Nonparametric Tests for Common Values In First-Price Sealed-Bid Auctions*

Philip A. Haile† Han Hong‡ Matthew Shum§

September 24, 2003

preliminary

Abstract

We develop tests for common values at first-price sealed-bid auctions. Our tests are nonparametric, require observation only of the bids submitted at each auction, and are based on the fact that the "winner's curse" arises only in common value auctions. The tests build on recently developed methods for using observed bids to estimate each bidder's conditional expectation of the value of winning the auction. Equilibrium behavior implies that in a private values auction these expectations are invariant to the number of opponents each bidder faces, while with common values they are decreasing in the number of opponents. This distinction forms the basis of our tests. We consider both exogenous and endogenous variation in the number of bidders. Monte Carlo experiments show that our tests can perform well in samples of moderate sizes. We apply our tests to two different types of U.S. Forest Service timber auctions and find little evidence of common values.

Keywords: first-price auctions, common values, private values, nonparametric testing, winner's curse, stochastic dominance, subsampling, endogenous participation, timber auctions

*We thank Don Andrews, Steve Berry, Ali Hortaçsu, Tong Li, Harry Paarsch, Isabelle Perrigne, Rob Porter, Quang Vuong, seminar participants at the Tow conference in Iowa, SITE, the World Congress of the Econometric Society in Seattle, and several universities for insightful comments. We thank Hai Che, Kun Huang, and Grigory Kosenok for research assistance. We thank the NSF, SSHRC and Sloan Foundation for financial support.

†Department of Economics, Yale University, P.O. Box 208264, New Haven, CT 06520, philip.haile@yale.edu

‡Department of Economics, Duke University, P.O. Box 90097, Durham, NC 27708, hanhong@duke.edu

§Department of Economics, Johns Hopkins University, 3400 N. Charles St., Baltimore, MD 21218, mshum@jhu.edu.

1 Introduction

At least since the influential work of Hendricks and Porter (1988), studies of auction data have played an important role in demonstrating the empirical relevance of economic models of strategic interaction between agents with asymmetric information. However, a fundamental issue remains unresolved: how to choose between private and common values models of bidders' information. In a common values auction, information about the value of the object for sale is spread among bidders; hence, a bidder would update his assessment of the value of winning if he learned the private information of an opponent. In a private values auction, by contrast, opponents' private information would be of interest to a bidder only for strategic reasons—learning an opponent's assessment of the good would not affect his beliefs about his own valuation.

In this paper we propose nonparametric tests to distinguish between the common value (CV) and private value (PV) paradigms based on observed bids at first-price sealed-bid auctions. The distinction between these paradigms is fundamental in the theoretical literature on auctions, with important implications for bidding strategies and the design of markets. While intuition is often offered for when one might expect a private or common values model to be more appropriate, a more formal approach would be valuable in many applications. In fact, discriminating between common and private values was the motivation behind Paarsch's (1992) pioneering work on structural estimation of auction models. More generally, models in which strategic agents' private information leads to adverse selection (a common values auction being just one example) have played a prominent role in the theoretical economics literature, yet the prevalence and significance of this type of informational asymmetry is not well established empirically. Because a first-price auction is a market institution particularly well captured by a tractable theoretical model, data from these auctions offer a promising opportunity to test for adverse selection using structure obtained from economic theory.

Several testing approaches explored previously rely heavily on parametric assumptions about the distribution functions governing bidders' private information (e.g. Paarsch (1992), Sareen (1999)). Such tests necessarily confound evaluation of the economic hypotheses of interest with evaluation of parametric distributional assumptions. Some prior work (e.g., Gilley and Karels (1981), Paarsch (1992)) has suggested examining variation in bid levels with the number of bidders as a test for common values. However, Pinkse and Tan (2002) have recently shown that this type of reduced-form test generally cannot distinguish CV from PV models in first-price auctions: in equilibrium, strategic behavior can cause bids to increase or decrease in the number of opponents under either paradigm. We overcome both of these limitations by taking a nonparametric structural approach, exploiting the relationships between observable bids and bidders' latent expectations implied by equilibrium bidding in a model that nests the private and common values frameworks. Unlike tests of particular PV or CV models (e.g., Paarsch (1992), Hendricks, Pinkse, and Porter (2003)), our approach enables testing a null hypothesis including all PV models within the standard affiliated values framework (Milgrom and Weber (1982)) against an alternative consisting of all CV models in that framework. The price we pay for these advantages is reliance on an assumption of equilibrium bidding. This is not an innocuous assumption. However, a first-price auction is a market institution that seems particularly well suited to this structural approach. Further, the equilibrium assumption itself imposes a testable overidentifying restriction (cf. Laffont and Vuong (1996), Guerre, Perrigne, and Vuong (2000)).

The importance of tests for common values to empirical research on auctions is further emphasized by recent results showing that CV models are identified only under strong conditions on the underlying information structure or on the types of data available (Athey and Haile (2002)). Hence, a formal method for determining whether a CV or PV model is more appropriate could offer an important diagnostic tool for researchers hoping to use demand estimates from bid data to guide the design of markets. Laffont and Vuong (1996) have pointed out that any common value model is observationally equivalent to some private values model, suggesting that such testing is impossible. However, they did not consider the possibilities of binding reserve prices or variation in the numbers of bidders, either of which could aid in distinguishing between the private and common value paradigms.

Our tests exploit variation in the number of bidders and are based on detecting the effects of the *winner's curse* on equilibrium bidding. The winner's curse is an adverse selection phenomenon arising in CV but not PV auctions. Loosely, winning a CV auction reveals to the winner that he was more optimistic about the object's value than were any of his opponents. This "bad news" (Milgrom (1981)) becomes worse as the number of opponents increases—having the most optimistic signal among many bidders implies (on average) even greater over-optimism than does being most optimistic among a few bidders. A rational bidder anticipates this bad news and adjusts his expectation of the value of winning (and, therefore, his bid) accordingly. In a PV auction, by contrast, the value a bidder places on the object does not depend on his opponents' information, so the number of bidders does not affect his expected value of the object conditional on winning. Relying on this distinction, our testing approach is based on detection of the adjustments rational bidders make in order to avoid the winner's curse as the number of competitors changes. This is nontrivial because we can observe only bids, and variation in the level of competition affects the aggressiveness of bidding even in a PV auction. However, economic theory enables us to separate this competitive response from responses (if any) to the winner's curse.

We consider several statistical tests, all involving the distributions of bidders' expected values (actually, particular conditional expectations of these values) in auctions with varying numbers of bidders. In a PV auction, these distributions should not vary with the number of bidders, whereas the CV alternative implies a first-order stochastic dominance relation. However, our testing problem is complicated by the fact that we cannot compare empirical distributions of bidders' expectations directly; rather, we can only compare empirical distributions of *estimates* of these expectations, obtained using nonparametric methods recently developed by Guerre, Perrigne, and Vuong (2000) (hereafter GPV) and extended by Li, Perrigne, and Vuong (2000, 2002)(hereafter LPV) and Hendricks, Pinkse, and Porter (2003) (hereafter HPP). This can significantly complicate the asymptotics of test statistics and makes use of the bootstrap difficult. A further complication arising in many applications is the endogeneity of bidder participation. After developing our tests for the base case of exogenous participation, we consider several standard models of endogenous participation and provide conditions under which our tests can be adapted.

While our testing approach is new, we are not the first to explore implications of the winner's curse as an approach for distinguishing PV and CV models. Hendricks, Pinkse and Porter (2003, footnote 2) suggest a testing approach applicable when there is a binding reserve price, in addition to several tests of a pure common values model that are applicable when one observes, in addition to bids, the *ex post* realization of the object's value. Although our tests are applicable when there is a binding reserve price, this is not required—an important advantage in many applications, including the drilling rights auctions studied by HPP and the timber auctions we study below. In addition, our tests require observation only of the bids—the only data available from most first-price auctions. For second-price and English auctions, Paarsch (1991) and Bajari and Hortacsu (2003) have considered tests for the winner's curse, using a simple regression approach. However, second-price sealed-bid auctions are rare, and the applicability of this approach to English auctions is limited by the fact that the winner's willingness to pay is never revealed (creating a missing data problem) and further by ambiguity regarding the appropriate interpretation of losing bids (e.g., Bikhchandani, Haile, and Riley (2002), Haile and Tamer (2003)). Athey and Haile (2002) have proposed an approach for discriminating private from common values at standard auctions, including first-price sealed-bid auctions. While their approach is related to ours, they focus on cases in which only a subset of the bids is observable, consider only exogenous participation, and do not develop formal statistical tests.

The remainder of the paper is organized as follows. The next section summarizes the underlying model, the method for inferring bidders' expectations of their valuations from observed bids, and the main principle of our testing approach. In section 3 we provide the details of two types of testing approaches and develop the necessary asymptotic theory. In section 4 we report the results of Monte Carlo experiments demonstrating the performance of our tests. In section 5 we show how the tests can be extended to environments with endogenous participation. Section 6 presents an approach for incorporating auction-specific covariates. Section 7 then presents the empirical application to U.S. Forest Service auctions of timber harvesting contracts, where we consider two data sets that differ in ways that seem likely *a priori* to affect the significance of any common value elements. We conclude in section 8.

2 Model and Testing Principle

The underlying theoretical framework is Milgrom and Weber's (1982) general affiliated values model. Throughout we denote random variables in upper case and their realizations in lower case. We use boldface to denote vectors. An auction has $N \in \{\underline{n} \ldots \bar{n}\}$ risk-neutral bidders, with $\underline{n} \geq 2$. Each bidder i has a valuation $U_i \in (\underline{u}, \overline{u})$ for the object and observes a private signal $X_i \in (\underline{x}, \overline{x})$ of this valuation. Valuations and signals have joint distribution $\tilde{F}_n(U_1, \ldots, U_n, X_1, \ldots, X_n)$, which is assumed to have a positive joint density on $(\underline{u}, \overline{u})^n \times (\underline{x}, \overline{x})^n$. We make the following standard assumptions (see Milgrom and Weber (1982)).

