# A Theory of Asset Prices based on Heterogeneous Information and Limits to Arbitrage


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## Motivation

- This paper: asset pricing theory based on heterogeneous info and limited arbitrage
  - **Parsimonious**: all results derive from these 2 elements
  - **General**: tractability allows analysis of wide class of securities
- Central message: **systematic departure** of prices from fundamentals
  - Beliefs are heterogeneous: private signal + price
  - Price = expectation of **marginal** trader
  - Noisy info aggregation $\Leftrightarrow$ prices $\neq$ expected dividends (cond. on public info)
    - $\Rightarrow$ Price over/undervaluation (depending on payoff structure)
    - $\Rightarrow$ Price volatility can exceed realized dividend volatility
- **Variety of applications**:
  - M&M capital structure irrelevance
  - Excess volatility of stock returns
  - Price reaction to public announcements

# Relation with Literature

1. **Information aggregation**
   - Grossman and Stiglitz (AER 80); Hellwig (JET 80); Diamond and Verrecchia (JFE 81); Wang (REStud 93).
2. **Heterogeneous Beliefs and Bubbles**
   - Harrison and Kreps (QJE 78); Scheinkman and Xiong (JPE 03); Abreu and Brunnermeier (ECT 03).
3. **Finance puzzles**
   - M&M Capital structure irrelevance: Myers (JF 84); Myers and Majluf (JFE 84).
   - Excess return volatility: Shiller (AER 81), Campbell and Shiller (RFS 88), Cochrane (RFS 92).
   - Stock price under/overreaction: Barberis et al. (JFE 98); Daniel et al. (JF 98); Hong and Stein (JF 99).

# Outline of Talk

1. Setup
2. Information Aggregation Wedge
3. Applications
4. Robustness

# Setup

# Environment

- Single risky asset in unit supply
- Pays $\pi(\theta)$;
  - Fundamental: $\theta \sim N(0, \sigma_\theta^2)$
  - Dividend function: $\pi'(\cdot) > 0$, otherwise unrestricted
- Two dates:
  - Trading in financial market $(t = 0)$
  - Payoffs realized $(t = 1)$

# Financial Market: $t = 0$

- Informed traders: $i \in [0, 1]$
  - Risk-neutral
  - Limits to arbitrage: Can buy at most 1 share, and cannot short-sell
  - Observe private signal $\mathbf{x_i} \sim N(\theta, \beta^{-1})$, share price $\mathbf{P}$
  - Buy ($\mathbf{d_i} = 1$)/don't buy ($\mathbf{d_i} = 0$):

$$\mathbf{d}(\mathbf{x}, \mathbf{P}) = \begin{cases} \mathbf{1} \text{ if } \mathbb{E}[\pi(\theta) \mid \mathbf{x_i}, \mathbf{P}] \geq \mathbf{P} \\ \mathbf{0} \quad \text{otherwise} \end{cases}$$

- Aggregate informed demand: $D(\theta, \mathbf{P}) = \int d(\mathbf{x}, \mathbf{P}) d\Phi(\sqrt{\beta}(\mathbf{x} - \theta))$
- Noisy demand: $\Phi(u)$; $u \sim N(0, \sigma_u^2)$

# Equilibrium Definition

**A Perfect Bayesian Equilibrium (PBE) consists of**

1. Price function $\mathbf{P}(\theta, u)$
2. Informed traders' demands $\mathbf{d}(\mathbf{x}, \mathbf{P})$
3. Posterior beliefs $\mathbf{H}(\theta|\mathbf{x}, \mathbf{P})$ for informed traders s.t.,
   (i) Informed traders demands are optimal (given beliefs)
   (ii) The asset market clears
   (iii) Posterior beliefs satisfy Bayes' rule

# Trader Optimality: Threshold Strategy

- Expected dividend, and demand $\mathbf{d}(\mathbf{x}, \mathbf{P})$: monotone in $\mathbf{x}$
  - Trading strategy: **signal threshold** $\tilde{\mathbf{x}}(\mathbf{P})$

$$\mathbf{d}(\mathbf{x}, \mathbf{P}) = \begin{cases} \mathbf{1} & \text{if } \mathbf{x_i} > \hat{\mathbf{x}}(\mathbf{P}) \\ \mathbf{0} & \text{if } \mathbf{x_i} < \hat{\mathbf{x}}(\mathbf{P}) \\ \in (\mathbf{0}, \mathbf{1}) & \text{if } \mathbf{x_i} = \hat{\mathbf{x}}(\mathbf{P}) \end{cases}$$

