the requirement (11) already analyzed) that for every ξ such that

(18) $$\xi_0 \geq 0 \qquad \xi_+ \text{ unrestricted}$$

we have

$$(19) \qquad \eta = \Gamma \xi \not\geqq 0.$$

For this to be true, it is necessary that for every $\xi_0 \geq 0$,

$$(20) \qquad p'\eta = p'\Gamma_0 \xi_0 \leqq 0.$$

To show this, suppose a $\xi_0 \geq 0$ existed such that

$$(21) \qquad p'\eta = p'\Gamma_0 \xi_0 > 0$$

Then, on account of (14), it would be possible to choose ξ_+ in such a way that

$$(22) \qquad \eta_n = (\Gamma_0 \xi_0 + \Gamma_+ \xi_+)_n = 0, \quad n = 2, \ldots, N,$$

and (21) would imply

$$(23) \qquad p_1 \eta_1 > 0$$

or, in view of (16),

$$(24) \qquad \eta_1 > 0,$$

which together with (22) contradicts (19).

Finally, for (20) to be true it is necessary that

$$(25) \qquad \boxed{p'\Gamma_0 \leqq 0}$$

This condition says that no unused process should permit an "accounting profit."

Conversely, it is easily seen that any point x satisfying (14) and (25), with p defined by (15), is mapped into an efficient point y. We thus have the <u>**Theorem:**</u> Necessary and sufficient that a point x satisfying (14) be mapped into an efficient point y is that (25) is satisfied, with p defined by (15).

So far we have not proved the existence of a single efficient point, and it is not difficult to construct matrices Γ for which (14) and (25) are incompatible. But the facts of life are that you cannot produce something out of nothing. Therefore actual matrices Γ are such that

(26) $$x \geq 0 \quad \text{excludes } y = \Gamma x \geq 0$$

In other words, Y does not contain any semi-positive points. It is not so easy to see what property of individual processes is involved here. Perhaps the law of conservation of energy. Or the fact that all processes require labor.

It should be noted further that Y is a convex set. Whenever $y^{(1)}$ and $y^{(2)}$ belong to Y, any point

(27) $$y = \lambda y^{(1)} + (1 - \lambda)y^{(2)} \qquad 0 \leqq \lambda \leqq 1$$

on the straight line between $y^{(1)}$ and $y^{(2)}$ also belongs to Y.

It should further be investigated under what conditions, as a result of these properties, there exists a transformation function

(28) $$f(y_1, \ldots, y_n) = 0$$

built up from facets of the type (17) or

(29) $$p'(y - y^{(0)}) = 0,$$

where $y^{(0)}$ is any point on the facet. The following picture seems plausible. If we define the partitioning $(x_{(0)}\ x_{(+)})$ as descriptive of one particular facet rather than of the point x, the border of that facet is made up of points where in addition to $x_{(0)}$ also one or more elements of $x_{(+)}$ vanish. Adjoining facets are those on which one element of $x_{(0)}$ is allowed to differ from zero. The points on one facet only are those for which (14) is satisfied. Points on the "borderlines" where two or more facets meet are points where Γ_+ sustains a corresponding reduction in rank. All first derivatives of the transformation function are of the same sign where they exist. At "borderline" points, some or all derivatives are different to the left and to the right.

It follows from the convexity of Y that, if y_1 is a primary factor

($y_1 < 0$) and y_2 a product, $\frac{\partial y_2}{\partial |y_1|}$ can only jump downwards as $|y_1|$ increases and $y_3 \ldots y_N$ are held constant. This establishes the presence of diminishing (better: non-increasing) returns with respect to the increase of any one factor toward the increase of any one output. The statement can be generalized with respect to the increase of any linear index number of output with respect to one factor, provided this is not the only factor.

In formulating the conditions for efficiency of a point $y = \Gamma x$, we have not needed to consider the number K_+ of elements in x_+. The only question for which this number is important is whether to a given efficient point y there corresponds one unique x or infinitely many. If

(30) $$L_+ \equiv K_+ - \rho(\Gamma_+)$$

is zero, there is a unique x; if (30) is a positive integer there is an L_+-parameter linear family of points x.

There is an alternative way of deriving (16) and (25) by the method of Lagrange parameters. It is noted that efficiency of y requires that

(31) $$y_1 = \Gamma_1 x$$

shall be a maximum for all x such that

(32) $$x \geqq 0, \qquad {}_1y = {}_1\Gamma x$$

where ${}_1y = \begin{pmatrix} y_2 \\ \vdots \\ y_N \end{pmatrix}$ is a given vector. One substitutes

(33) $$x_k = z_k^2, \qquad z_k \text{ real}$$

to satisfy $x \geqq 0$ automatically and requires

(34) $$\delta\left(\sum_{k=1}^{K} \gamma_{1k} z_k^2 + \sum_{n=2}^{N} p_n \sum_{k=1}^{K} \gamma_{nk} z_k^2\right) = 0$$

the Lagrange parameters, p_m being such as to satisfy

(35) $$y_n = \sum_k \gamma_{nk} z_k^2, \qquad n = 2, \ldots, N.$$

Introducing

(36) $$p_1 = 1,$$

the problem is seen to be symmetric with respect to the various goods $n = 1, \ldots, N$. The first order condition derivable from (34) is

(37) $$z_k \sum_{n=1}^{N} p_n \gamma_{nk} = 0 \qquad k = 1, \ldots, K$$

which implies that, for any k either

(38) $$z_k = 0, \; x_k = 0$$

or

(39) $$\sum_n p_n \gamma_{nk} = 0,$$

which is equivalent to (15). The second order condition leads, with somewhat greater effort, to (25). While this approach seems quite elegant, it brings in the somewhat irrelevant technique of differential calculus, and is less "visual" than the previous derivation.

