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TAXATION AND ENTREPRENEURS' CHOICE

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(with acknowledgements to K. Arrow)

1. Proposition on the irrelevance of tax schedules.

Write G ("gross") = anticipated profit before taxes; $T = T(G)$ = profit tax. If $G = G(x)$ depends on the quantity x of a certain production factor, and $\hat{x}$ is the quantity of x that maximizes the net profit $G-T(G)$ then,

$$G'(\hat{x})\cdot(1-T'(G)) = 0; \quad G''(\hat{x})(1-T'(G)) - G'(\hat{x})T''(G)) < 0.$$

This condition is satisfied if $G'(\hat{x}) = 0$, $G''(\hat{x}) < 0$, $0 < T'(G) < 1$; that is, if x maximizes $G(x)$ and if the marginal tax rate $T'(G)$ is smaller than 1. Thus the firm's decision to employ more or less of a factor does not depend on the tax schedule provided $T'(G) < 1$. The latter provision is usually satisfied, with a progressive as well as a linear (or, more particularly, proportional) tax schedule: the profit increment is not fully taken away. L. R. Klein [1] gave and proved this proposition as outlined. (Arrow suspects that already Smith said so!) This proposition seems to contradict the often made statement that stiff progressive taxes affect business activities.

2. Two complications.

However, the proposition does not take account of two complications (Lerner, [2] seems to have in mind these complications or some combination of them): a) by assuming the tax schedule $T(G)$ differentiable, it does not take into account losses (except when they are credited against another year's profits), or the existence of the exemption limit, or the discontinuous nature of "income brackets" in certain types of schedules; b) it neglects uncertainty. We shall show (in Sec. 4) that complication (a) is not essential, but (in Sec. 5 and following) complication (b) is essential.

3. The set of mutually exclusive decisions.

Instead of considering single production factors it is convenient to define, more generally, the set $\underline{Z}$ of all mutually exclusive decisions open to the firm and affecting its gross profit G. The set $\underline{Z}$ can be approximately regarded as the set of all balance sheets that the firm may have, account being taken of physical and market conditions, the firm's capacity to borrow, etc. More generally, some of the elements Z', Z'' of $\underline{Z}$ may be distinct with respect to qualitative properties only (e.g., 2 different production methods may involve identical balance-sheets.) In sec. 5 and following, however, $\underline{Z}$ will be a set of balance sheets. If the balance sheet consists of two items whose ratio, or whose combined money value, is fixed, the set $\underline{Z}$ becomes a single, possibly continuous, variable z, viz. the money value of one of the assets; and the single elements of $\underline{Z}$ become single values of z--say, z', $\hat{z}$ etc. If there are m assets subject to n restrictions, the firm has m-n degrees of freedom and $\underline{Z}$ becomes a vector, say $\underline{z}$, with m-n components, viz. the decision-variables $z_1, \ldots, z_{m-n}$ which may be asset-values, ratios etc. (Hurwicz [3] has a somewhat different terminology).

4. Non-differentiable tax schedules in absence of uncertainty. Write N="net" profit = $G-T(G) = N(G)$. If the profit increment is never fully taxed away,

$$(4.1)\quad N(G_1) > N(G_2) \text{ if and } [\text{because } T(G) \text{ is single-valued}] \text{ only if } G_1 > G_2.$$

That is, $N(G)$ monotomically increasing; it need not be differentiable. The case of losses and of tax-exempt profits is shown on the diagram. Now let $\hat{Z}$ in $\underline{Z}$ be the best, i.e. N-maximizing decision when the net profit schedule is $N^*(G)$; and let Z' in $\underline{Z}$ be some other decision, $Z' \neq \hat{Z}$. Then

$$N^*(G(\hat{Z})) > N^*(G(Z')); \text{ and by (4.1)}$$
$$G(\hat{Z}) > G(Z')$$
$$N^{**}(G(\hat{Z})) > N^{**}(G(Z')),$$

where N^{**} is some net profit schedule other than $N^*(G)$. Thus, in spite of the change in the tax-schedule, $\hat{Z}$ will remain the firm's best decision.