- Price = dividend expectation of *marginal* trader ($x_i = \hat{\mathbf{x}}(\mathbf{P})$)

$$\mathbf{P} \quad = \quad \mathbb{E}[\pi(\theta)|x_i = \hat{\mathbf{x}}(\mathbf{P}), \mathbf{P}] = \int \pi(\theta) dH(\theta|\hat{\mathbf{x}}(\mathbf{P}), \mathbf{P})$$

# Market Clearing

- $D(\theta, \mathbf{P}) + \Phi(u) = 1$;
$$\Phi(\sqrt{\beta}(\tilde{\mathbf{x}}(\mathbf{P}) - \theta)) = \Phi(u)$$
$$\hat{\mathbf{x}}(\mathbf{P}) = \theta + 1/\sqrt{\beta} \cdot u \equiv \mathbf{z}$$
- $\mathbf{P}$: aggregates private info
  - $\mathbf{P}$ *informationally equivalent* to $\hat{\mathbf{x}}(\mathbf{P}) = \mathbf{z}$
- $\mathbf{z}$: endogenous public signal
  - Increasing in fundamental $\theta$, noisy demand $u$
  - $\theta|\mathbf{z} \sim N(\mathbf{z}, \sigma_{\mathbf{u}}^2/\beta)$; $\longrightarrow$ precision of $\mathbf{z}$: $\beta/\sigma_{\mathbf{u}}^2$
  - $\beta$: private info precision; $\sigma_{\mathbf{u}}^2$: noisy demand variance

## Proposition: Asset Market Equilibrium

- Unique equilibrium: price $P_\pi(\mathbf{z})$ and traders' threshold $\hat{x}(p) = \mathbf{z} = P_\pi^{-1}(p)$,

$$P_\pi(\mathbf{z}) = \int \pi(\theta) d\Phi\left(\sqrt{\sigma_\theta^{-2} + \beta + \beta\sigma_u^{-2}}\left(\theta - \underbrace{\frac{\beta + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \beta + \beta\sigma_u^{-2}}}_{\equiv\gamma_P}\mathbf{z}\right)\right)$$

$$= \int \pi(\gamma_P \mathbf{z} + \sigma_\theta\sqrt{1-\gamma_P}\,u)\phi(u)du$$

  - Marginal trader pricing share conditions on private signal $\mathbf{x_i} = \mathbf{z}$; public signal $\mathbf{z}$
  - Bayesian weight $\gamma_P$ on signal $\mathbf{z}$; residual uncertainty $= 1 - \gamma_P$

- Expected dividend, conditional on public signal $\mathbf{z}$ only

$$V_\pi(\mathbf{z}) = \int \pi(\theta) d\Phi\left(\sqrt{\sigma_\theta^{-2} + \beta\sigma_u^{-2}}\left(\theta - \underbrace{\frac{\beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \beta\sigma_u^{-2}}}_{\equiv\gamma_V}\mathbf{z}\right)\right)$$

$$= \int \pi(\gamma_V \mathbf{z} + \sigma_\theta\sqrt{1-\gamma_V}\,u)\phi(u)du$$

  - Bayesian weight $\gamma_V (< \gamma_P)$ on signal $\mathbf{z}$; residual uncertainty $= 1 - \gamma_V$

# Information Aggregation Wedge

# Information Aggregation Wedge

- Information aggregation wedge: $\mathbf{W}_\pi(\mathbf{z}) \equiv \mathbf{P}_\pi(\mathbf{z}) - \mathbf{V}_\pi(\mathbf{z})$
  - Marginal trader puts higher weight on market signal $\mathbf{z}$ than "outsider" who only observes the price ($\gamma_P > \gamma_V$)

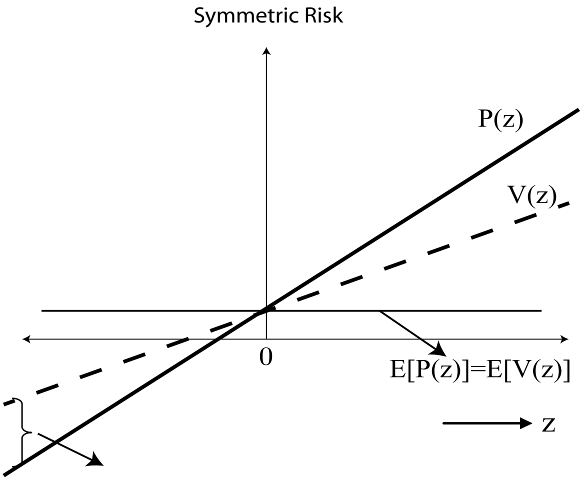
$$P_\pi(\mathbf{z}) = \int \pi(\theta) d\Phi \left( \sqrt{\sigma_\theta^{-2} + \beta + \beta\sigma_u^{-2}} \left( \theta - \underbrace{\frac{\beta + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \beta + \beta\sigma_u^{-2}}}_{\equiv \gamma_P} \mathbf{z} \right) \right)$$