Assumption 1 *(Symmetry)* $\tilde{F}_n(U_1, \ldots, U_n, X_1, \ldots, X_n)$ *is exchangeable with respect to the indices* $1, \ldots, n$.[1]

Assumption 2 *(Affiliation)* $U_1, \ldots, U_n, X_1, \ldots, X_n$ *are affiliated.*

Assumption 3 *(Nondegeneracy)* $E[U_i | X_i = x, \mathbf{X}_{-i} = \mathbf{x}_{-i}]$ *is strictly increasing in* x $\forall \mathbf{x}_{-i}$.

Initially, we also assume that the number of bidders is not correlated with bidder valuations or signals:

Assumption 4 *(Exogenous Participation) For each* $n < \bar{n}$ *and all* $(u_1, \ldots, u_n, x_1, \ldots, x_n)$, $\tilde{F}_n(u_1, \ldots, u_n, x_1, \ldots, x_n) = \tilde{F}_{\bar{n}}(u_1, \ldots, u_n, \infty, \ldots, \infty, x_1, \ldots, x_n, \infty, \ldots, \infty)$.[2]

[1] We discuss relaxation of the symmetry assumption in section 8.

[2] This assumption is not made by Milgrom and Weber (1982) since they consider fixed n.

Such exogenous variation in the number of bidders across auctions will arise naturally in some applications but not others (cf. Athey and Haile (2002) and section 5 below). After developing our tests for this base case, we will also consider endogenous participation.

A seller conducts a first-price sealed-bid auction for a single object; i.e., sealed bids are collected from all bidders, and the object is sold to the high bidder at a price equal to his own bid.[3] Given Assumptions 1–3, in an n-bidder auction there exists a unique symmetric Bayesian Nash equilibrium in which each bidder employs a strictly increasing strategy $s_n(\cdot)$. As shown by Milgrom and Weber (1982), the first-order condition characterizing this equilibrium bid function is

$$v(x,x,n) = s_n(x) + \frac{s_n'(x)F_n(x|x)}{f_n(x|x)} \quad \forall x \tag{1}$$

where

$$v(x,x',n) \equiv E\left[U_i|X_i = x, \max_{j\neq i} X_j = x'\right] \tag{2}$$

$F_n(\cdot|x)$ is the distribution of the maximum signal among a given bidder's opponents conditional on his own signal being x, and $f_n(\cdot|x)$ is the corresponding conditional density.

Our testing approach is based on the fact that the conditional expectation $v(x,x,n)$ in (2) is decreasing in n whenever valuations contain a common value element. To show this, we first formally define private and common values.[4]

Definition 1 *Bidders have* **private values** *iff* $E[U_i|X_1,\ldots,X_n] = E[U_i|X_i]$*; bidders have* **common values** *iff* $E[U_i|X_1,\ldots,X_n]$ *strictly increases in* X_j *for* $j \neq i$*.*

Note that the definition of common values incorporates a wide range of models with a common value component, not just the special case of *pure common values*, where the value of the object is unknown but identical for all bidders.[5]

[3] We describe the auction as one in which bidders compete to buy. The translation to the procurement setting, where bidders compete to sell, is straightforward.

[4] Affiliation implies that $E[U_i|X_1,\ldots,X_n]$ is nondecreasing in all X_j. For simplicity our definition of common values rules out cases in which the winner's curse arises for some realizations of signals but not others. Without this, the results below would still hold but with weak inequalities replacing some strict inequalities. Up to this simplification, our PV and CV definitions define a partition of Milgrom and Weber's (1982) framework.

[5] Our terminology corresponds to that used by, e.g., Klemperer (1999) and Athey and Haile (2002), although it is not the only one used in the literature. Some authors reserve the term "common values" for the special case we call pure common values and use the term "interdependent values" (e.g., Krishna (2002)) or the less accurate "affiliated values" for the class of models we call common values. Additional confusion sometimes arises because the partition of the Milgrom-Weber framework into CV and PV models is only one of two partitions that might be of interest, the other being defined by whether bidders' signals are independent. Note in particular that dependence of bidders' information is neither necessary nor sufficient for common values.

The following theorem gives the key result enabling discrimination between PV and CV models.

Theorem 1 *Under Assumptions 1–4, $v(x, x, n)$ is invariant to n for all x in a PV model but strictly decreasing in n for all x in a CV model.*

Proof: Given symmetry, we focus on bidder 1 without loss of generality. With private values, $E[U_1|X_1, \ldots, X_n] = E[U_1|X_1]$, which does not depend on n. With common values

$$\begin{aligned} v(x, x, n) &\equiv E\left[U_1|X_1 = X_2 = x, X_3 \leq x, \ldots, X_{n-1} \leq x, X_n \leq x\right] \\ &= E_{X_n \leq x} E\left[U_1|X_1 = X_2 = x, X_3 \leq x, \ldots, X_{n-1} \leq x, X_n\right] \\ &< E_{X_n} E\left[U_1|X_1 = X_2 = x, X_3 \leq x, \ldots, X_{n-1} \leq x, X_n\right] \\ &= E\left[U_1|X_1 = X_2 = x, X_3 \leq x, \ldots, X_{n-1} \leq x\right] \\ &\equiv v(x, x; n-1) \end{aligned}$$

with the inequality following from the definition of common values. □

Informally, the realized value of the object affects a bidder's utility only when he wins. In a CV auction a rational bidder must therefore adjust his unconditional (on winning) expectation $E[U_i|X_i = x_i]$ downward to reflect the fact that he wins only when his own signal was higher than those of all opponents. The size of this adjustment depends on the number of opponents, since the information that the maximum signal among n is equal to x implies a higher expectation of U_i than the information that the maximum among $m > n$ is equal to x. Hence, the conditional expectation $v(x, x, n)$ decreases in n.

2.1 Structural Interpretation of Observed Bids

To use Theorem 1 to test for common values, we must be able to infer or estimate the latent expectations $v(x_i, x_i, n)$ for bidders in auctions with varying numbers of participants. We assume that for each n, the researcher observes the bids $B_1, \ldots, B_n$ from T_n n-bidder auctions. We let $T = \sum_n T_n$ and assume that for all n, $\frac{T_n}{T} \to \zeta_n \in (0, 1)$ as $T \to \infty$. Below we will add the auction index $t \in \{1, \ldots, T\}$ to the notation defined above as necessary. For simplicity we initially assume an identical object is sold at each auction. As shown by GPV, standard nonparametric techniques can be applied to control for auction-specific covariates. Below we will also suggest a more parsimonious alternative that may be more useful in applications with many covariates. We assume throughout that each auction is independent of all others.[6]

[6]This is a standard assumption, but one that serves to qualify almost all empirical studies of bidding, where data are taken from auctions in which bidders compete repeatedly over time. An exception is the recent paper of Jofre-Bonet and Pesendorfer (forthcoming).

As pointed out by GPV, in equilibrium the joint distribution of bidder signals is related to the joint distribution of bids through the relations

$$\begin{aligned} F_n\left(y|x\right) &= G_n\left(s_n(y)|s_n(x)\right) \\ f_n\left(y|x\right)\times &= g_n\left(s_n(y)|s_n(x)\right)s_n'\left(y\right) \end{aligned} \tag{3}$$

where $G_n(\cdot|s_n(x))$ is the equilibrium distribution of the highest bid among i's competitors conditional on i's equilibrium bid $s_n(x)$, and $g_n\left(\cdot|s_n\left(x\right)\right)$ is the corresponding conditional density. Due to the monotonicity of $s_n(\cdot)$, the highest bid among i's opponents will be the bid of the opponent whose signal is highest. Since in equilibrium $b_i = s_n(x_i)$, the differential equation (1) can then be rewritten

$$v(x_i, x_i, n) = b_i + \frac{G_n(b_i|b_i)}{g_n(b_i|b_i)} \equiv \xi(b_i; n). \tag{4}$$

For simplicity we will refer to the expectation $v(x_i, x_i, n)$ on the left side of (4) as bidder i's "value." Although these values are not observed directly, the joint distribution of bids is. Hence, the ratio $\frac{G_n(\cdot|\cdot)}{g_n(\cdot|\cdot)}$ is nonparametrically identified. Since $x_i = s_n^{-1}(b_i)$, (4) implies that each $v\left(s_n^{-1}\left(b_i\right), s_n^{-1}\left(b_i\right), n\right)$ is identified as well.

To address estimation, let B_{it} denote the bid made by bidder i at auction t, and let B_{it}^* represent the highest bid among i's opponents. GPV and LPV suggest nonparametric estimates of the form

$$\begin{aligned} \hat{G}_n(b;b) &= \frac{1}{T_n \times h_G \times n}\sum_{t=1}^{T}\sum_{i=1}^{n} K\left(\frac{b-b_{it}}{h_G}\right)\mathbf{1}\left(b_{it}^* < b, N_t = n\right) \\ \hat{g}_n(b;b) &= \frac{1}{T_n \times h_g^2 \times n}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbf{1}(N_t = n)K\left(\frac{b-b_{it}}{h_g}\right)K\left(\frac{b-b_{it}^*}{h_g}\right). \end{aligned} \tag{5}$$

Here h_G and h_g are bandwidths and $K(\cdot)$ is a kernel. $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ are nonparametric estimates of

$$G_n(b;b) \equiv G_n(b|b)g_n\left(b\right) = \frac{\partial}{\partial b}\Pr(B_{it}^* \leq m, B_{it} \leq b)|_{m=b}$$

and

$$g_n(b;b) \equiv g_n(b|b)g_n\left(b\right) = \frac{\partial^2}{\partial m \partial b}\Pr(B_{it}^* \leq m, B_{it} \leq b)|_{m=b}$$

where $g_n(\cdot)$ is the marginal density of the bids in equilibrium. Since

$$\frac{G_n(b;b)}{g_n(b;b)} = \frac{G_n(b|b)}{g_n(b|b)} \tag{6}$$

$\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)}$ is a consistent estimator of $\frac{G_n(b|b)}{g_n(b|b)}$.[7] Hence, by evaluating $\hat{G}_n(\cdot,\cdot)$ and $\hat{g}_n(\cdot,\cdot)$ at each observed bid, we can construct a pseudo-sample of consistent estimates of the realizations of each $V_{it} \equiv v(X_{it}, X_{it}, n)$ using (4):

$$\hat{v}_{it} \equiv \hat{\xi}(b_{it}; n_t) = b_{it} + \frac{\hat{G}_n(b_{it}; b_{it})}{\hat{g}_n(b_{it}; b_{it})}. \tag{7}$$

This possibility was first articulated for the independent private values model by Laffont and Vuong (1993) and GPV, and has been extended to affiliated values models by LPV and HPP. Following this literature, we refer to each estimate $\hat{v}_{it}$ as a "pseudo-value."

