Semiparametric Modelling in Financial Time Series and Panel Data

Oliver Linton
London School of Economics

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- There is an important class of nonparametric/semiparametric models where the quantity of interest is a function $m(.)$ that is only implicitly defined but is known to satisfy a linear Fredholm integral equation of the second kind in the space $L_2(p)$ for some density p,

$$m(x) = m^*(x) + \int \mathcal{H}(x,y) m(y) p(y) dy,$$

  where the function $m^*(x)$ and the operator $\mathcal{H}(x,y)$ are defined explicitly in terms of the distribution of some observable quantities.
- This sort of structure arises in many settings and we shall give some examples below. Carrasco, Florens, and Renault (2006) HofE. ET special issue.

- We write this equation in short hand

$$m = m^* + \mathcal{H}m,$$

where $\mathcal{H}$ is operator and m^* is intercept

- Key questions:
  - Does there exist a unique solution to this equation
  - Is the solution continuous in some sense
  - How to compute estimators of $m(x)$ in practice given noisy observations $\widehat{m}^*(x)$ and $\widehat{\mathcal{H}}(x, y)$ are available on $m^*(x)$ and $\mathcal{H}(x, y)$.
  - Asymptotic distributions and inference
  - Optimality. There may be many such equations

The key properties are to do with the nature of the operator or family of operators $\mathcal{H}$. We will make use of the following condition.
Assumption A1. The operator $\mathcal{H}(x, y)$ is Hilbert-Schmidt

$$\int\int \mathcal{H}(x, y)^2 p(x)p(y)dxdy < \infty.$$

Under Assumption A1, $\mathcal{H}$ is a continuous compact operator. If it is also self-adjoint it has a countable number of real eigenvalues:

$$\infty > |\lambda_1| \geq |\lambda_2| \geq ....,$$

$$\sum_{j=1}^{\infty} \lambda_j^2 < \infty.$$

This condition is satisfied in many cases under quite weak conditions.

- Another key condition is that for a constant $\gamma < 1$

$$\lambda_j < \gamma$$

  for $j \geq 1$.
- To verify this condition requires some special arguments.
- If this is true we get that $I - \mathcal{H}$ has eigenvalues bounded from below by $1 - \gamma > 0$. Therefore $I - \mathcal{H}$ is invertible and $(I - \mathcal{H})^{-1}$ has only positive eigenvalues that are bounded by $(1 - \gamma)^{-1}$.
- So we can directly solve the integral equation and write

$$m = (I - \mathcal{H})^{-1} m^*.$$

If also

$$|\lambda_1| < 1, \text{ then } m = \sum_{j=0}^{\infty} \mathcal{H}^j m^*.$$

In this case, the sequence of successive approximations

$$m^n = m^* + \mathcal{H} m^{n-1}, n = 1, 2, \ldots$$

converges to m from any starting point.
There are some situations where $|\lambda_1| \geq \cdots \geq |\lambda_k| \geq 1$, and so the conditions that guarantee convergence of the successive approximations method are not satisfied. In that case, one has to transform the integral equation in order to obtain an equation which is more regular.

EXAMPLES

1. Additive Nonparametric Regression
2. Semiparametric Volatility News Impact curve*
3. Yield curve fitting
4. Semiparametric Fama French Panel model*
5. Nonparametric Euler (time series) equation†
6. Dynamic Discrete Choice (time series)†
7. Second-price, private-values auctions/Generalized Competing Risk model††

Additive Nonparametric Regression

We observe Y_i, X_i from

$$Y_i = m_0 + m_1(X_{1i}) + \ldots + m_d(X_{di}) + \varepsilon_i,$$

where $E(\varepsilon_i|X_i) = 0$ and $E(\varepsilon_i^2|X_i) < \infty$.
The functions $m_1(.), \ldots, m_d(.)$ are unknown but smooth.
Let H be the space $L_2(F_X)$ and H_j the space $L_2(F_{X_j})$. Define the space H_{add} to be the space of additive functions in H, so that

$$H_{add} = \{m : m(x) = m_0 + m_1(x_1) + \ldots + m_d(x_d)\}$$

for square integrable m_j. In this case the projections P_j onto H_j are just the conditional expectation operators $E(\cdot|X_j)$ but the projection P on H_{add} is more complicated and not available in "closed form" - it is the solution of a linear inverse problem.
Iterative projections, Von Neumann (1933).