5. Introducing Uncertainty.

Consider the gross profit G as a random variable with distribution density

$$(5.1)\quad \phi\,(G;\, \underline{\theta}(\underline{Z}))$$

where each element of the parameter vector $\underline{\theta} = \theta_1,\ldots,\theta_I$ is a function of the set of firm's decisions $\underline{Z}$. In studying the effect of the tax-schedule upon the choice of decision, we shall assume that the firm maximizes the mean anticipated net profit

$$(5.2)\quad M \equiv EN(G) \equiv \int_{-\infty}^{\infty} N(G)\,\phi\,(G;\,\theta)\,dG;$$

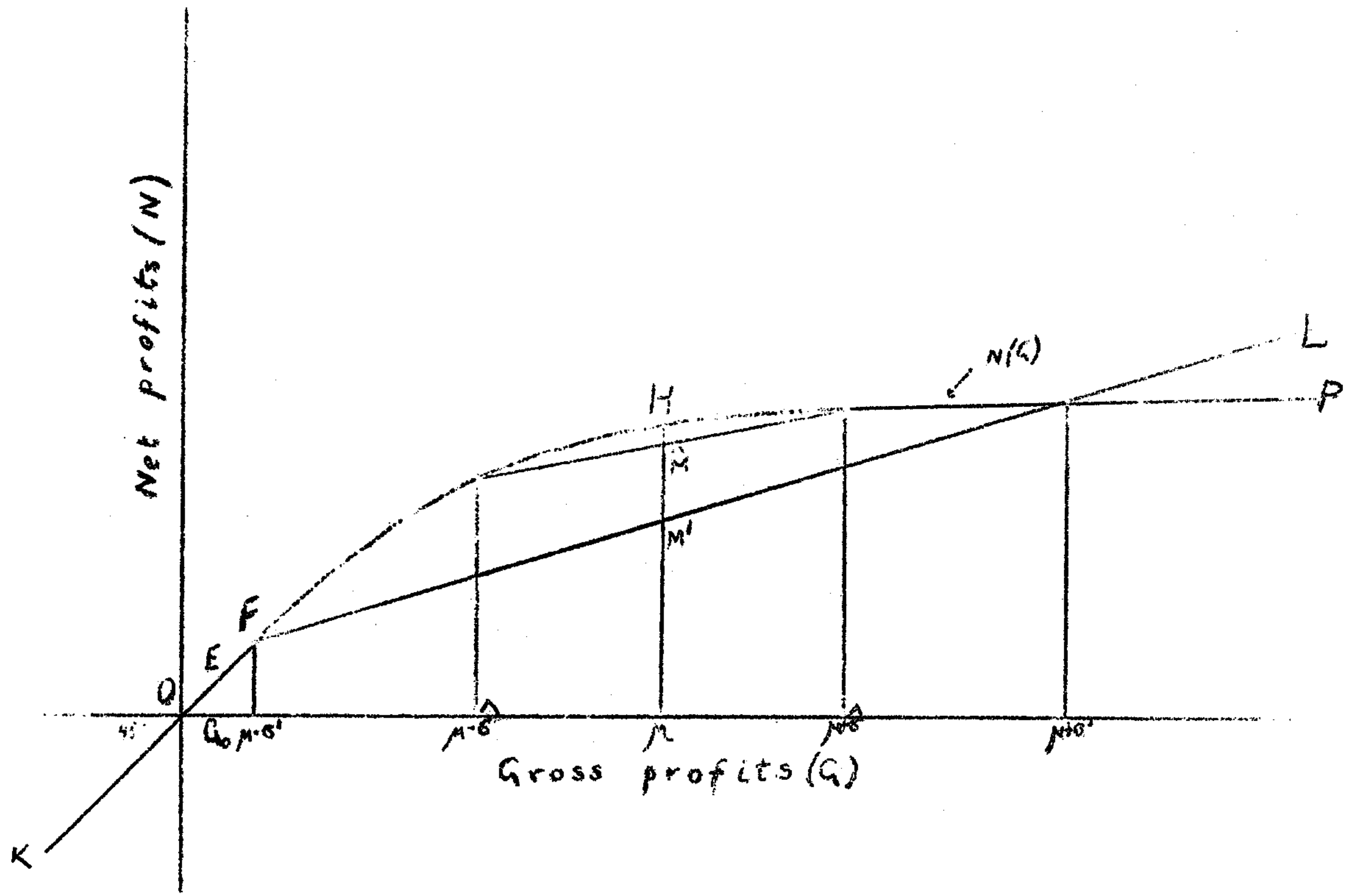
We assume that the firm is not concerned with any other distribution parameters of the net profit (a random variable), e.g. with its variance. Thus, unlike Domar and Musgrave [4], we do not introduce "risk" as an argument of the utility function. If M is at its maximum, utility is at its maximum, too.*) The implications of assuming that the firm maximizes some function $U(M, E(M-N)^2)$ may be studied later.