$$V_\pi(\mathbf{z}) = \int \pi(\theta) d\Phi \left( \sqrt{\sigma_\theta^{-2} + \beta\sigma_u^{-2}} \left( \theta - \underbrace{\frac{\beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \beta\sigma_u^{-2}}}_{\equiv \gamma_V} \mathbf{z} \right) \right)$$

- But $V_\pi(z)$ is the **correct metric** for valuing the unconditional dividend:

$$\implies \mathbb{E}[\pi(\theta)] = \mathbb{E}[V_\pi(z)]$$

# Information Wedge in Linear Case




Price more responsive to $z$ than expected dividend

# Intuition: Shift in Marginal Trader's Identity

- Key intuition: for each realization of $\mathbf{z}$, **marginal trader** is a **different agent**
- Higher $\mathbf{z}$ (due to $\theta$, and/or $u$) has two effects on beliefs
  - Higher $\theta$ shifts up distribution of signals $x's$ : higher demand $\rightarrow$ higher $\hat{x}(P)$
  - Higher $u$ lowers net supply $\rightarrow$ higher $\hat{x}(P)$ to deter buying by informed
- Expectations of new marginal trader pricing shares raised through **both** effects
  - Higher expectations due to market signal (just like anyone else)
  - Higher expectations due to **shift** in identity (this is the extra kick)

$\Rightarrow$ "Double weighting" of market info $\mathbf{z}$ is **rational** (Bayesian updating)

# Unconditional Wedge

**Lemma (unconditional wedge)**: The unconditional wedge is given by

$$W_\pi(\sigma_P) \equiv \mathbb{E}[W(z)] = \int_0^\infty \underbrace{\left(\pi'(\theta) - \pi'(-\theta)\right)}_{\text{sign}} \underbrace{\left(\Phi\left(\frac{\theta}{\sigma_\theta}\right) - \Phi\left(\frac{\theta}{\sigma_P}\right)\right)}_{\text{magnitude}} d\theta,$$

- Sign: related to curvature of $\pi(\cdot)$
- Magnitude given by **informational frictions** $\sigma_P^2$
  - Marginal trader's posterior $\theta|z$: $\sim N(\gamma_P z, (1-\gamma_P)\sigma_\theta^2)$
  - Prior: $z \sim N(0, \sigma_\theta^2/\gamma_V)$
  - Compounded distribution: $\theta \sim N(0, \underbrace{(1-\gamma_P)\sigma_\theta^2 + \gamma_P^2\sigma_\theta^2/\gamma_V}_{\sigma_P^2 > \sigma_\theta^2})$
- Marginal trader **overweights tails** of $\theta$ distribution (overreacts to $z$)
  - Pricing of shares as if $\theta \sim N(0, \sigma_P^2)$, rather than $\sim N(0, \sigma_\theta^2)$ (fatter tails)

# Results: Over/Under-Valuation

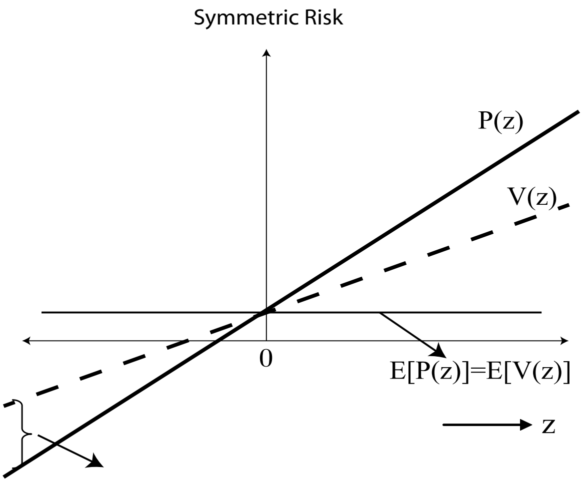
- **Definition (risk type)**: a dividend function $\pi(\theta)$, $\forall \theta > 0$,
  - (i) Has symmetric risks if $\pi'(\theta) = \pi'(-\theta)$,
  - (ii) Has upside risks if $\pi'(\theta) \geq \pi'(-\theta)$,
  - (iii) Has downside risks if $\pi'(\theta) \leq \pi'(-\theta)$,
  - (iv) If $\pi_1'(\theta) - \pi_1'(-\theta) \leq \pi_2'(\theta) - \pi_2'(-\theta)$,