We have:

$$\begin{aligned} \mathrm{E}\left[Y_i \middle| X_{ji} = x_j\right] &= m_0 + m_j(x_j) + \sum_{r \neq j} \mathrm{E}\left[m_r(X_{ri}) \middle| X_{ji} = x_j\right] \\ &= m_0 + m_j(x_j) + \sum_{r \neq j} \int m_r(x_r) \frac{p_{j,r}(x_j, x_r)}{p_j(x_j) p_r(x_r)} p_r(x_r) dx_r \end{aligned}$$

with $p_{j,r}$ (p_j) density of (X_{ji}, X_{ri}) (or X_{ji}, resp.).

$$m_j^* = m_j + \sum_{r \neq j} \mathcal{H}_j m_r,$$

where $\mathcal{H}_j$ are conditional expectations operators and $m_j^* = E(Y|X_j) - E(Y)$.

This can be written in the form

$$m^* + \mathcal{H}m = m.$$

$$\begin{pmatrix} I & \mathcal{H}_1 & \cdots & \mathcal{H}_1 \\ \mathcal{H}_2 & I & \mathcal{H}_2 & \cdots \\ \vdots & & \ddots & \\ \mathcal{H}_d & \cdots & & I \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_d \end{pmatrix} = \begin{pmatrix} m_1^* \\ m_2^* \\ \vdots \\ m_d^* \end{pmatrix}.$$

Mammen, Linton, and Nielsen, AS 1999. Self adjointness follows from projection starting point. Iterative smoothing method with fast convergence. Pointwise distribution theory.

Semiparametric GARCH Model. Linton and Mammen (2005, Ec)

- Suppose that the observed process $\{y_t\}_{t=-\infty}^{\infty}$ is a stationary martingale difference sequence with

$$E(y_t^2 \,|\mathcal{F}_{t-1}) \equiv \sigma_t^2 = \theta\sigma_{t-1}^2 + m(y_{t-1}) = \sum_{j=1}^{\infty} \theta^{j-1} m(y_{t-j}).$$

  The parameters θ_0 and the function $m(.)$ are unknown and to be estimated.

- This model is of interest in financial applications, and is a natural generalization of the GARCH(1,1) model where m is quadratic. Engle (1982), Bollerslev (1986), Engle and Ng (1993). The generalization allows for flexibility in the 'news impact curve', i.e., the function m, which is interpreted as the channel through which news affects volatility in financial markets.

Here one can obtain the equation

$$m^* + \mathcal{H}m = m,$$

$$m^*(y) = (1-\theta^2)\sum_{j=1}^{\infty}\theta^{j-1}\,\mathrm{E}\left[y_0^2 \middle| y_{-j}\right],$$

$$\mathcal{H}_\theta(y,x) = -\sum_{j=\pm 1}^{\pm\infty}\theta^{|j|}\frac{p_{0,j}(y,x)}{p(y)p(x)},$$