While it has been suggested that in a wide class of cases (with at least one exception noted below) a single transformation function exists when (4) holds, the transformation surface has a lower dimensionality when

(40) $$\rho(\Gamma) \leqq N - 1,$$

because in that case the dimensionality of Y is already less than N, and that of Y_3 may be expected to be still lower. A case of this kind is that where two goods can only be produced in the same proportion by any of the processes. In such cases the desirability of admitting free goods in the analysis is even stronger.

In the case where Γ partitions as follows

(41) $$\Gamma = \begin{pmatrix} \Gamma_{I} & 0 \\ 0 & \Gamma_{II} \end{pmatrix}$$

the only efficient points where (14) holds are those where

(42) $$\text{either } y_I = 0 \text{ or } y_{II} = 0,$$

because for any other efficient points we must have two independent restrictions

(43) $$p'_I \Gamma_{I+} = 0, \qquad p'_{II} \Gamma_{II+} = 0$$

on neighboring efficient points. This is the case where products and factors split up into two sets without any possibility of substitution or transformation between sets, and there are no relative prices between goods of different sets. In this case, therefore, the conjecture (28) is not valid.

4. Optimum allocation in the static case with limited capital budget

The foregoing model has been described as having saturation of capital because our analysis is entirely in terms of rates of output (or input), without introducing a "penalty" on the use of capital intensive processes. Therefore, among processes being compared, the most productive ones will be used, irrespective of their capital requirements.

Capital can be introduced by partitioning y as follows

$$y = \begin{pmatrix} y_- \\ y_0 \\ y_+ \end{pmatrix}, \qquad \begin{array}{l} y_- \leq 0, \\ y_0 = 0, \\ y_+ \geq 0, \end{array}$$

where y_- represents rates of use of primary factors, y_0 net rates of output of intermediate goods, and y_+ rates of consumption of final goods. Besides the net flows y_0, which vanish in the static case, we consider a stock vector $z_0 \geq 0$ of the same intermediate goods. Each process P_k now carries with it coefficients δ_{km} indicating what stocks of intermediate goods are required by it.

The fact that y_0 is constrained to the value 0 does not prevent the intermediate goods from having a vector p_0 of prices. These prices can always be established by relaxation of the constraint and variation in the small of y_0. Similarly the stocks of intermediate goods carry with them a vector q_0 of "rentals", which we define free of depreciation, because the latter is a rate of input. The relevant equations derivable as before are thus

$$(45)\quad \begin{cases} y_- = \Gamma_{-*}\, x \\ y_0 = \Gamma_{0*}\, x = 0 \\ y_+ = \Gamma_{+*}\, x \\ z_0 = \Delta_{0*}\, x \\ p_-'\Gamma_{-+} + p_0'\Gamma_{0+} + p_+'\Gamma_{++} + q_0'\Delta_{0+} = 0 \\ p_-'\Gamma_{-0} + p_0'\Gamma_{00} + p_+'\Gamma_{+0} + q_0'\Delta_{00} \leqq 0 \end{cases}$$

Besides those strictly static conditions, there is a new condition for optimal allocation of resources, deriving from the fact that a positive net output of an intermediate good adds to its stock: It should not be possible, at the going prices p, q_0, to make small changes in the stocks z_0 of intermediate goods that preserve the total value

$$(46)\qquad C \equiv q_0' z_0$$

of capital, and at the same time "increase" the vector y. Such changes could, of course, be made only by temporary relaxation of the restraint $y_0 = 0$, that is by a dynamic process. It is easily seen that this new condition requires the existence of an interest rate, i.e. a scalar λ such that

$$(47)\qquad q_0 = \lambda\, p_0 .$$

Defining the value of the outputs and inputs

$$(48)\qquad V_+ \equiv p_+' y_+ , \qquad V_- \equiv p_-' y_-$$

the quantity

$$(49)\qquad \left(\frac{\partial V_+}{\partial C}\right)_{V_-\ \text{constant}}$$

is, for small variations of C, independent of the particular direction in which production is expanded. In an economy as here considered, therefore, there is no aggregation problem involved in defining the marginal productivity of capital. Before an aggregate production function in the large can be established, it is necessary to make further specifications on the demand side, so as to make possible a determination of the direction η_{+} in which the added income from a greater capital stock will be spent.

References

(1) Karl Schlesinger, Über die Produktionsgleichungen der ökonomischen Wertlehre, Ergebnisse Eines Mathematischen Kolloquiums, Heft 6, 1933-1934.

(2) A. Wald, Über die eindeutige positive Lösbarkeit der neuen Produktionsgleichungen, Ergebnisse Eines Mathematischen Kolloquiums, Heft 6, 1933-1934.

(3) A. Wald, Über die Produktionsgleichunger der ökonomischen Weltlehre. II., Ergebnisse Eines Mathematischen Kolloquiums, Heft 7, 1934-1935.

(4) J. von Neumann, Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, Ergebnisse Eines Mathematischen Kolloquiums, Heft 8, 1935-1936. English translation by G. Morgenstern, A model of general economic equilibrium, The Review of Economic Studies, Vol. XIII (1), No. 33, 1945-1946.

(5) Oscar Lange, The foundations of welfare economics, Econometrica, Vol. 10, July-October, 1942.