2.2 Main Principle of the Test

Each pseudo-value $\hat{v}_{it}$ is an estimate of $v(x_{it}, x_{it}, n_t)$. If we have pseudo-values from auctions with different numbers of bidders, we can exploit Theorem 1 to develop a test. Let $F_{v,n}(\cdot)$ denote the distribution of the random variable $V_{it} = v(X_{it}, X_{it}, n)$. Since $F_{v,n}(v) = \Pr(v(X_{it}, X_{it}, n) \leq v)$, Theorem 1 and Assumption 4 immediately imply that under the PV hypothesis, $F_{v,n}(\cdot)$ must be the same for all n, while under the CV alternative, $F_{v,n}(v)$ must strictly increase in n for all v.

Corollary 1 *Under the private values hypothesis*

$$F_{v,\underline{n}}(v) = F_{v,\underline{n}+1}(v) = \ldots = F_{v,\bar{n}}(v) \quad \forall v. \tag{8}$$

Under the common values hypothesis

$$F_{v,\underline{n}}(v) < F_{v,\underline{n}+1}(v) < \ldots < F_{v,\bar{n}}(v) \quad \forall v. \tag{9}$$

3 Tests for Stochastic Dominance

Corollary 1 suggests that a test for stochastic dominance applied to estimates of the distributions $F_{v,n}(\cdot)$ would provide a test for common values. If the values $v_{it} = v(x_{it}, x_{it}, n)$ were directly observed, a wide variety of existing tests from the statistics and econometrics literature could be used (e.g, McFadden (1989), Barrett and Donald (2003), Davidson and Duclos (2000) or Anderson (1996)).[8] The empirical distribution function

$$\hat{F}_{v,n}(y) = \frac{1}{T_n}\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n} 1(v_{it} \leq y, N_t = n).$$

[7] Strict monotonicity of the equilibrium bid function and the assumption that $\tilde{F}_n(\cdot)$ has a positive density ensures that the denominator in (6) is nonzero for b in the range of $s_n(\cdot)$.

[8] Some of these tests are consistent against all deviations from the null hypothesis that $F_{v,n}(\cdot) \equiv F_{v,m}(\cdot) \forall n, m$. These include the Kolmogorov-Smirnov type tests of, e.g., McFadden (1989), which uses the

is commonly used to form test statistics.

Our testing problem has the complication that the realizations $v_{it} = v(x_{it}, x_{it}, n_t)$ are not directly observed but estimated. Hence, the empirical distributions we can construct are

$$\hat{F}_{\hat{v},n}(y) = \frac{1}{T_n}\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbf{1}\left(\hat{v}_{it} \leq y, N_t = n\right).$$

Although we can formulate consistent tests using these empirical distributions and the testing principles above, deriving the approximate large sample distributions for inference purposes is difficult here for several reasons. First is the fact that in finite samples estimates of $v(x, x, n)$ and $v(x', x', n)$ are dependent for x' near x, due to smoothing. Standard tests for FOSD typically assume independent draws from the distributions in question. A second complication is the trimming used to handle the boundaries of the supports of the pseudo-value distributions. Trimming creates difficulties both through the need to trim in a way that does not lead to inconsistency, and through the presence of local nonparametric kernel density estimate $\hat{g}_n(b_{it}|b_{it})$ in the denominator of (7).[9] A third difficulty is that although using the bootstrap for inference may seem natural here, doing so requires resampling under the null hypothesis. Because we must estimate pseudo-values before constructing test statistics, this would mean developing a scheme for resampling *bids* that imposes the null hypothesis concerning distributions of bidders' *values*.

To deal with these difficulties we consider two general approaches. The first involves testing the implications of stochastic dominance for finite sets of functionals of each $F_{v,n}(\cdot)$. This approach enables us to apply multivariate one-sided hypothesis tests based on tractable asymptotic approximations. The second approach uses a version of familiar Kolmogorov-Smirnov type statistics, with critical values approximated by subsampling.

---

sup norm in constructing the test statistics $\sup_y \left(\hat{F}_{v,n}(y) - \hat{F}_{v,m}(y)\right)$, and related Von-Mises type statistics. Another consistent test of this sort is the rank test of stochastic dominance (Hajek, Sidak, and Sen (1999)), which uses the statistics $R = \frac{1}{nT_n}\frac{1}{mT_m}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\sum_{s=1}^{T_m}\sum_{j=1}^{m} 1(y_{it} - y_{jt}) - \frac{1}{2}$. For $n < m$, $R < 0$ suggests evidence against the null hypothesis of no stochastic dominance. Other tests detect deviations from H_0 in given directions. For example Anderson (1996) compared $\hat{F}_{v,n}(y)$ and $\hat{F}_{v,m}(y)$ at a fixed grid of points. See Linton, Massoumi, and Whang (2002) for a recent example of stochastic dominance tests using regression residuals.

[9]The assumption used for boundary trimming in Lavergne and Vuong (1996) does not hold in the current context. The trimming problem might be alleviated if we instead consider global nonparametric estimation of the conditional inverse hazard function $\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}$ using sieve based methods (see, for example, Chen and Shen (1998)).

3.1 Multivariate One-Sided Hypothesis Tests for Stochastic Dominance

Let γ_n denote a finite vector of functionals of the distribution $F_{v,n}(\cdot)$. We will consider tests of hypotheses of the form[10]

$$\begin{aligned} H_0 \text{ (PV)}: &\ \gamma_{\underline{n}} = \gamma_{\underline{n}+1} = \cdots = \gamma_{\bar{n}} \\ H_1 \text{ (CV)}: &\ \gamma_{\underline{n}} > \gamma_{\underline{n}+1} > \cdots > \gamma_{\bar{n}} \end{aligned}$$

or, letting $\delta_{m,n} \equiv \gamma_m - \gamma_n$ and $\delta \equiv \left(\delta_{\underline{n},\underline{n}+1}, , \ldots , \delta_{\bar{n}-1,\bar{n}}\right)'$,

$$\begin{aligned} H_0 \text{ (PV)}: &\ \delta = \mathbf{0} \\ H_1 \text{ (CV)}: &\ \delta > \mathbf{0}. \end{aligned} \tag{10}$$

We consider two types of functionals γ_n. The first is a vector of quantiles of $F_{v,n}(\cdot)$. The second is the mean. In the next two subsections we show that for both cases consistent estimators of each γ_n (or the difference vector δ) are available and have approximate multivariate normal distributions in large samples. These results rely on the following assumptions:

Assumption 5 *1. $G_n(b;b)$ is $R+1$ times differentiable in its first argument and R times differentiable in its second argument. $g_n(b;b)$ is R times differentiable in both arguments. The derivatives are bounded and continuous.*

2. $\int K(u)\,du = 1$ and $\int u^r K(u)\,du = 0$ for all $r < R$. $\int |u|^R K(u)\,du < \infty$.

3. $h_G = h_g = h$. As $T_n \to \infty$, $h \longrightarrow 0$, $T_n h^2 \big/ \log T_n \longrightarrow \infty$, $T_n h^{2+2R} \longrightarrow 0$.

3.1.1 Tests based on Quantiles

Let $\hat{b}_{\tau,n}$ denote the τth quantile of the empirical distribution of bids from all n-bidder auctions, i.e.,

$$\hat{b}_{\tau,n} = \hat{G}_n^{-1}(\tau) \equiv \inf\{b : \hat{G}_n(b) \geq \tau\}$$

where $\hat{G}_n(b) = \frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n} 1(b_{it} \leq b, N_t = n)$. Similarly, $b_{\tau,n}$ will denote $G^{-1}(\tau)$ and $x_\tau = F_x^{-1}(\tau)$ will denote the τth quantile of the marginal distribution $F_x(\cdot)$ of a bidder's signal. Equation (4) and monotonicity of the equilibrium bid function $s_n(\cdot)$ imply that the τth quantile of $F_{v,n}(\cdot)$ can be estimated by

$$\hat{v}_{\tau,n} = \hat{b}_{\tau,n} + \frac{\hat{G}_n(\hat{b}_{\tau,n};\hat{b}_{\tau,n})}{\hat{g}_n(\hat{b}_{\tau,n};\hat{b}_{\tau,n})}.$$

[10] Because each null hypothesis we consider consists of a single point in the space of the "parameter" δ, the difficulties discussed in Wolak (1991) do not arise here.

Since sample quantiles of the bid distribution converge to population quantiles at rate $\sqrt{T_n}$, the sampling variance of $\hat{v}_{\tau,n} - v(x_\tau, x_\tau, n)$ will be governed by the slow pointwise nonparametric convergence rate of $\hat{g}_n(\cdot;\cdot)$.[11] As shown in GPV, for fixed b, $\hat{g}_n(b;b)$ converges at rate $\sqrt{T_n h_g^2}$ to $g_n(b;b)$. Theorem 2 then describes the limiting behavior of each $\hat{v}_{\tau,n}$. The proof is given in the appendix.