where $p_{0,j}$ (p) density of (y_0, y_j) (y_0, respectively.)
Can prove that

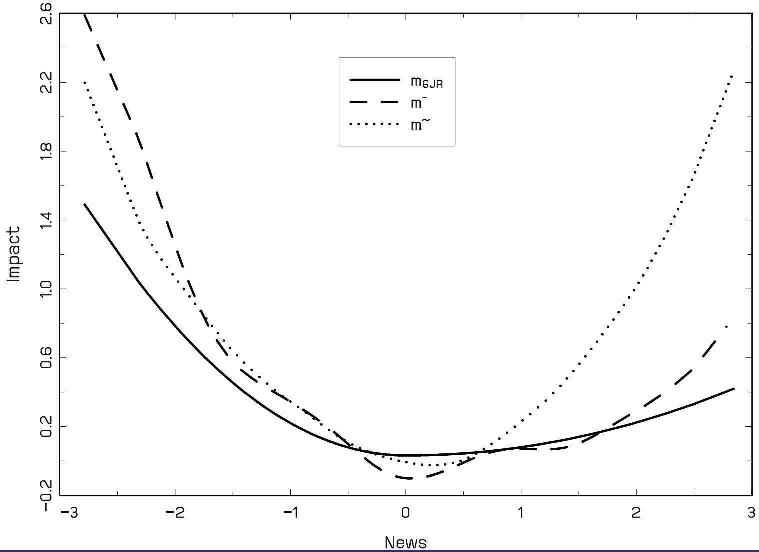
$$\lambda_j(\mathcal{H}) < 1$$

(for all θ, j) but

$$|\lambda_j(\mathcal{H})| > 1$$

(at least for some θ, j)

S&P500 Daily Return



Estimation of Yield Curves from coupon bonds (LMNT, 2001)

Coupon bonds generate several payments at future dates, and in an efficient bond market, the present value of these future payments should, apart from a small error, be equal to the trading price, p_i. Let the payments to the owner of bond i at time τ_{ij} be $c_i(\tau_{ij})$, where for bond i, $\tau_{i1} < \ldots < \tau_{im_i}$ are the possible payment dates. Note that the time to maturity τ_{im_i} varies across bonds.

The statistical model we adopt is

$$p_i = \sum_{j=1}^{d_i} c_i(\tau_{ij}) m(\tau_{ij}) + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where ε_i is a random sequence satisfying $E[\varepsilon_i] = 0$, $i = 1, \ldots, n$. The problem is to extract the unknown, but smooth, discount function $m(.)$ from information $\{p_i, d_i, \tau_{i1}, \ldots, \tau_{id_i}, c_i(\tau_{i1}), \ldots, c_i(\tau_{id_i}), \; i = 1, \ldots, n\}$ on a sample of bonds.

It can be shown that m can be characterized as the solution of a linear type 2 integral equation.

Semiparametric Fama French Model for Financial panel data (Connor, Linton, and Hagmann (2008))

- The model

$$y_{i,t+1} = f_{u,t+1} + \sum_{j=1}^{d} m_j(X_{jit}) f_{j,t+1} + \varepsilon_{i,t+1},$$

where:

- X_{jit} are observed (continuous) covariates; y_{it} observed returns.
- f_t is an unobserved strictly exogenous stochastic process with unspecified dynamics and at least

$$E\left[\varepsilon | X, f\right] = 0$$

- We treat vector $f \in \mathbb{R}^{T(d+1)}$ as unknown fixed parameters (independent of all).
- $m_1(.), \ldots, m_d(.)$ are unknown but smooth functions
- cross-section dimension n and time series length T are large

- First take the exposure functions m as given and characterize f - least squares regression. In the next step characterize $m(\cdot)$ given f
- First order conditions wrt $m_j(\cdot)$:

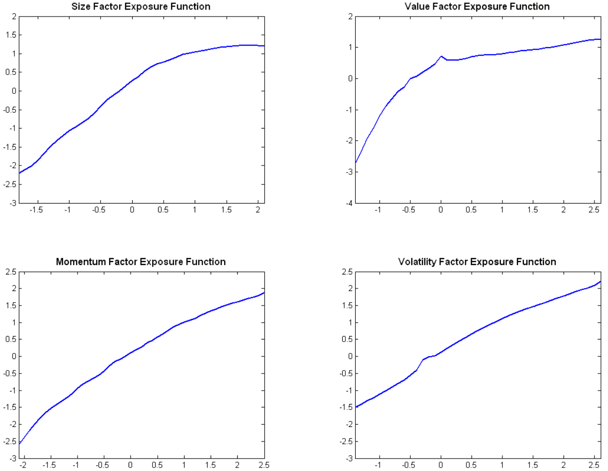
$$\frac{1}{T}\sum_{t=1}^{T} f_{jt} E\left[y_{it} | X_{ji} = x_j\right] = \frac{1}{T}\sum_{t=1}^{T} f_{jt} f_{ut} + m_j(x_j) \frac{1}{T}\sum_{t=1}^{T} f_{jt}^2 + \frac{1}{T}\sum_{t=1}^{T}\sum_{k \neq j} f_{jt} f_{kt} E\left[m_k(X_{ki}) | X_{ji} = x_j\right]$$