*) It is interesting to note that the results that follow admit formally of an alternative economic interpretation. In (5.1) we may rename some of the symbols as follows: G = net income of an individual, $N(G)$ = (measurable)utility; and the tax schedule determines parameters of the function $\theta(\underline{Z})$. Then M = mean value of utility. If the derivatives of the utility functions are (as they are usually supposed to be) $N'(G) > 0$, $N''(G) < 0$, the properties of the net-profit schedule based on progressive tax schedule, will apply to the utility function. Thus "risk aversion" is derived: Comp. Marshall [5]. To derive "love of long odds" (i.e. of skewness), properties of $N'''(G)$ must be introduced.

Denote by $\underline{a} \equiv (a_1,\ldots,a_K)$ the parameters of the net-profit schedule: $N=N(G;\underline{a})$. Then M in (5.2) depends on $\underline{Z}$ and $\underline{a}$

$$(5.3)\quad M \equiv M\,(\underline{a},\, \theta\,(\underline{Z}));$$

hence the element $\hat{Z}$ in $\underline{Z}$ which maximizes M is a function of $\underline{a}$; and the parameters $\hat{\theta}_1,\ldots,\hat{\theta}_I$ corresponding to the optimum decision are also functions of $\underline{a}$. If the firm has H degrees of freedom, so that $\underline{Z} = z_1,\ldots,z_H$, we may derive the properties of the derivatives $\partial\hat{z}_h/\partial a_k$ ($h = 1,\ldots, H$; $k = 1,\ldots,K$) by writing the conditions


Net profits (N)
Gross profits (G)
L
P
N(G)
H
M̂
M'
F
E
O
K
45°
G_0
$\mu-\sigma'$
$\mu-\hat{\sigma}$
μ
$\mu+\hat{\sigma}$
$\mu+\sigma'$

for maximum M with respect to $\underline{Z}$, and differentiating the resulting equations with respect to $\underline{a}$.

6. Uncertainty but no losses; one decision variable.

Assume that, for an interval $G'' > G > G'$ $(G' > 0)$, the cubic and higher terms of the Taylor expansion for $N(G)$ are negligible and that, approximately

(6.1) $\int_{G'}^{G''} \phi(G)dG = 1$. Write

(6.2) $N = a + bG - 1/2\, cG^2, \quad G'' > G > G'$.

If the tax is progressive, $N''(G) \equiv -c < 0$; if the tax is linear, $c = 0$. The case $N''(G) < 0$ (regressive tax) will not be considered. Further since no profit increment is fully taxed away

(6.3) $0 < N'(G) = b - cG$,
hence $b > 0$*).

*) In fact, $0 < b < 1$ (as is also seen from the diagram). Let $G_o (0 \leq G_o \leq G')$ be the exemption limit; then $G_o = a + bG_o - 1/2\, cG_o^2$; subtracting from (6.2),

$$N = bG - (1-b)\, G_o - 1/2\, c(G^2 - G_o^2) < G,$$
$$b + 1/2c\, (G + G_o) < 1, \quad b < 1$$

Write $E(N) \equiv M$ (average net profit); $E(G) = \mu$ (average gross profit); $E(G-\mu)^2 = \sigma^2$. Then by (6.1), (6.2),

(6.4) $M \equiv a + b\mu - 1/2c\, (\mu^2 + \sigma^2)$,

and by (6.3) $b - c\mu > 0$; We shall define a new tax parameter

(6.5) $r \equiv c/b$; we may call it progressiveness.