Theorem 2 *Suppose Assumption 5 holds. Then as* $T_n \longrightarrow \infty$,
(i) $\hat{b}_{\tau,n} - s_n\left(F_x^{-1}(\tau)\right) = O_p\left(\frac{1}{\sqrt{T_n}}\right)$.
(ii) For each b *such that* $g_n(b;b) > 0$,

$$\sqrt{T_n h^2}\left[\hat{\xi}(b;n) - v\left(s_n^{-1}(b), s_n^{-1}(b), n\right)\right] = \sqrt{T_n h^2}\left(\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)} - \frac{G_n(b|b)}{g_n(b|b)}\right)$$
$$\xrightarrow{d} N\left(0, \frac{1}{n}\frac{G_n(b|b)^2}{g_n(b|b)^3 g_n(b)}\left[\int\int K(x)^2 K(y)^2\, dxdy\right]\right)$$

(iii) For distinct quantile values $\tau_1, \ldots, \tau_L$ *in* $(0,1)$, *the L-dimensional vector of elements* $\sqrt{T_n h^2}\left(\hat{\xi}\left(\hat{b}_{\tau_l,n}; n\right) - v\left(F_x^{-1}(\tau_l), F_x^{-1}(\tau_l), n\right)\right)$ *converges in distribution to* $Z \sim N(0,\Omega)$, *where* Ω *is a diagonal matrix with* l*th diagonal element*

$$\Omega_l = \frac{1}{n}\frac{G_n(s_n(x_\tau)|s_n(x_\tau))^2}{g_n(s_n(x_\tau)|s_n(x_\tau))^3 g_n(s_n(x_\tau))}\left[\int\int K(x)^2 K(y)^2\, dxdy\right].$$

3.1.2 Tests based on Means

An alternative to comparing quantiles is to compare means of the pseudo-value distributions. We can estimate

$$E_x[v(x,x,n)] = \int v\, dF_{v,n}(v)$$

[11] Note that $G_n(b;b)$ is estimated more precisely than $g_n(b;b)$ for all bandwidth sequences h. For simplicity, in Assumption 5 we have chosen $h_G = h_g$, in which case the sampling variance will be dominated by that from estimation of $g_n(b;b)$. We have assumed undersmoothing rather than optimal smoothing to avoid estimating the asymptotic bias term for inference purposes. An alternative is to choose different sequences for h_G and h_g. If we have chosen h_G and h_g close to their optimal range, the sampling variance will still be dominated by that of $\hat{g}_n(b;b)$ and the result of the theorem will not change. On the other hand if $h_G \approx h_g^2$ so that $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ share the same magnitude of variance, then the convergence rate for $G_n(b;b)$ will be far from optimal.

with the sample average of the pseudo-values in all n-bidder auctions:

$$\hat{\mu}_n = \frac{1}{n \times T_n} \sum_{t=1}^{T_n} \sum_{i=1}^{n} \hat{v}_{it}. \tag{11}$$

By Corollary 1, $E_x[v(x, x, n)]$ is the same for all n under private values but strictly decreasing in n with common values.

One difficulty in implementing a test, however, arises from boundary trimming typically involved in kernel density estimates such as those appearing in (5). Unlike partial mean nonparametric regression problems where fixed exogenous trimming is possible (cf. Newey (1994)), here it is more difficult to preserve consistency of the test statistics with trimming at the boundary.[12] In particular, we must avoid trimming scheme that will truncate different ranges of signals in auctions with different numbers of bidders (e.g., that suggested by GPV), since doing so could create differences in the mean pseudo-values when the null was true, or hide differences when the null is false.

To overcome this problem, we suggest a trimming method that equalizes the quantiles trimmed from $F_{\hat{v},n}(\cdot)$ across all n. Because equilibrium bid functions are strictly monotone, the pseudo-value at the τth quantile of $F_{v,n}(\cdot)$ is that of the bidder with signal at the τth quantile of $F_x(\cdot)$. Hence, trimming bids at the same quantile for all values of n also trims the same bidder types from all distributions, enabling consistent testing based on Corollary 1.

Let $\hat{b}_{\tau,n}$ and $\hat{b}_{1-\tau,n}$ denote the τth and $(1-\tau)$th quantiles of $\hat{G}_n(\cdot)$. The *quantile-trimmed mean* is defined as

$$\mu_n \equiv E\left[v(x, x, n)\ \mathbf{1}\left(x_\tau \le x \le x_{1-\tau}\right)\right]$$

with sample analog

$$\hat{\mu}_{n,\tau} \equiv \frac{1}{n \times T_n} \sum_{t=1}^{T} \sum_{i=1}^{n} \hat{v}_{it}\ \mathbf{1}\left(\hat{b}_{\tau,n} \le b_{it} \le \hat{b}_{1-\tau,n}, N_t = n\right).$$

We can then test the modified hypotheses

$$H_0 : \mu_{\underline{n},\tau} = \cdots = \mu_{\bar{n},\tau} \tag{12}$$

$$H_1 : \mu_{\underline{n},\tau} > \cdots > \mu_{\bar{n},\tau} \tag{13}$$

[12] One might attempt to mimic the stochastic trimming used in, for example, Lavergne and Vuong (1996) and Lewbel (1998). However, this approach usually requires smoothness conditions on $g_n(b; b)$ to reduce the order bias. Since each pseudo-value involves the inverse of $g_n(b; b)$, the asymptotic variance may not be finite when $g_n(b; b)$ is smooth.

which are implied by (8) and (9), respectively. The next theorem shows the consistency and asymptotic distribution of $\hat{\mu}_{n,\tau}$.

Theorem 3 *Suppose Assumption 5 holds,* $\frac{(\log T)^2}{Th^3} \longrightarrow 0$ *and* $T_n h^{1+2R} \longrightarrow 0$*. Then*
(i) (Consistency) $\hat{\mu}_{n,\tau} \xrightarrow{p} \mu_{n,\tau}$.
(ii) (Asymptotic distribution) $\sqrt{T_n h}\left(\hat{\mu}_{n,\tau} - \mu_{n,\tau}\right) \xrightarrow{d} N\left(0, \omega_n\right)$ *where*

$$\omega_n = \left[\int \left(\int K(v) K(u+v)\, dv\right)^2 du\right] \left[\frac{1}{n} \int_{F_b^{-1}(\tau)}^{F_b^{-1}(1-\tau)} \frac{G_n(b;b)^2}{g_n(b;b)^3} g_n(b)^2\, db\right] \tag{14}$$

and the integration is over the support of the kernel function $K(\cdot)$*.*

The proof is given in the appendix. Note that the convergence rate of each $\hat{\mu}_n$ is $\sqrt{T_n h}$. While this is slower than the parametric rate $\sqrt{T_n}$, it is faster than the $\sqrt{T_n h^2}$ rate of the quantile differences described above. Intuitively, the intermediate $\sqrt{T_n h}$ rate of convergence arises because $\hat{g}_n(b;b)$ is an estimated bivariate density function, but in constructing the estimate $\hat{\mu}_n$ we average along the one-dimensional 45^o line (b_{ij}, $b_{ij}^* = b_{ij}$) (cf. Newey (1994)).[13]

3.1.3 Test Statistics

A likelihood ratio (LR) test (e.g., Bartholomew (1959), Wolak (1989)) or the weighted power test of Andrews (1998) provide possible approaches for formulating test statistics based on the asymptotic normality results above. Since we do not have a good a prior choice of the weighting function for Andrews' weighted power test, we have chosen to use the LR test. In fact, Monte Carlo results in Andrews (1998) comparing the LR test to his more general tests for multivariate one-sided hypotheses, which are optimal in terms of a "weighted average power," suggests that the LR tests are "close to being optimal for a wide range of [average power] weighting functions" (pg. 158).

In this section we focus on a test for differences in the means of the pseudo-value distributions. An analogous test can be constructed using quantiles; however, because of the faster rate of convergence of the means test and its superior performance in Monte Carlo simulations, we focus on this approach.

[13] While the test based on averaged pseudo-values converges faster than that based on fixed number of quantiles, the improvement of the convergence rate is not proportional since the conditions on bandwidths for the partial mean case are more stringent than those on the pointwise estimates. However, there are still improvements after taking this into account, and this advantage of the means-based test is evident in (unreported) Monte Carlo simulations. Details of the quantile-based tests are available from the authors on request.

Let ω_n denote the asymptotic variance given in (14) for each value of $n = \underline{n}, \ldots, \bar{n}$ and define $a_n \equiv \frac{T_n h}{\omega_n}$. Then the asymptotic covariance matrix of the vector $\left(\hat{\mu}_{\underline{n},\tau} \ \ldots \ \hat{\mu}_{\bar{n},\tau}\right)'$ is given by

$$\Sigma = \begin{bmatrix} \frac{1}{a_{\underline{n}}} & 0 & 0 & 0 \\ 0 & \frac{1}{a_{\underline{n}+1}} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{a_{\bar{n}}} \end{bmatrix}.$$

The restricted maximum-likelihood estimate of the (quantile-trimmed) mean pseudo-value under the null hypothesis (12) is given by

$$\bar{\mu} = \frac{\sum_{n=\underline{n}}^{\bar{n}} a_n \, \hat{\mu}_{n,\tau}}{\sum_{n=\underline{n}}^{\bar{n}} a_n}.$$

To test against the alternative (13) let $\mu^*_{\underline{n}}, \ldots, \mu^*_{\bar{n}}$ denote the solution to

$$\min_{\mu_{\underline{n}}, \ldots, \mu_{\bar{n}}} \sum_{n=\underline{n}}^{\bar{n}} a_n \left(\hat{\mu}_{n,\tau} - \mu_n\right)^2 \quad s.t. \quad \mu_{\underline{n}} \geq \mu_{\underline{n}+1} \geq \cdots \geq \mu_{\bar{n}}. \tag{15}$$

This solution can be found using the well-known “pool adjacent violators” algorithm (Ayer, et al. (1955)), using the weights a_n. Let $K \in \{1, \ldots, \bar{n} - \underline{n} + 1\}$ denote the number of distinct values in $\mu^*_{\underline{n}}, \ldots, \mu^*_{\bar{n}}$. Now define the test statistic

$$\bar{\chi}^2 = \sum_{n=\underline{n}}^{\bar{n}} a_n \left(\mu^*_{n,\tau} - \bar{\mu}\right)^2.$$

The following corollary states that, under the null hypothesis, the LR statistic $\bar{\chi}^2$ is asymptotically distributed as a mixture of Chi-square random variables. The proof is given in Bartholmew (1959, Section 3).