- Replace the unknown expectations by nonparametric estimates (kernels). Iteratively solve. Theory is related to additive nonparametric regression. Also important behaviour of "portfolio returns"

$$\beta_{jk} = \frac{\sum_{t=1}^{T} f_{jt} f_{kt}}{\sum_{t=1}^{T} f_{jt}^2}$$

Application

- We apply this methodology to CRSP monthly stock returns 1965-2005 with four characteristics:
  - Size=log of previous June cap
  - Value=market value/book value at previous June
  - Momentum=cumulative 12m return upto and including previous month
  - Volatility=realized std over last 12m
- $n_t \in [467, 6003]$. $T = 480/504.$ Total 1.8million observations.


Size Factor Exposure Function
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-1.5
-1
-0.5
0
0.5
1
1.5
2
Value Factor Exposure Function
2
1
0
-1
-2
-3
-4
-1
-0.5
0
0.5
1
1.5
2
2.5
Momentum Factor Exposure Function
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
Volatility Factor Exposure Function
2.5
2
1.5
1
0.5
0
-0.5
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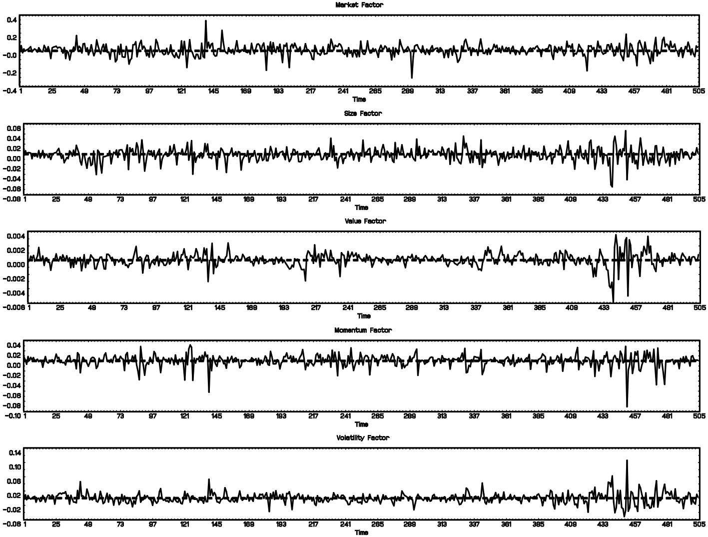
Market Factor
Time
Size Factor
Time
Value Factor
Time
Momentum Factor
Time
Volatility Factor
Time

Nonparametric Euler Equation (Linton and Lewbel (2008))

Let C_t be expenditures on consumption. Let $m(C_t)$ denote a time t marginal utility function. Let R_{jt} be the gross return in time period t of owning one unit of asset j in period $t-1$. For a consumer with time separable utility and a rate of time preference β that saves by owning assets j, the Euler equation for maximizing utility is usually represented as

$$E_t\left(S_{t+1}R_{jt+1}\right) = E\left(\beta\frac{m(C_{t+1})}{m(C_t)}R_{jt+1} \mid C_t\right) = 1$$

where S_t is the time t pricing kernel.

Many previous works have parameterized U, e.g., $m(c) = c^{-\gamma}$; one incidental advantage of this has been that consumption only enters Euler equation through a ratio.

Habits and Recursive preferences. Chen and Ludvigson (2009).

Define $\mathcal{H}_j(C_{t+1}, C_t) = E\left(R_{jt+1} \mid C_{t+1}, C_t\right) f\left(C_{t+1}|C_t\right)$. Then the Euler equation can be rewritten as

$$m(c) = \beta \int m(c')\mathcal{H}_j(c', c)dc'$$

$$(\mathcal{H}_j - \lambda I)m = 0$$

This is a homogeneous linear Fredholm integral equation of the second kind (eigenequation). The scale of m is not identified, so an arbitrary scale normalization (which does not affect the resulting estimate of S_t) will need to be imposed. Countable number of "solutions" $m_j(.), \lambda_j$.
Nonstationary consumption enters not necessarily through a ratio form, need to deal with nonstationarity. Wang and Phillips (2009)