Denote by z a (single) decision variable and write $M = M(z)$ and also

$$\mu = \mu(z), \quad \sigma = \sigma(z), \text{ corresponding to } \theta\ (\underline{Z}) \text{ in } (5.1).$$

We shall study the effect of tax-parameters a, b, c, upon the choice of $\hat{z}$, the best (i.e. M-maximizing) value of z, and upon the corresponding values of μ, σ. We shall write $M_1, \mu_1, \mu_{11}, \sigma_1, \sigma_{11}$ for dM/dz, $d\mu/dz$, $d^2\mu/dz^2$ etc; when $z = \hat{z}$ we shall sometimes (but not always) write $\hat{M}, \hat{\mu}, \hat{\mu}_1$ etc for $M(\hat{z})$, $\mu(\hat{z})$, $\mu_1(\hat{z})$.. $\hat{\mu}_1, \hat{\sigma}_1$ may be called, respectively, the "marginal contribution of z to the gross profit" and "to the riskiness", at the point of equilibrium.

Under a linear tax, $c = 0$, $M = a + b\mu$, and the firm will choose the decision which maximizes the average gross profit μ regardless of the decision's effect upon the standard deviation σ ("risk") of the gross profit:

(6.7) $c = 0$, $\hat{\mu} = \mathrm{Max}_z \mu$; $\hat{\mu}_1 = 0$; $\hat{\mu}_{11} < 0$.

For $c > 0$, maximize (6.5) with respect to z.

(6.8) $M_1(\hat{z}) \equiv \hat{M}_1 = 0 = b\hat{\mu}_1 - c(\hat{\sigma}\hat{\sigma}_1 + \hat{\mu}\hat{\mu}_1)$;

(6.9) $0 = \hat{\mu}_1 - r\,(\hat{\sigma}\hat{\sigma}_1 + \hat{\mu}\hat{\mu}_1)$ where
$r = c/b > 0$.

Thus decision z depends only on the ratio of two of the three tax-parameters a,b,c. Further,

$$(6.10)\quad 0 > M_{11}/b = \hat{\mu}_{11} - r\,(\hat{\sigma}\hat{\sigma}_{11} + \hat{\sigma}_1^{\,2} + \hat{\mu}\hat{\mu}_{11} + \hat{\mu}_1^{\,2}) = K_{11} \text{ (say).}$$

Differentiate (6.9) with respect to r

$$(6.10a)\quad 0 = K_{11} \cdot d\hat{z}/dr - (\hat{\sigma}\hat{\sigma}_1 + \hat{\mu}\hat{\mu}_1) = K_{11} \cdot dz/dr - \mu_1/r$$

$$(6.11)\quad d\hat{z}/dr = \hat{\mu}_1/rK_{11};\ \text{and } d\hat{\mu}/dr = \mu_1 \cdot dz/dr = \hat{\mu}_1^{\,2}/rK_{11};\ d\hat{\sigma}/dr = \hat{\sigma}_1\hat{\mu}_1/rK_{11}$$

Apart from the above case of linear tax ($r = 0, \hat{\mu}_1 = 0$), $d\hat{\mu}/dr < 0$ since $K_{11} < 0$. i.e., the increase in r leads to a fall in the average gross profit. To see whether the riskiness $\hat{\sigma}$ is also decreased solve (6.9) for $\hat{\mu}_1$:

$$(6.12)\quad \hat{\mu}_1 = r\hat{\sigma}\hat{\sigma}_1/(1-r\hat{\mu});\ \text{hence}$$

$$(6.13)\quad d\hat{\sigma}/dr = \hat{\sigma}\hat{\sigma}_1^{\,2}/K(1-r\hat{\mu});$$