Corollary 2 *Under the null PV hypothesis,*

$$\Pr\left(\bar{\chi}^2 \geq c\right) = \sum_{k=2}^{\bar{n}-\underline{n}+1} \Pr\left(\chi^2_{k-1} \geq c\right) w(k;\Sigma) \qquad \forall c > 0.$$

where χ^2_j denotes a standard Chi-square distribution with j degrees of freedom, and each mixing weight $w(k;\Sigma)$ is the probability that the solution to (15) has exactly k distinct values when the vector $\{\hat{\mu}_{\underline{n},\tau}, \ldots, \hat{\mu}_{\bar{n},\tau}\}$ has a multivariate $N(0,\Sigma)$ distribution.

3.2 A Sup-Norm Test Using Subsampling

A second testing approach is based on a Kolmogorov-Smirnov-type statistic for a k-sample test of equal distributions against an alternative of strict first-order stochastic dominance. Consider the sum of supremum distances between successive empirical distributions of pseudo-values:[14]

$$\delta_T = \sum_{n=\underline{n}}^{\overline{n}-1} \sup_{v \in [v_\tau, v_{1-\tau}]} \left\{ \hat{F}^T_{\hat{v},n+1}(v) - \hat{F}^T_{\hat{v},n}(v) \right\}.$$

Uniform consistency of each $\hat{F}^T_{\hat{v},n}(\cdot)$ on the compact set $[v_\tau, v_{1-\tau}]$ implies that $\delta_T \to 0$ as $T \to \infty$ under H_0, while $\delta_T \to \Delta > 0$ under H_1. This forms a basis for testing. In particular, define the test statistic

$$S_T = \sigma_T \delta_T$$

where σ_T is a normalizing sequence proportional to $\left(\frac{Th^2}{\log T}\right)^{1/2}$, the rate of uniform convergence of each $\hat{F}^T_{\hat{v},n}(\cdot)$ to the corresponding $F_{v,n}(\cdot)$.

To approximate the asymptotic distribution of the test statistic, we use a subsampling approach.[15] Recall that the observables consist of the set of bids $B_t = (B_{1t}, \ldots, B_{n_t t})$ from each auction $t = 1, \ldots, T$. So we can write

$$\delta_T = \delta_T \left(B_1, \ldots, B_{T_{\underline{n}}}, \ldots, B_T \right).$$

Let R_T denote a sequence of subsample sizes. For each n, let R_{nT} be a sequence proportional to R_T. Let

$$\eta_T = \sum_{n=\underline{n}}^{\overline{n}} \binom{T_n}{R_{nT}}$$

denote the number of subsets $\left(B^*_1, \ldots B_{R_{\underline{n}T}}, \ldots B^*_{R_{\overline{n}T}}\right)$of $(B_1, \ldots B_T)$ consisting of all bids from R_{nT} of the original T_n n-bidder auctions, $n = \underline{n}, \ldots \overline{n}$. Let $\delta_{T,R_T,i}$ denote the statistic $\delta_{R_T}\left(B^*_1, \ldots B^*_{R_{\underline{n}T}}, \ldots B^*_{R_{\overline{n}T}}\right)$ obtained using the ith such subsample of bids. The sampling distribution Φ_T of the test statistic S_T is then approximated by

$$\Phi_{T,R_T}(x) = \frac{1}{\eta_T} \sum_{i=1}^{\eta_T} 1 \left\{ \sigma_{R_T} \delta_{T,R_T,i} \leq x \right\} \tag{16}$$

The critical value for a test at level α is taken to be the $1-\alpha$ quantile, $\Phi^{1-\alpha}_{T,R_T}$, of Φ_{T,R_T}.

[14] Wallenstein (1980) proposed a version of this statistic for testing in the case of iid draws from k different distributions. In that special case, the exact sampling distribution can be derived.

[15] See Linton, Massoumi and Whang (2002) for a recent application of subsampling to tests for stochastic dominance in a different context.

Theorem 4 *(i) Assume that under* H_0*,* Φ_{T,R_T} *converges weakly to a distribution* Φ *which is continuous at its* $1-\alpha$ *quantile,* $\Phi^{1-\alpha}$*. If* $R_T \to \infty$ *and* $\frac{R_T}{T} \to 0$ *as* $T \to \infty$*, then under* H_0*,* $\Phi^{1-\alpha}_{T,R_T} \to \Phi^{1-\alpha}$ *in probability and* $\Pr\left(S_T > \Phi^{1-\alpha}_{T,R_T}\right) \to \alpha$*.*
(ii) Assume that as $T \to \infty$*,* $R_T \to \infty$*,* $\frac{R_T}{T} \to 0$*, and* $\liminf_T (\sigma_T/\sigma_{R_T}) > 1$*. Then under* H_1*,* $\Pr\left(S_T > \Phi^{1-\alpha}_{T,R_T}\right) \to 1$ *as* $T \to \infty$*.*

The proof is omitted since the result follows from Theorem 2.6.1 of Politis, Romano and Wolf (1999), given the discussion above. As usual, in practice the distribution in (16) is approximated using random subsampling.

4 Monte Carlo Simulations

Here we summarize the results of Monte Carlo experiments performed to evaluate our testing approaches. We consider data generated by two PV models and two CV models:

(PV1) independent private values, $x_i \sim u[0,1]$;

(PV2) independent private values, $\ln x_i \sim N(0,1)$;

(CV1) common values, i.i.d. signals $x_i \sim u[0,1]$, $u_i = \alpha x_i + (1-\alpha)\frac{\sum_{j\neq i} x_j}{n-1}$;[16]

(CV2) pure common values, $u_i = u \sim u[0,1]$, conditionally independent signals x_i uniform on $[0,u]$.[17]

Before examining the results from the Monte Carlo experiments, we turn to Figure 1. Here we illustrate the empirical distributions of pseudo-values obtained from one simulation draw from each of the four models. We do this for $n = 2,\ldots,5$, with $T_n = 200$. For the PV models, the estimated distributions are very close to each other. For the CV models these distributions clearly suggest the first-order stochastic dominance relation implied by CV models.

Note that in both model CV1 and model CV2, the effect of a change in n on the distribution appears to be largest when n is small. This is the case in most CV models and is quite intuitive: the difference between $E[U_1|X_1 = \max_{j\in\{2,\ldots,n\}} X_j = x]$ and $E[U_1|X_1 =$

[16] For the case $\alpha = \frac{1}{2}$ considered below, one can show that $v(x,x;n) = \frac{3n-2}{4(n-1)}x$, leading to the equilibrium bid function $s_n(x) = \frac{3n-2}{4n}x$. In this example it is easy to see that although $v(x,x;n)$ is strictly decreasing in n, $s_n(x)$ strictly increases in n

[17] The symmetric equilibrium bid function for this model, given in Matthews (1984), is $s_n(x) = \int_0^x \hat{v}(t,n)\left(\frac{n-1}{x}\right)\left(\frac{t}{x}\right)^{n-2} dt$ where $\hat{v}(t,n) = \int_x^1 c\, g(c|x,n)\, dc$ and $g(c|x,n) = \frac{\frac{n}{c}\left(\frac{x}{c}\right)^{n-1}}{\int_x^1 \frac{n}{w}\left(\frac{x}{w}\right)^{n-1} dw}$.

$\max_{j\in\{2,\dots,n+1\}} X_j = x]$ typically shrinks as n grows. This is important since most auction data sets contain relatively few observations for n large but many observations for n small—exactly where the effects of the winner's curse are most pronounced.

We first consider the LR test, based on quantile-trimmed means. Tables 1 and 2 summarize the test results, using tests with nominal size 5% and 10%. The last two rows in Table 1 indicate that in the two PV models there is a tendency to over-reject. For example, for tests with nominal size 10% and data generated by the PV1 model, we reject 20.5% of the time when the range of bidders is 2–4, and 39% of the time when the range of bidders is 2–5. The tests do appear to have good power properties, rejecting the CV models in 70 to 100 percent of the replications. However, the over-rejection under the null is a concern.

One possible reason for the over-rejections is that the asymptotic approximations of the variances of the average pseudo-values (Σ in Corollary 3) may be poor at the modest sample sizes we consider. We have considered an alternative of using bootstrap estimates of Σ. The results, reported in Table 2, indicate that the tendency towards over-rejection is attenuated when we estimate these variances with the bootstrap. For a test with nominal size 10%, we now reject no more than 13.5% of the time when the range of n is 2–4, and 18% of the time when the range of n is 2–5. With a 5% nominal size, our rejection rates range between 4% and 11.5%. The power properties remain very good. These results are encouraging and suggest use of the bootstrap in practice.

Finally, Table 3 provides results for the Kolmogorov-Smirnov test using random subsampling to obtain critical values.[18] This test appears to perform extremely well. The rejection rates for the two PV models are very close to the nominal sizes in all cases. The rejection rates for the CV models are also extremely high, particularly when more than two distributions are compared.

5 Endogenous Participation

Thus far we have assumed that variation in participation across auctions is exogenous to the joint distribution of bidders' valuations and signals. Such exogenous variation could arise, for example, from exogenous shocks to bidders' costs of participation, exogenous variation in bidder populations across markets, or seller restrictions on participation—e.g.,

[18] Here we have incorporated the recentering approach suggested by Chernozhukov (2002). In each subsample, the test statistic was recentered by the original full-sample test statistic: $\mathcal{L}^s \equiv \sqrt{\frac{Bh_B^2}{\log B}}\left[\sum_{n=\underline{n}}^{\bar{n}-1} \sup_x \left(\hat{F}_{n+1}^s(x) - \hat{F}_n^s(x)\right) - \mathcal{K}\right]$ where $K \equiv \sum_{\underline{n}}^{\bar{n}-1} \sup_x \left(\hat{F}_{n+1}(x) - \hat{F}_n(x)\right)$, the full-sample statistic. The subsampled p-value is computed as $\frac{1}{S}\sum_{s=1}^{S} \mathbf{1}\left(\mathcal{L}^s > \sqrt{\frac{Th_T^2}{\log T}}\mathcal{K}\right)$.

in government auctions (McAfee and McMillan (1987)) or field experiments (Engelbrecht-Wiggans, List, and Lucking-Reiley (1999)). However, in many applications participation may be endogenous. Here we explore adaptation of our testing approach to such situations, considering several different models of participation.