    $\Rightarrow$ $\pi_1(\cdot)$ has more downside (less upside) risk than $\pi_2(\cdot)$

- **Theorem (value bias)**:
  - (i) If $\pi(\cdot)$ has symmetric risk, $W_\pi(\sigma_P) = 0$
  - (ii) If $\pi(\cdot)$ has upside risk, $W_\pi(\sigma_P) > 0$
  - (iii) If $\pi(\cdot)$ has downside risk, $W_\pi(\sigma_P) < 0$
  - (iv) $||W_\pi(\sigma_P)||$ increasing in info frictions $\sigma_P$
  - (v) If $\pi_1(\cdot)$ has more downside (less upside) risk than $\pi_2(\cdot)$,

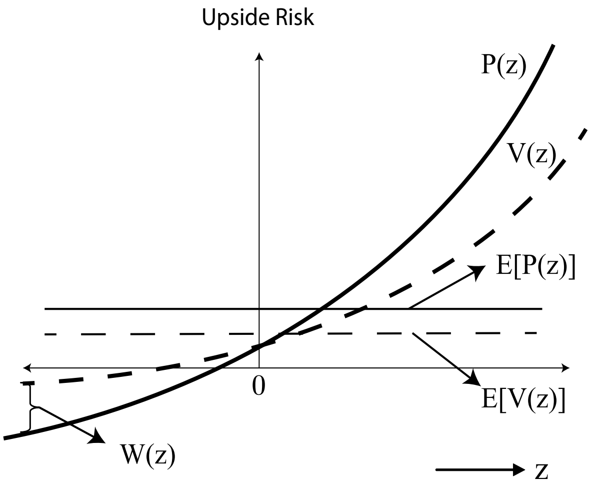
    $\Rightarrow$ $W_{\pi_2}(\sigma_P) - W_{\pi_1}(\sigma_P)$ increasing in $\sigma_P$

# Risk Types and Information Wedges



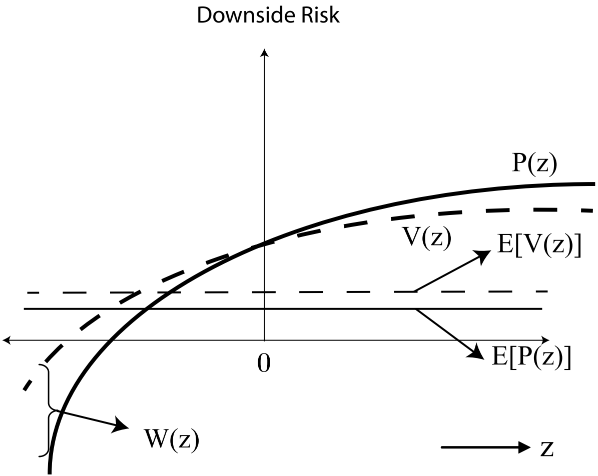

Expected wedge $= 0$ (as in CARA-normal)

# Risk Types and Information Wedges




Expected wedge $> 0$: Overpriced security (on expectation)

# Risk Types and Information Wedges




Expected wedge $< 0$: Underpriced security (on expectation)

# Formal Results: Volatility

**Theorem (excess variability)**:

For any payoff function $\pi(\cdot)$ with symmetric, upside or downside risks,

(i) $\mathbb{E}((\pi(\theta) - \pi(0))^2) > \mathbb{E}\left((V_\pi(z) - V_\pi(0))^2\right)$

(ii) $\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2\right) > \mathbb{E}\left((V_\pi(z) - V_\pi(0))^2\right)$

- Prices more volatile than expected dividends

(ii) $\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2\right) > \mathbb{E}\left((\pi(\theta) - \pi(0))^2\right)$, if $\sigma_u^2$ and/or $\beta$ high enough

- Prices more volatile than realized dividends, in the absence of a SDF
- Compare with West (Ect, 1988):

$\Rightarrow$ variability of posterior expectation $<$ variability of realized dividends