And since $K_{11} < 0$ and $1-r\mu > 0$ (by (6.10), (6.6)),

$$(6.14)\quad \partial\hat{\sigma}/\partial r < 0.$$

To study the influence of tax on the amount of the decision variable used, let us call a decision variable <u>risky</u> (at the equilibrium point $z = \hat{z}$) if $\hat{\sigma}_1 > 0$. Since $d\hat{\sigma}/dr = \hat{\sigma}_1 \cdot dz/dr$, we have, for a risky decision,

$$\operatorname{sgn} dz/dr = \operatorname{sgn} d\hat{\sigma}/dr.$$

Using (6.14) we conclude that the risk $\hat{\sigma}$, of the entrepeneur's best plan, and the (best) quantity $\hat{z}$ of a "risky" decision variable, falls with increasing r.

If the tax is linear we have (6.7); and by (6.12), (6.10)

$$d\hat{\sigma}/dr = \hat{\sigma}\hat{\sigma}_1^{\,2}/\mu_{11} < 0;$$

so that if $\hat{\sigma}_1 > 0$, $dz/dr = \hat{\sigma}\hat{\sigma}_1/\mu_{11} < 0$; i.e., the introduction of a (slightly) progressive tax instead of a linear one, decreases the risk and the use of risky decisions. As to the effect of such action on the average profit, we have $d\hat{\mu}/dr = dz/dr \cdot \hat{\mu}_1 = \hat{\sigma}\hat{\sigma}_1/\hat{\mu}_{11} \cdot \hat{\mu}_1 = 0$. The second derivative is

$$d^2\hat{\mu}/dr^2 = d^2z/dr^2 \cdot \hat{\mu}_1 + (dz/dr)^2 \cdot \hat{\mu}_{11} = 0 + (\hat{\sigma}\hat{\sigma}_1)^2/\hat{\mu}_{11} < 0.$$

That is,

$$(\hat{\mu})_{r=0} = \operatorname{Max}_r \hat{\mu},$$

i.e. linear tax entails higher best average gross profit than does progressive tax.

7/ <u>A simple case.</u>

On the diagram the line KEFHP represents a net profit schedule $N(G)$, based on progressive income tax with exemption limit G_0. Let (G) be as follows: μ is constant, and

$$(7.1)\quad \text{Prob}(G = \mu + k) = 1/2 = \text{Prob}\,(G = \mu - k);\ \mu - k > G_0\ ;$$

we have $\sigma = k = \sigma(z)$. Let $\hat{\sigma}$ be the lowest possible value of σ, and σ' some other value. Then the ordinate $M = 1/2\, N(\mu + \hat{\sigma}) + 1/2\, N(\mu - \hat{\sigma})$ represents the maximum average net profit; it is higher than, e.g., $M' = 1/2N(\mu + \sigma') + 1/2N(\mu - \sigma')$. The firm will therefore choose a value of z that would yield the lowest risk, viz. $\hat{\sigma}$. This best value of $\hat{z}$ will remain the same under any progressive income tax since $\hat{\sigma}$ does not depend on the tax-schedule. If however, the tax schedule is linear--e.g. KEFL, the maximum average net profit becomes independent of σ, hence σ becomes indeterminate. The replacing of linear by <u>any</u> progressive income tax leads to choose the least risky arrangement. This is in line with the equations of sec. 6, extended to the case of any concave $N(G)$ and immediately applicable to the case of several decision variables. To study in the next section, the case of several variables under a general distribution $\phi(G)$ instead of (7.1), we have to introduce again the Taylor approximation (6.2).