5.1 Binding Reserve Prices

The most common model of endogenous participation is one in which the seller uses a binding reserve price r, so that only bidders with sufficiently favorable signals bid. We continue to let N denote the number of potential bidders and will now let A denote the number of *actual* bidders—those submitting bids of at least r. Variation in the number of potential bidders is still taken to be exogenous; i.e., Assumption 4 is maintained in this case. However, A will be determined endogenously. Because we consider sealed bid auctions with private information, it is natural to assume that bidders know the realization of N but not that of A when choosing their bids, since A is determined by the realizations of the signals.[19] We assume the researcher also can observe N.[20] As before, we let $F_{v,n}(\cdot)$ denote the distribution of the values $v(X, X, n)$ of the n potential bidders.

As shown by Milgrom and Weber (1982), given r and n, a bidder participates if and only if his signal exceeds the "screening level"

$$x^*(r, n) = \inf \left\{ x : E\left[U_i | \, X_i = x, \max_{j \neq i} X_j \leq x \right] \geq r \right\}. \tag{17}$$

That is, a bidder participates only if he would be willing to pay the reserve price for the good even when no other bidder were. In a PV auction, we may assume without loss of generality that $E[U_i|X_i = x] = x$. Since in a PV model $E\left[U_i| \, X_i = x, \max_{j \neq i} X_j \leq x\right] = E[U_i|X_i = x]$, (17) then implies $x^*(r, n) = r$. In a common values model, however, $E\left[U_i| \, X_i = x, \max_{j \neq i} X_j \leq x\right]$ decreases in n (the proof follows that of Theorem 1), implying that $x^*(r, n)$ increases in n. This gives the following lemma, which will imply that our baseline testing approach must be modified in this case.

Lemma 1 *The screening level $x^*(r, n)$ is invariant to n in a PV model but strictly increasing in n in a CV model.*

[19] Schneyerov (2002) considers a different model in which bidders observe a signal of the number of actual bidders after the participation decision but before bids are made.

[20] See HPP for an example. If this is not the case, not only is testing difficult, but the more fundamental identification of bidders' values $v(x, x, n)$ generally fails. This is because bidding is based on a first-order condition for bidders who condition on the realization of N when constructing their beliefs $G_n(b; b)$ regarding the most competitive opposing bid. Identification based on this first-order condition requires that the researcher condition on the same information available to bidders.

For both PV and CV models, the equilibrium participation rule implies that the marginal distribution of the signals of actual bidders is the truncated distribution

$$F(x|r,n) = \frac{F(x) - F(x^*(r,n))}{1 - F(x^*(r,n))}.$$

Hence, letting

$$v^*(r,n) = v\left(x^*\left(r,n\right), x^*\left(r,n\right), n\right)$$

the distribution of values for actual bidders is given by

$$F^A_{v,n}(v) = \frac{F_{v,n}(v) - F_{v,n}(v^*(r,n))}{1 - F_{v,n}(v^*(r,n))} \quad \forall v \geq v^*(r,n). \tag{18}$$

In a PV model, $F(x|r,n) = \frac{F(x)-F(r)}{1-F(r)}$. Since neither this distribution nor the expectation $v(x,x,n)$ varies with n in a PV model, it is then still the case that the distribution $F_{v,n}(\cdot)$ is invariant to n in a PV model, implying that $F^A_{v,n}(\cdot)$ is too. However, the CV case does not give a clean prediction about $F^A_{v,n}(\cdot)$. Because $x^*(r,n)$ increases with n under common values, changes in n affect the marginal distribution of actual bidders' values in two ways: first by the fact that $v(x,x,n)$ decreases in n for fixed x; second by the fact that as n increases, only higher values of x are in the sample. The first effect creates a tendency toward the FOSD relation derived in Theorem 1 for CV models, while the second effect works in the opposite direction. This leaves the effect on $F^A_{v,n}(v)$ of an exogenous change in n ambiguous in a CV model. However, we can obtain unambiguous predictions under both the PV and CV hypotheses by exploiting the following result.

Lemma 2 $F_{v,n}\left(v^*\left(r,n\right)\right)$ *is identified.*

Proof: Let $\tilde{F}_{x,n}(\cdot)$ denote the joint distribution of signals $X_1,\ldots,X_n$ in an n-bidder auction. Nondegeneracy and exchangeability imply

$$F_{v,n}\left(v^*(r,n)\right) = F_x(x^*(r,n)) = \tilde{F}_{x,n}\left(x^*(r,n), \infty, \ldots, \infty\right). \tag{19}$$

Observe that

$$\Pr(A=0|N=n) = \tilde{F}_{x,n}\left(x^*(r,n), x^*(r,n), \ldots, x^*(r,n)\right)$$

while

$$\Pr\left(0 < A < n|N=n\right) = n\left[\tilde{F}_{x,n}\left(x^*(r,n), \infty, \ldots, \infty\right) - \tilde{F}_{x,n}\left(x^*(r,n), x^*(r,n), \ldots, x^*(r,n)\right)\right].$$

Hence, using (19),

$$F_{v,n}\left(v^*(r,n)\right) = \frac{\Pr\left(0 < A < n|N=n\right)}{n} + \Pr(A=0|N=n). \tag{20}$$

□

With $F_{v,n}\left(v^*(r,n)\right)$ known for each n, we can then reconstruct $F_{v,n}(v)$ for all $v \geq v^*(r,n)$. In particular, from (18) we have

$$F_{v,n}(v) = \left[1 - F_{v,n}\left(v^*\left(r,n\right)\right)\right] F^A_{v,n}(v) + F_{v,n}\left(v^*\left(r,n\right)\right) \quad \forall v \geq v^*(r,n). \tag{21}$$

Theorem 5 *$F_{v,n}(v)$ is identified for all $v \geq v^*(r,n)$.*

Proof: Noting that with a binding reserve price

$$G_n(b|b) = \Pr(A=1|B_i=b, N=n) + \sum_{j=2}^{n} \Pr(A=j, \max_{k\in\{1,2,\ldots,j\}\backslash i} B_k \leq b | B_i = b, N=n)$$

the observables and the first-order condition (4) uniquely determine $v(s_n^{-1}(b), s_n^{-1}(b), n)$ for all n and $b \geq s_n(x^*(r,n))$. This determines the distribution $F^A_{v,n}(\cdot)$. Lemma 2 and (21) then give the result. □

Testable implications of the PV and CV models for the distribution $F_{v,n}(v)$ were established in Corollary 1, and estimation is easily adapted from that for the baseline case using sample analogs of the distributions in the identification results above. However, note that we cannot compare the distributions $F_{v,n}(v)$ in their (truncated) left tails, but rather only on regions of common support of the distributions $F^A_{v,n}(\cdot)$. In particular, since $x^*(r,n)$ is nondecreasing in n we can perform a test (using the estimation and testing approaches developed above) of[21]

$$H_0 : F_{v,\underline{n}}(v) = F_{v,3}(v) = \cdots = F_{v,\bar{n}}(v) \quad \forall v \geq v^*(r,\bar{n}) \tag{22}$$

against

$$H_1 : F_{v,\underline{n}}(v) < F_{v,3}(v) < \cdots < F_{v,\bar{n}}(v) \quad \forall v \geq v^*(r,\bar{n}) \tag{23}$$

which are implied by (8) and (9), respectively.[22] While this provides an approach for consistent testing, the fact that we must restrict the region of comparison could be a limitation of this approach in finite samples, particularly if the true model is one in which the effects

[21] $v^*(r,n)$ is just the lowest value of $v(x,x,n)$ for an actual bidder and is therefore easily estimated from the pseudovalues.

[22] We have assumed here that r is fixed across auctions. This is not necessary. Indeed, as GPV have suggested, variation in r can enable one to trace out more of the distribution $F_{v,n}(\cdot)$ by extending methods from the statistics literature on random truncation. A full development of this extension, however, is a topic unto itself and not pursued here.

of the winner's curse are most pronounced for bidders with signals in the left tail of the distribution. However, note that a significant difference between $v^*(r, \bar{n})$ and $v^*(r, n)$ for $n < n^*$ (the reason a test of (22) vs. (23) would involve a substantially restricted support) is *itself* evidence inconsistent with the PV hypothesis but implied by the CV hypothesis. Hence, a complementary testing approach is available based on the following theorem.

Theorem 6 *Under the PV hypothesis, $F_{v,n}(v^*(r,n))$ is identical for all n. Under the CV hypothesis, $F_{v,n}(v^*(r,n))$ is strictly increasing in n.*

Proof: Since $F_{v,n}(v^*(r,n)) = F(x^*(r,n))$, the result follows from Lemma 1. □

Consistent estimation of $F_{v,n}(v^*(r,n))$ for each n is easily accomplished with sample analogs of the probabilities on the right-hand side of (20).[23] A multivariate one-sided hypothesis test similar to those developed above could then be applied.

5.2 Costly Participation

Endogenous participation also arises when it is costly for bidders to participate. In some applications preparing a bid may be time consuming. In others, learning the signal X_i might require estimating costs based on detailed contract specifications, soliciting bids from subcontractors for a construction project, or analyzing seismic surveys of offshore oil tracts. Because bidders must recover these costs on average, for N large enough it is not an equilibrium for all bidders to participate in the auction, even if bidders are certain to place a value on the good strictly above the reserve price (if any). We consider two standard models of costly participation from the theoretical literature.

5.2.1 Bid Preparation Costs

Samuelson (1985) studied a model in which bidders first observe their signals and the reserve price r, then decide whether to incur a cost c of preparing a bid. Samuelson considered only the independent private values model; however, for our purposes this model of costly participation is equivalent to one in which the seller charges a participation fee c (a bidder's participation decision and first-order condition are the same regardless of whether the fee is paid to the seller, to an outside party, or simply "burned"). The case of a participation fee paid to the seller has been treated by Milgrom and Weber (1982) for the general affiliated values model.[24]

[23] The definitions of $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ would require the obvious modifications to account for the fact that only a bids, not n, are observed in each auction with n potential bidders.