Dynamic Discrete Choice Problem (Linton and Srisuma (2010))

Consider the following optimization problem:

$$V\left(s_t\right) = \max_{\{a_\tau(s_\tau)\}_{\tau=t}^{T}} E\left[\sum_{\tau=t}^{T} \beta^\tau u\left(a_\tau\left(s_\tau\right), s_\tau\right) \middle| s_t\right] \quad \text{for all } \tau \geq t.$$

where

- a_t : discrete control variable, *discrete decisions*
- s_t : continuous state variable, *capital, wealth, productivity*
- u : payoff function, *utility, profit*
- β : discounting factor

Examples: Patents renewal (Pakes, 1986), Machine replacement (Rust, 1987), Labor decisions (Rust and Phelan, 1997) and more recently in estimation of discrete choice games. Pakes, Ostrovsky, and Berry (2007). Bajari et al. (2009).

Econometric Model

- $a_t \in A = \{1, \ldots, K\}$ and $s_t = (x_t, \varepsilon_t) \in X \times \mathcal{E}$ with $X \subset \mathbb{R}^L$ and $\mathcal{E} \subset \mathbb{R}^K$
- **Conditional Independence**:

$$p\left(x_{t+1}, \varepsilon_{t+1} \middle| a_t, x_t, \varepsilon_t\right) = q\left(\varepsilon_{t+1} \middle| x_{t+1}\right) f_{x_{t+1}|x_t, a_t}\left(x_{t+1} \middle| a_t, x_t\right)$$

- $q\left(\varepsilon_t \middle| x_t\right) = q\left(\varepsilon_t\right)$ is known
- **Additive Separability**:

$$u\left(a_t, x_t, \varepsilon_t\right) = \pi\left(a_t, x_t\right) + \sum_{k=1}^{K} \varepsilon_{a_k, t} \mathbf{1}\left[a_t = k\right]$$

- $\pi_\theta\left(a, x\right)$ is known upto $\theta \in \Theta \subset \mathbb{R}$ $(\pi_{\theta_0} \equiv \pi)$
- $\left\{a_t, x_t\right\}_{t=1}^{T}$ is a controlled Markovian process

The value function, continuation value, and conditional value:

$$V\left(s_t\right) = \max_{a \in A}\left\{u\left(a, s_t\right) + \beta E\left[V\left(s_{t+1}\right)\left|s_t, a_t = a\right.\right]\right\}$$

$$E\left[V\left(s_{t+1}\right)\left|s_t, a_t\right.\right] = E\left[V\left(s_{t+1}\right)\left|x_t, a_t\right.\right]$$

$$\underbrace{E\left[V\left(s_{t+1}\right)\left|s_t, a_t\right.\right]}_{\text{continuation value}} = E\left[\underbrace{E\left[V\left(s_{t+1}\right)\left|x_{t+1}\right.\right]}_{\text{conditional value}}\left|x_t, a_t\right.\right]$$

Conditional Value Function $m_\theta(x) = E[V_\theta(s_t) | x_t = x]$ is a solution of the equation

$$m_\theta(x) = r_\theta(x) + \beta \int m_\theta(x') f_{x_{t+1}|x_t}(x'|x)\, dx'$$

$$m_\theta = r_\theta + \mathcal{L} m_\theta$$

where for any $x \in X$

$$r_\theta(x) = E[u_\theta(a_t, s_t) | x_t = x].$$

Since discount factor $\beta < 1$ the operator $\mathcal{L}$ is a contraction and

$$m_\theta = (I - \mathcal{L})^{-1} r_\theta = \sum_{j=0}^{\infty} \mathcal{L}^j r_\theta$$

Continuation Value Function $g_\theta(a_t, x_t) = E\left[V_\theta\left(s_{t+1}\right) | a_t, x_t\right]$ satisfies the relation

$$E\left[V_\theta\left(s_{t+1}\right) | a_t, x_t\right] = E\left[m_\theta\left(x_{t+1}\right) | a_t, x_t\right].$$