8. Several decision variables.

Let $\underline{z} = z_1, \ldots, z_{11}$, $\mu = \mu(\underline{z})$, $\sigma = \sigma(\underline{z})$: Approximate $N(G)$ by

$$(6.2)\quad N = a + bG - 1/2cG^2,\ \text{hence}$$
$$(6.4)\quad M = a + b - c(\mu^2 + \sigma^2)$$

Using notations similar to those of Sec. 6,

$$(7.1)\quad \hat{M}_h/b = 0 = \hat{\mu}_h - r(\hat{\sigma}\hat{\sigma}_h + \hat{\mu}\hat{\mu}_h),\ h = 1,\ldots,H;$$

$$(7.2)\quad \partial\hat{M}_h/(b\,\partial z_i) = \hat{M}_{hi}/b = K_{hi} = K_{ih}.$$

Differentiate (7.1) with respect to r, denoting

$d\hat{z}_h/dr$ by $\bar{z}_h$; we get

$$(7.3)\quad 0 = \sum_i K_{hi}\bar{z}_i - (\hat{\sigma}\hat{\sigma}_h + \hat{\mu}\hat{\mu}_h);\ \text{and by (7.1)}$$
$$(7.4)\quad \sum_i K_{hi}\,\bar{z}_i = \hat{\mu}_h/r;\ \text{solving,}$$
$$(7.5)\quad \bar{z}_h = \sum_i \hat{\mu}_i K^{hi}/r,\ \text{where } \|K^{hi}\| \text{ and } \|K_{hi}\| \text{ are both negative}$$

definite because $\hat{M}$ is a maximum. Therefore

$$(7.6)\quad d\hat{\mu}/dr = \sum_h \bar{z}_h \hat{\mu}_h = \sum_{h,i} \hat{\mu}_h K^{hi} \hat{\mu}_i < 0;$$

$$(7.7)\quad d\hat{\sigma}/dr = \sum_h \bar{z}_h \hat{\sigma}_h = \sum_{h,i} \hat{\sigma}_h K^{hi} \hat{\mu}_i/r.$$

The quadratic form in (7.6) is negative definite; to determine the sign of the bilinear form in (7.7) we notice that by (7.1)

$$\hat{\sigma}_h = \hat{\mu}_h(1 - r\hat{\mu})/r\hat{\sigma},\quad h = 1,\ldots,H.$$

$$(7.8)\quad d\hat{\sigma}/dr = (1-r\hat{\mu})/r^2\hat{\sigma} \cdot \Sigma_{h,i}\hat{\mu}_h K^{hi}\hat{\mu}_i;$$

$$= (1-r\hat{\mu})/r^2\hat{\sigma} \cdot d\hat{\mu}/dr; \text{ hence } d\hat{\mu}/dr < 0, \text{ by } (6.6),(7.6).$$

Thus, an increase in r decreases the optimal average gross profit $\hat{\mu}$ as well as the optimal average gross riskiness $\hat{\sigma}$.

9. Introducing losses.

Drop the assumption $G > G_o > 0$. Also, for simplicity, put $G_o = 0$ (a trivial detail) so that $a = 0$,

$$(8.1)\quad \begin{aligned} N &= bG-(1/2)c\, G^2, \quad G > 0 \\ N &= G, \quad G < 0 \end{aligned}$$

Introduce new distribution parameters

$\nu = \int_{-\infty}^{0} G\phi\, dG$; $\rho = \int_{0}^{\infty} G\phi\, dG$: the negative and positive "part-means"; and $\tau = \int_{0}^{\infty} (G-\rho)^2 \phi\, dG$ (standard deviation of positive profits). Then

$$(8.2)\quad M = E(N) = \nu + b\rho - 1/2c(\tau^2 + \rho^2/p),$$

where $p = \int_{0}^{\infty} \phi\, dG$ = probability of positive profits. We shall not pursue this further but assume a linear tax (following Arrow's suggestion). We have

$$(8.2)\quad M = \nu + b\rho$$

$$(8.3)\quad 0 = (dM/dz)_{z=\hat{z}} = \hat{\nu}_1 + b\hat{\rho}_1, \text{ where } \hat{\nu}_1 = d\nu(\hat{z})/dz \text{ etc.}$$

$$(8.4)\quad 0 > \hat{\nu}_{11} + b\hat{\rho}_{11}.$$

Differentiate (8.3) with respect to b and solve:

$$d\hat{z}/db = -\hat{\rho}_1/(\hat{\nu}_{11} + b\hat{\rho}_{11})$$

$$(8.5)\quad d\hat{\rho}/db = (d\hat{z}/db)\colon \hat{\rho}_1 = -1/(\nu_{11} + b\hat{\rho}_{11}) > 0, \text{ by } (8.4).$$

$$(8.6)\quad d\hat{\nu}/db = (d\hat{z}/db)\colon \nu_1;\quad \text{sgn}\, d\hat{\nu}/db = \text{sgn}(\hat{\rho}_1\ \nu_1)..$$