[24] We will assume that their "regular case" (pp. 1112–1113) obtains.

Given r and n, participation is again determined by the realization of signals and a screening level

$$x^*(r,c,n) = \inf\left\{x : \int_{-\infty}^{x} [v(x,y,n) - r]\, dF_n(y|x)] \geq c\right\}.$$

Unlike the model in the preceding section, here the screening level $x^*(r,c,n)$ varies with n even with private values, since $F_n(x|x)$ varies with n. However, a valid testing approach can nonetheless be developed in a manner nearly identical to that above. In particular, the argument used to prove Lemma 2 also implies the following result.

Lemma 3 *With a reserve price r and bid preparation cost c, $F_{v,n}(v(x^*(r,c,n), x^*(r,c,n), n))$ is identified from observation of A and N.*

Letting $v^*(r,n)$ now denote $v(x^*(r,c,n), x^*(r,c,n), n)$, the first-order condition (4) and equation (21) can then be used to construct $F_{v,n}(v)$ for all $v \geq x^*(r,c,\bar{n})$, enabling testing of the hypotheses (22) vs. (23) as in the preceding section.

5.2.2 Signal Acquisition Costs

A somewhat different model is considered by Levin and Smith (1994).[25] In that model, each bidder chooses whether to incur cost c in order to learn (or to process) his signal X_i and submit a bid. Bidders know N and observe the number of actual bidders before they bid. Levin and Smith derive a symmetric mixed strategy equilibrium in which each potential bidder's participation decision is a binomial randomization. With no reserve price, this leads to exogenous variation in A. Because A is observed by bidders prior to bidding in their model, our analysis for the case of exogenous variation in N then carries through directly, substituting A for N.[26]

5.3 Unobserved Heterogeneity

The last model of endogenous participation we consider is the most challenging empirically. Here participation is determined in part by unobserved factors that also affect the distribution of bidders' valuations and signals. Intuitively, if auctions with large numbers

[25] Li (2002) has considered parametric estimation of the symmetric IPV model for first-price auctions under this entry model.

[26] If, in addition, there were a binding reserve price, only bidders who paid the signal acquisition cost *and* observed sufficiently high signals would participate. The mixed strategies determining signal acquisition, however, still result in exogenous variation in the number of "informed bidders," I, a subset A of whom would obtain sufficiently high signals to bid. In this case testing would be possible following the approach in the preceding section, but with I replacing N.

of bidders tend to be those in which the good is known by bidders to be of particularly high (or low) value, tests based on an assumption that variation in participation is exogenous can give misleading results. In general, unobserved heterogeneity introduces serious challenges to the nonparametric identification of the first-price auction model that underlies our approach (recall footnote 20).[27] However, in some cases this problem can be overcome ifinstrumental variables are available.

Suppose that the number of actual bidders at each auction is determined by two scalar factors, Z and W. Bidders observe both, but the researcher observes only Z. While we will refer to Z as the instrument, it may in fact be a function (known or estimated) of a vector of instruments $\mathbf{I}$. W summarizes the effects of unobservables on participation and may be correlated with bidders' valuations. We make the following assumptions:

Assumption 6 *Z is independent of* $(U_1, \ldots, U_{\bar{n}}, X_1, \ldots, X_{\bar{n}}, W)$.

Assumption 7 $A = \phi(Z, W)$*, with* ϕ *nondecreasing in* Z *and strictly increasing in* W.

Assumption 6 allows the possibility that W is correlated with $(U_1, \ldots, U_{\bar{n}}, X_1, \ldots, X_{\bar{n}})$, but requires that the instrument Z is not. In Assumption 7, monotonicity of ϕ in Z is the requirement that the instrument be positively correlated with the endogenous variable A. Weak monotonicity of $\phi(\cdot)$ in the unobservable W would be only a normalization. The strict monotonicity assumed here is a restriction that requires W to be discrete. The important implication is that conditioning on (A, Z) is then equivalent to conditioning on (A, Z, W).[28] More precisely, letting $\tilde{F}_a(\cdot)$ denote the joint distribution of $U_1, \ldots, U_a, X_1, \ldots, X_a$, etc.,

$$\begin{aligned}&\tilde{F}_a\left(U_1, \ldots, U_a, X_1, \ldots, X_a | A = \phi(z, w) = a, Z = z, W = w\right) = \\ &\tilde{F}_a\left(U_1, \ldots, U_a, X_1, \ldots, X_a | A = a, W = w\right).\end{aligned}$$

While Z might just be the number of potential bidders N, it need not be. For example, one structure consistent with Assumptions 7 and 6 is the model

$$A = \eta(\mathbf{I}) + W$$

where $Z = \eta(\mathbf{I})$ is a function (possibly unknown) of a vector of instruments $\mathbf{I}$, and W is independent of $\mathbf{I}$.

[27]Krasnokutskaya (2003) has recently shown that methods from the literature on measurement error can be used to enable estimation of a particular private values model in which unobserved heterogeneity enters multiplicatively (or additively) and is independent of the idiosyncratic components (themselves independently distributed) of bidders' values.

[28]If the relationship between A and W were only weakly monotone, conditioning on (A, Z) would be equivalent to conditioning on (A, Z) and the event $W \in \mathcal{W}$ for some set $\mathcal{W}$. In some applications this may be sufficient to enable the use of the first-order condition (4) as a useful approximation.

The following example illustrates the problem that this type of model can create for our basic testing approach.

Example. Consider the simple linear model

$$\begin{aligned} U_i &= U + \epsilon_i^1 \\ X_i &= U_i + \epsilon_i^2 \\ A &= \phi(Z, W) \end{aligned}$$

where $\phi(\cdot)$ is strictly increasing in both W and Z, which are discrete random variables; the disturbance terms ϵ_i^1 and ϵ_i^2 are mean zero and independent of Z and U; W and ϵ_i^2 are independent; but ϵ_i^1 and W are correlated. Assuming U is not degenerate, one obtains a PV model if ϵ_i^2 is degenerate and a CV model otherwise. Letting $v(X_i, X_i; a) = E[U_i|X_i, \max_{j\neq i} X_j = X_i, A = a]$, under the PV hypothesis,

$$\begin{aligned} Pr(v(X_i, X_i; a) \leq v) &= Pr(X_i \leq v|A = a) \\ &= Pr(U + \epsilon_i^1 \leq v|\phi(Z, W) = a) \\ &\neq Pr(U + \epsilon_i^1 \leq v|\phi(Z, W) = a + 1) \end{aligned}$$

where the inequality follows from the correlation of ϵ_i^1 and W. Hence, under the PV null the distribution of $v(X_i, X_i; a)$ varies with a. Likewise, under the CV alternative, the stochastic dominance relation of Corollary 1 need not hold. $\diamond$

To see how this problem can be overcome, first define

$$v(x, x; a, z) \equiv E\left[U_i|X_i = \max_{j\neq i} X_j = x, \phi(Z, W) = a, Z = z\right]. \tag{24}$$

We can consistently estimate each $v(x_i, x_i; a, z)$ by first conditioning on a and z to construct estimates of

$$G_{a,z}(b^*|b) = \Pr(\max_{j\neq i} B_j \leq b^*|B_i = b, A = a, Z = z)$$

and the corresponding conditional density $g_{a,z}(b^*|b)$ in order to exploit the first-order condition (the analog of (4))

$$v(x_i, x_i; a, z) = b_i + \frac{G_{a,z}(b_i|b_i)}{g_{a,z}(b_i|b_i)}. \tag{25}$$

As before, this first-order condition enables recovery of estimates of each $v(x_i, x_i; a, z)$

from the observed bids. Now, observe that

$$\begin{aligned}\Pr\left(v\left(X,X;A,z\right)\leq v\right) &= \Pr\left(v\left(X,X;\phi(z,W),z\right)\leq v\right)\\ &= \Pr\left(E\left[U_1|X_1=\max_{j\in\{2,\ldots,\phi(z,W)\}}X_j=X,W,Z=z\right]\leq v\right)\\ &= \Pr\left(E\left[U_1|X_1=\max_{j\in\{2,\ldots,\phi(z,W)\}}X_j=X,W\right]\leq v\right)\end{aligned}$$

where the final equality follows from Assumption 6.

Assumption 7 and the proof of Theorem 1 imply that the last expression above is strictly increasing in z in a CV auction (for any W, X) but invariant to z in a PV auction. Hence our testing approaches are still applicable if we rely on exogenous variation in the instrument Z rather than exogenous variation in N or A. In particular, after estimating pseudo-values using equation (25), one can pool pseudo-values across all values of a while holding z fixed to then compare the empirical distributions of the pseudo-values across values of z. We emphasize that while the *comparison* of distributions of pseudo-values forming the test for common values is done pooling over a, the *estimation* of pseudo-values must be done conditioning on both a and z.

6 Observable Heterogeneity

While we assumed above that data were available from auctions of identical goods, in practice this is rarely the case. In our application below, as in many others, we observe auction-specific characteristics that are likely to shift the distribution of bidder valuations. The results above can be extended to incorporate observables using standard nonparametric techniques. Let $\mathbf{Y}$ be a vector of observables and define $G_{n,\mathbf{y}}(b|b) = \Pr(\max_{j\neq i} B_j \leq b|B_i = b, N = n, \mathbf{Y} = \mathbf{y})$ etc. One simply substitutes $\frac{G_{n,\mathbf{y}}(b|b)}{g_{n,\mathbf{y}}(b|b)}$ for $\frac{G_n(b|b)}{g_n(b|b)}$ on the right-hand side of the first-order condition (4) and

$$v(x,x,n,\mathbf{y}) \equiv E[U_i|X_i = \max_{j\neq i} X_j = x, \mathbf{Y} = \mathbf{y}]$$

on the left-hand side. Standard smoothing techniques can be used to estimate $\frac{G_{n,\mathbf{y}}(b|b)}{g_{n,\mathbf{y}}(b|b)}$.