This can be summarized by

$$g_\theta = \mathcal{H} m_\theta,$$

where $\mathcal{H}$ denotes the conditional expectation operator such that, for any $(a, x) \in A \times X$

$$\begin{aligned} g_\theta(a, x) &= \mathcal{H} m_\theta(a, x) \\ &= E\left[m_\theta\left(x_{t+1}\right) | a_t = a, x_t = x\right] \\ &= \int m_\theta\left(x'\right) f_{x_{t+1}|x_t, a_t}\left(x' | a, x\right) dx' \end{aligned}$$

Hotz and Miller's Inversion (1993). In our case

$$r_\theta(x) = E\left[\pi(a_t, x_t) + \sum_{k=1}^{K} \varepsilon_{a_k,t}\mathbf{1}[a_t = k] \,|x_t = x\right].$$

From Hotz and Miller, there exists a map $\Psi(\cdot; q)$

$$E\left[\sum_{k=1}^{K} \varepsilon_{a_k,t}\mathbf{1}[a_t = k]\middle| x_t = x\right] = \Psi\left(\{P(a|x)\}_{a\in A}; q\right),$$

e.g. in the logit case

$$E\left[\sum_{k=1}^{K} \varepsilon_{a_k,t}\mathbf{1}[a_t = k]\middle| x_t = x\right] = \gamma + \sum_{k=1}^{K} P(a|x)\log(P(a|x)).$$

We replace $\{P(a|x)\}_{a\in A}$ by nonparametric estimates.

Second-price, private-values auctions / Generalized competing risks, Athey and Haile (2002)

- Komarova (2008). Observe winners bid and identity only. Consider three person case. Private value distributions F_i. For each bidder i, we observe the function G_i on $[0, T]$:

$$G_i(t) = \Pr(\text{price} \leq t,\ i \text{ wins}), \quad i = 1, 2, 3.$$

- Assuming the independence of bidders' values, functions G_i can be expressed through F_i as follows. Let b_i, $i = 1, 2, 3$, indicate the submitted bids. Then

$$\begin{aligned} G_1(t) &= \Pr(\max\{b_2, b_3\} < b_1, \max\{b_2, b_3\} \leq t) \\ &= \Pr(\max\{X_2, X_3\} < X_1, \max\{X_2, X_3\} \leq t) \\ &= \int_0^t (F_2 F_3)'(1 - F_1)\, ds, \end{aligned}$$

- Therefore, unknown distribution functions F_i are related to observable functions G_i by means of this system of integral-differential equations:

$$
\begin{aligned}
G_1(t) &= \int_0^t (F_2 F_3)'(1 - F_1) ds \\
G_2(t) &= \int_0^t (F_1 F_3)'(1 - F_2) ds \\
G_3(t) &= \int_0^t (F_1 F_2)'(1 - F_3) ds.
\end{aligned}
$$

- Differentiating we obtain the following system of differential equations almost everywhere (a.e.) on $[0, T]$:

$$g_1 = (F_2 F_3)'(1 - F_1)$$
$$g_2 = (F_1 F_3)'(1 - F_2)$$
$$g_3 = (F_1 F_2)'(1 - F_3),$$

where g_i stands for the a.e. derivative of G_i. Distribution functions F_i in this system must satisfy the following initial conditions:

$$F_i(0) = 0, \quad i = 1, 2, 3.$$

- Write $H_1 = F_2F_3$, $H_2 = F_1F_3$, and $H_3 = F_1F_2$. H_j are distribution functions, $H_j(0) = 0$, and $F_1 = \sqrt{H_2H_3/H_1}$, $F_2 = \sqrt{H_1H_3/H_2}$, and $F_3 = \sqrt{H_1H_2/H_3}$. Then

$$
\begin{aligned}
h_1(t) &= \frac{g_1(t)}{1 - \sqrt{\frac{\int_0^t h_2(s)ds \int_0^t h_3(s)ds}{\int_0^t h_1(s)ds}}} \\
h_2(t) &= \frac{g_2(t)}{1 - \sqrt{\frac{\int_0^t h_1(s)ds \int_0^t h_3(s)ds}{\int_0^t h_2(s)ds}}} \\
h_3(t) &= \frac{g_3(t)}{1 - \sqrt{\frac{\int_0^t h_2(s)ds \int_0^t h_1(s)ds}{\int_0^t h_3(s)ds}}}.
\end{aligned}
$$

- This is a nonlinear system of Volterra integral equations.
- Komarova (2008) has proved identification of F under weak conditions. She also proposed an estimation method based on sieves and established consistency thereof.
- We propose an alternative method based on kernels. We make much stronger assumptions regarding smoothness of density functions etc.