Thus, if losses are not impossible, a decrease in b -- i.e., an increase in the marginal tax rate (1-b) --drives the firm to a decision involving a fall in the average of (positive) gross profits. And if ventures promising large average profits imply as a rule large losses, the increase in the marginal tax rate leads to cutting down on such ventures. This is true even without the tax schedule being progressive. In the case of progressive tax and possible losses, the effect of the tax-parameters b, c can be seen on the simple example of Sec. 7 (but with both μ and σ variable) , with

(8.7) $G_2 = \mu + \sigma > 0, \quad G_1 = \mu - \sigma < 0;$

$2M = N(G_1) + N(G_2)$; substitute into (8.1)

(8.8) $2\,M = b(\mu + \sigma) - \frac{1}{2}\,c(\mu + \sigma)^2 + (\mu - \sigma)$

(8.9) $0 = 2\hat{M}_1 = b(\hat{\mu}_1 + \hat{\sigma}_1) - c(\hat{\mu} + \hat{\sigma})(\hat{\mu}_1 + \hat{\sigma}_1) + (\hat{\mu}_1 - \hat{\sigma}_1);$

(8.10) $2\hat{M}_{11} < 0$; differentiate (8.8) with respect to b, c:

$0 = 2\hat{M}_{11} \cdot d\hat{z}/db + \hat{\mu}_1 + \hat{\sigma}_1 = \hat{M}_{11}\ d\hat{z}/dc - (\hat{\mu} + \hat{\sigma})\ (\hat{\mu}_1 + \hat{\sigma}_1)$; hence

$d\hat{z}/db = (\hat{\mu}_1 + \hat{\sigma}_1)/ -\hat{M}_{11}\ ;\ d\hat{z}/dc = -\ G_2\ .\ d\hat{z}/db$

$d\hat{\mu}/db = d\hat{z}/db : \hat{\mu}_1 = \left(1 + \frac{\hat{\sigma}_1}{\hat{\mu}_1}\right)/-\hat{M}_{11};\ \ d\hat{\mu}/dc = -G_2 \cdot d\hat{\mu}/db$

$d\hat{\sigma}/db = d\hat{z}/db : \hat{\sigma}_1 = \frac{\hat{\mu}_1}{\hat{\sigma}_1} + 1/-\hat{M}_{11}\ ;\ \ d\hat{\sigma}/dc = -G_2\ d\hat{\sigma}/db\ ;\ \ \hat{M}_{11} < 0$

Thus if z is a decision causing a rise in both μ and σ, the raise in b increases $\hat{\mu}$ and $\hat{\sigma}$, while the raise in c (and thus also in r = c/b) has the opposite effect.

10. Maximizing median net profit. If the firm maximizes median net profit (or any percentile), no change in a monotonically increasing net profit schedule can have any effect on the firm's decisions. (T.W.Anderson).

1 L. R. Klein. Economic Fluctuations in the United States 1921-1941. Second Draft). Multigraphed, June 1947. Chapter II, p. 23.

2 Abba P. Lerner. The Economics of Control. New York, 1944. p. 238.

3 Leonid Hurwicz. Theory of the Firm and of Investment, Econometrica 1946, esp. p. 116, 125-127.

4 E. Domar and R. Musgrave. Proportion Income Taxation and Risk Taking. Quarterly Journal of Economics, 1943.

5 Alfred Marshall. Principles of Economics. Mathematical Appendix, Note IX. (8th ed. p. 843)