$\Rightarrow$ our model: **change in identity** delivers the extra volatility

# Recap: Key Results thus far

- Parsimonious model of info aggregation
  - Applies to arbitrary (monotone) payoff functions
  - Tractability arises from risk-neutral setup with limited arbitrage
- Main result: Information aggregation wedge
  - Prices overreact to market info due to *identity* effect
  - Leads to average over/undervaluation (depending on curvature of $\pi(\cdot)$)
  - Leads to excess volatility of prices

# Applications

## Application 1: M&M Dividend Split Irrelevance

- Suppose dividend is split in 2 and sold in separate markets
  - $\pi(\cdot) = \pi_1(\cdot) + \pi_2(\cdot)$
  - $\pi_1(\cdot)$ has downside risk, $\pi_2(\cdot)$ has upside risk
- Market characteristics
  - Informed traders active in one market only, observe $x_{i,j} \sim \mathcal{N}\left(\theta, \beta_j^{-1}\right)$
  - Noisy demands:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{u,1}^2 & \rho\sigma_{u,1}\sigma_{u,2} \\ \rho\sigma_{u,1}\sigma_{u,2} & \sigma_{u,2}^2 \end{pmatrix}\right)$$

- Consider informationally segmented markets
  - Traders in mkt $j$ don't observe price $P_{-j}$
  - Results also hold in info connected markets (see paper)
  - Market characterized fully by info frictions $\sigma_{P,j}$

# Application 1: M&M Dividend Split Irrelevance

**Proposition**:

(i) Seller's revenue is independent of split iff $\sigma_{P,1} = \sigma_{P,2}$

$\Rightarrow$ Markets have **identical** information frictions

(ii) Total expected revenue from $\pi(\cdot)$ maximized by following split:

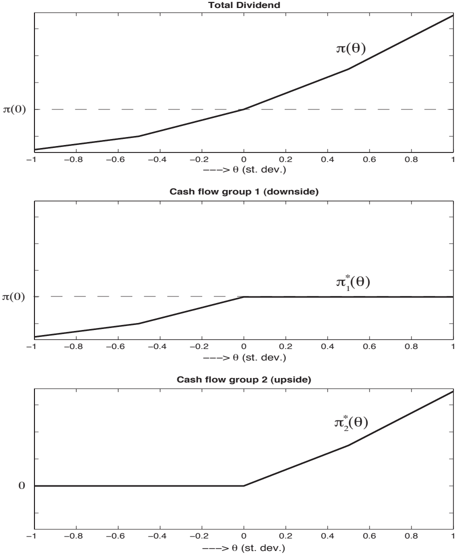
- $\pi_1^* (\theta) = \min \{\pi (\theta), \pi (0)\}$, and $\pi_2^* (\theta) = \max \{\pi (\theta) - \pi (0), 0\}$
- Assign $\pi_1^*$ to investor pool with lower informational friction ($\sigma_{P,1}$)

- Intuition
  - $\pi_1^*$ has more downs. risk than any other $\pi_1$,
  - $\pi_2^*$ has more ups. risk than any other $\pi_2$,

  $\Rightarrow$ Any alternative split $\{\pi_1, \pi_2\}$ transfers ups. risk from $\sigma_{P,2}$ to $\sigma_{P,1}$ investors

  $\Rightarrow$ ...resulting in a net loss of revenue (lower overall wedge)

# Splitting Cash Flows for Arbitrary $\pi(\theta)$




$\Rightarrow \pi_1^*$ has max. downside risk; $\pi_2^*$ max. upside risk

# Application 2: Dynamic Trading

- Dynamic extension:
  - Dividend each period: $\pi(\theta_t)$, $\theta_t$ i.i.d.
  - Traders infinitely lived, discount future at fixed rate $\delta \in (0,1)$
- Price and expected dividend satisfy recursive expression:

$$\begin{aligned} P_\pi(z_t) &= \mathbb{E}\left(\pi(\theta_t) + \delta P_\pi(z_{t+1}) | x = z_t, z_t\right) \\ V_\pi(z_t) &= \mathbb{E}\left(\pi(\theta_t) + \delta V_\pi(z_{t+1}) | z_t\right) \end{aligned}$$

- And so does the wedge:

$$\begin{aligned} W_\pi(z_t) &= w_\pi(z_t) + \delta \mathbb{E}\left(W_\pi(z)\right) = w_\pi(z_t) + \frac{\delta}{1-\delta}\mathbb{E}\left(w_\pi(z)\right), \\ \text{where } w_\pi(z_t) &= \mathbb{E}\left(\pi(\theta_t) | x = z_t, z_t\right) - \mathbb{E}\left(\pi(\theta_t) | z_t\right) \end{aligned}$$