Lets suppose we have an estimator $\widehat{h}$ that satisfies

$$\widehat{h}_1(t) = \frac{\widehat{g}_1(t)}{1 - \sqrt{\frac{\int_0^t \widehat{h}_2(s)ds \int_0^t \widehat{h}_3(s)ds}{\int_0^t \widehat{h}_1(s)ds}}}$$

$$\widehat{h}_2(t) = \frac{\widehat{g}_2(t)}{1 - \sqrt{\frac{\int_0^t \widehat{h}_1(s)ds \int_0^t \widehat{h}_3(s)ds}{\int_0^t \widehat{h}_2(s)ds}}}$$

$$\widehat{h}_3(t) = \frac{\widehat{g}_3(t)}{1 - \sqrt{\frac{\int_0^t \widehat{h}_2(s)ds \int_0^t \widehat{h}_1(s)ds}{\int_0^t \widehat{h}_3(s)ds}}},$$

where $\widehat{g}$ is a consistent estimator with some further properties.

- We are going to express the estimation error $\widehat{h} - h$ in terms of the estimation error of $\widehat{g} - g$.
- This is accomplished by a functional Taylor expansion (linearization) around the truth.
- Therefore, we have the linearized system of equations

We can write this system as a linear Volterra system

$$\phi(t) = \gamma(t) + A(t) \int_0^t \phi(s) ds,$$

where ϕ is the estimation error $\widehat{h} - h$, while γ, A are "known" vectors/matrices of functions.

$$\gamma(t) = \begin{bmatrix} \frac{1}{1-\sqrt{\frac{H_2(t)H_3(t)}{H_1(t)}}}(\widehat{g}_1 - g_1)(t) \\ \frac{1}{1-\sqrt{\frac{H_1(t)H_3(t)}{H_2(t)}}}(\widehat{g}_2 - g_2)(t) \\ \frac{1}{1-\sqrt{\frac{H_1(t)H_2(t)}{H_3(t)}}}(\widehat{g}_3 - g_3)(t) \end{bmatrix}.$$

$$
\begin{aligned}
A(t) &= \frac{1}{2}\sqrt{\frac{1}{H_1(t)H_2(t)H_3(t)}} \\
&\times \begin{bmatrix} \frac{1}{1-\sqrt{\frac{H_2(t)H_3(t)}{H_1(t)}}} & 0 & 0 \\ 0 & \frac{1}{1-\sqrt{\frac{H_1(t)H_3(t)}{H_2(t)}}} & 0 \\ 0 & 0 & \frac{1}{1-\sqrt{\frac{H_1(t)H_2(t)}{H_3(t)}}} \end{bmatrix} \\
&\times \begin{bmatrix} h_1(t)\frac{H_2(t)H_3(t)}{H_1(t)} & -h_1(t)H_3(t) & -h_1(t)H_2(t) \\ -h_2(t)H_3(t) & h_2(t)\frac{H_1(t)H_3(t)}{H_2(t)} & -h_2(t)H_1(t) \\ -h_3(t)H_2(t) & -h_3(t)H_1(t) & h_3(t)\frac{H_1(t)H_2(t)}{H_3(t)} \end{bmatrix}
\end{aligned}
$$

This has a solution

$$\phi(t) = \gamma(t) - \int_0^t K(t,s)\gamma(s)ds,$$

where K is the resolvent kernel, which can be calculated recursively. This shows that stochastic estimation error of h only depends on the estimation error of g, since the integration in $\int_0^t K(t,s)\gamma(s)ds$ reduces the variance by an order of magnitude.

THANK-YOU