# Application 2: Dynamic Trading

**Proposition (Dynamic Wedge)**:

Suppose that $\pi(\cdot)$ is bounded below, increasing, and convex:

- For any $\sigma_P > \sigma_\theta$, $\exists\ \hat{\delta} < 1$ s.t. $\forall\ \delta > \hat{\delta}$, $W(z_t) > 0$, for all $z_t$.
- Dynamic model implies:
  - Future expected wedges raise current share price (if $\pi(\cdot)$ has upside risk)
  - For high enough $\delta$, share might **always** be overpriced

# Application 3: Public Disclosures

- How does exogenous public info about $\theta$ affect wedge?
  - Let $y \sim N(\theta, \alpha^{-1})$ be a public disclosure on $\theta$
- Same eq. characterization as before, but with extra info

$$P_\pi(y, z) = \int \pi(\theta) d\Phi \left( \sqrt{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} \left( \theta - \frac{\alpha y + (\beta + \beta\sigma_u^{-2}) z}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} \right) \right)$$

$$V_\pi(y, z) = \int \pi(\theta) d\Phi \left( \sqrt{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \left( \theta - \frac{\alpha y + \beta\sigma_u^{-2} z}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \right) \right)$$

- Public info crowds out impact of $z$ on price and expected dividend
  - In the limit $\alpha \to \infty$, wedge dissapears
  - But for finite levels of precision $\alpha$, impacts are more subtle...

## Application 3: Public Disclosures

**Proposition (Public Disclosures)**: Consider linear dividend $\pi(\cdot)$ (holds more generally)

( i) $Var\left(V_{\pi}\left(y, z\right)\right)$ increasing in $\alpha$

- Standard Blackwell

( ii) For $\sigma_u^{-2} \geq 2$, $Var\left(P_{\pi}\left(y, z\right)\right)$ increasing in $\alpha$;

Otherwise, $Var\left(P_{\pi}\left(y, z\right)\right)$ increasing in $\alpha$ iff $\alpha \geq \alpha'$

- If noisy demand too volatile, $\alpha$ reduces price overreaction to $z$
- But for large enough $\alpha$, price vol increasing (more responsive to $\theta$)

( iii) $Var\left(W\left(y, z\right)\right)$ is decreasing in $\alpha$ iff $\alpha \geq \alpha''$ (and increasing otherwise),

- For low $\alpha$, an increase reduces impact of $z$ on $V_{\pi}\left(y, z\right)$ more than on $P_{\pi}\left(y, z\right)$
  - $\Rightarrow$ Larger wedge
- But for large enough $\alpha$, both $V_{\pi}\left(y, z\right)$ and $P_{\pi}\left(y, z\right)$ hardly respond to $z$
  - $\Rightarrow$ Wedge vanishes

# Robustness

# Robustness

- Alternative distributional assumptions: let
  - $\theta \sim$ on arbitrary smooth prior on $\mathbb{R}$, $x_i \sim$ iid cdf $F(\cdot|\theta)$ satisfying MLRP,
  - Noisy demand $D \sim$ according to cdf $G(\cdot)$ on $[0, 1]$
  - $\Rightarrow$ Can always characterize wedge in this environment
- Price impact of information
  - Let noisy demand be elastic: $D(u, P) = \Phi\left(u + \eta\left(\mathbb{E}\left(\pi\left(\theta\right)|P\right) - P\right)\right)$
  - Wedge is inversely related to elasticity $\eta$
  - $\Rightarrow$ Noise traders arbitrage away the wedge
- Wedge in CARA-normal setup (noisy REE)
  - Can only solve in the linear case: $\pi'(\cdot) = k > 0$
  - Wedge has two components
    - A constant reflecting discount (premium) for average shares held
    - A symmetric information aggregation wedge
  - Unconditional returns driven by the average compensation for risk

# Conclusions

- Tractable noisy REE framework
  - Heterogeneous beliefs, risk neutrality and limited arbitrage
  - Useful to analyze more general payoff structures
- Key result: **information aggregation wedge**
  - Prices overreact to market information
  - Generates excess price/return volatility
  - Generates over/undervaluation on average (depending on shape of payoffs)
- Applications in finance
  - M&M capital structure irrelevance
  - Excess volatility puzzle
  - Impact of public disclosures