With many covariates, however, estimation will require large data sets. An alternative that may be more useful in many applications can be applied if we assume

$$v(x,x,n,\mathbf{y}) = v(x,x,n) + \Lambda(\mathbf{y}) \tag{26}$$

with $\mathbf{Y}$ independent of $X_1, \ldots, X_n$. This additively separable structure is particularly useful because it is preserved by equilibrium bidding.[29]

Lemma 4 *Suppose that* $\mathbf{Y}$ *is independent of* $\mathbf{X}$ *and (26) holds. Then the equilibrium bid function, conditional on* $\mathbf{Y} = \mathbf{y}$*, has the additively separable form* $s(x; n, \mathbf{y}) = s(x; n) + \Lambda(\mathbf{y})$*.*

The proof follows the standard derivation of the equilibrium bid function for a first-price auction (only the boundary condition for the differential equation (1) changes) and is therefore omitted. An important implication of this result is that we can account for observable heterogeneity in a two-stage procedure that avoids the need to condition on (smooth over) $\mathbf{Y}$ when estimating distributions and densities of bids. First note that, letting

$$s_0(n) = E_x[s(x; n)]$$

and

$$\Lambda_0 = E_{\mathbf{y}}[\Lambda(\mathbf{y})]$$

we can write the equilibrium bidding strategy as

$$s(x; n, \mathbf{y}) = s_0(n) + \Lambda_0 + \Lambda_1(\mathbf{y}) + s_1(x; n)$$

where $s_1(x; n)$ has mean zero conditional on $(n, \mathbf{y})$. Now observe that

$$\beta_{it} \equiv s_0(n_t) + \Lambda_0 + s_1(x_{it}; n_t) \tag{27}$$

is the bid that bidder i *would have submitted* in equilibrium in a generic (i.e., $\Lambda_1(\mathbf{y}) = 0$) n-bidder auction. Our tests can then be applied using the "homogenized" bids constructed using estimates of (27).

To implement this approach, in the first stage we regress all observed bids on the covariates $\mathbf{Y}$ and a set of dummy variables for each value of n. The sum of each residual and the corresponding intercept estimate provides an estimate of each β_{it}. In the second stage, these estimates are treated as bids in a sample of auctions of homogeneous goods. Note that the function $\Lambda_1(\cdot)$ is estimated in the first stage regression using all bids in the sample rather than separately for each value of n. This can make it possible to incorporate a large set of covariates and can make a flexible (or even nonparametric) specification of $\Lambda_1(\cdot)$ feasible.

Adapting this approach to the models of endogenous participation discussed above is straightforward. The case requiring modification is that in which instrumental variables are

[29]If the covariates enter multiplicatively rather than additively, an analogous approach to that proposed below can be applied.

used. There the intercept of the equilibrium bid functions $s(x; a, \mathbf{y}, w)$ will now vary with both a and w (or, equivalently, with both a and z). Under the assumptions of section 5.3, one needs only to include separate intercepts $s_0(a, z) + \Lambda_0$ (replacing $s_0(n) + \Lambda_0$ above) for each combination of a and z in the first stage. One then treats the sum of the (a, z)-specific intercept and the residuals from the corresponding auctions as the homogenized bids in the second stage.

Finally, note that the asymptotic properties of the ultimate test statistic are not affected by the first stage as long as the first-stage estimates converge at a faster rate than the pseudo-value estimates, as is guaranteed if the first stage is parametric.

7 Application to U.S. Forest Service Timber Auctions

7.1 Data and Background

We apply our tests to timber auctions run by the United States Forest Service (USFS). In each sale, a contract for timber harvesting on federal land was sold by first-price sealed bid auction.[30] Detailed descriptions of the auctions can be found in Baldwin (1995), Baldwin, Marshall, and Richard (1997), Athey and Levin (2001), Haile (2001), or Haile and Tamer (2003). We discuss only a few key features that are particularly relevant to our analysis.

Despite considerable attention to these auctions in the literature, there has been disagreement about whether they should be viewed as common or private value auctions. There are, in fact, two very different types of Forest Service auctions, for which the significance of common value elements may be different.

The first type of auction is known as a *lumpsum* sale. As the term suggests, here bidders submit a total bid for the entire volume of timber on the tract. The winning bidder pays his bid regardless of the volume actually realized at the time of harvest. Bidders, therefore, may face considerable common uncertainty over the volume of timber on the tract, since this can only be estimated *ex ante*. More significant, bidders often conduct their own "cruises" of tract before the auction to form their own estimates. Of course, private cruises may provide information about common or private value features of the tract. Furthermore, before each sale, the Forest Service conducts its own cruise of the tract to provide bidders with estimates of (among other things) timber volumes by species, harvesting costs, costs of manufacturing end products from the timber, and selling prices of these end products. This creates a great deal of common knowledge information about the tract. Whether sufficient scope remains

[30] The forest service also conducts English auctions, although we do not consider these here.

for private information regarding features common to all bidders is uncertain, although our *a priori* belief was that lumpsum sales were likely possess common value elements.

The second type of auction is known as a "scaled sale." Here, bids are made on a per unit (thousand board-feet of timber) basis, with the winner selected based on these unit prices *ex ante* estimates of timber volumes. However, actual payments to the Forest Service are based on *actual* volumes, measured by a third party at the time of harvest. As a result, the importance of common uncertainty regarding tract values may be reduced. In fact, bidders are less likely to send their own cruisers to assess the tract value (National Resources Management Corporation (1997)). This may leave less scope for private information regarding any shared determinants of bidders' valuations and, therefore, less scope for common values. Bidders may, however, have private information of an idiosyncratic (PV) nature regarding their own sales and inventories of end products, contracts for future sales, or inventories of uncut timber from private timber sales. This has led several authors (e.g., Baldwin, Marshall, and Richard (1997), Haile (2001), Haile and Tamer (2003)) to assume a private values model for scaled sales.[31] However, this is not without controversy; Baldwin (1995) and Athey and Levin (2001) argue for a common values model even for scaled sales.[32]

We will separately consider lumpsum sales and scaled sales. With our formal tests, we hope to evaluate both the question of whether common value elements are present in these auctions, and the question of whether *a priori* intuition regarding this question is reliable.

The auctions in our samples took place between 1982 and 1990 in Forest Service regions 1 and 5. Region 1 covers Montana, eastern Washington, Northern Idaho, North Dakota, and northwestern South Dakota. The Region 5 data consist of sales in California. The restriction to sales after 1981 is made due to policy changes in 1981 that (among other things) reduced the significance of subcontracting as a factor affecting bidder valuations, since resale opportunities can alter bidding in ways that confound the empirical implications of the winner's curse (cf. Haile (2001), Bikhchandani and Huang (1989), and Haile (1999)). For the same reason, we restrict attention to sales with no more than 12 months between the auction and the harvest deadline.[33] For consistency, we consider only sales in which the Forest Service provided *ex ante* estimates of the tract values using the predominant method

[31]Other studies assuming private values at timber auctions (USFS and others) include Cummins (1994), Elyakime, Laffont, Loisel, and Vuong (1994), Hansen (1985), Hansen (1986), Johnson (1979), Paarsch (1991), and Paarsch (1997).

[32]Other studies assuming common values models for Forest Service timber auctions include Chatterjee and Harrison (1988), Lederer (1994), and Leffler, Rucker, and Munn (1994).

[33]This is the same rule used by Haile and Tamer (2003) and the opposite of that used by Haile (2001) to focus on sales with significant resale opportunities.

of this time period, known as the "residual value method."[34] We also exclude salvage sales, sales set aside for small firms, and sales of contracts requiring the winner to construct roads.

Table 4 describes the resulting sample sizes for auctions with each number of bidders $n = 2, 3, \ldots, 12$. There are fairly few auctions with more than 4 bidders, particularly in the sample of lumpsum sales. However, the unit of observation, both for estimation of the pseudo-values and estimation of the distribution of pseudo-values, is a bid. Our data set contains 75 or more bids for auctions of up to seven bidders in both samples.

Our data set includes all bids[35] for each auction, as well as a large number of auction-specific observables. These include the year of the sale, the appraised value of the tract, the acreage of the tract, the length (in months) of the contract term, the volume of timber sold by the USFS in the same region over the previous six months, and USFS estimates of the volume of timber on the tract, harvesting costs, costs of manufacturing end products, selling value of the end products, and an index of the concentration of the timber volume across species.[36] All dollar values are in constant 1983 dollars per thousand board-feet of timber. Table 5 provides summary statistics.

7.2 Results

We first perform the tests on each sample under the assumption of exogenous participation. We consider comparisons of auctions with up to 7 bidders, although we look at ranges of 2–3, 2–4, 2–5, and 2–6 bidders as well. We use the method described in section 6 to eliminate the effects of observable heterogeneity with a first-stage regression of bids on the covariates listed above. Figures 2 and 3 show the estimated distributions of pseudo-values for each of these comparisons. The distributions compared appear to be roughly similar, although there is certainly some variation. Table 6 reports the formal test results. For each specification we report the R^2 from the first-stage regression of bids on auction covariates, the means of each estimated distribution of pseudovalues, and the p-value associated with each test of the private values null hypothesis. The fit of the first-stage bid regressions are generally very good (recall that bids are already normalized by the size of the tract). Both the means test and the K-S test fail to reject the null hypothesis of private values at

[34] See Baldwin, Marshall, and Richard (1997) for details.

[35] In practice separate prices are bid for each identified species on the tract. Following, e.g., Baldwin, Marshall, and Richard (1997), Haile (2001), and Haile and Tamer (2003), we consider only the total bid of each bidder, which is also the statistic used to determine the auction winner. See Athey and Levin (2001) for an analysis of the distribution of bids across species.

[36] This concentration index is equal to the sum of the squared shares of each species on the tract. Because sawmills typically are highly specialized, a tract consisting primarily of a single species may be more valuable than another with the same volume spread over many species.

standard levels in any specification.

One possible reason for a failure to reject the null is the presence of unobserved heterogeneity correlated with the number of bidders.[37] If tracts of higher value in unobserved dimensions also attracted more bidders, for example, there would be a tendency for the distributions compared to shift in the direction opposite that predicted by the winner's curse, and there is some suggestion of this in the graphs. Hence, we also perform the test using the model of endogenous participation with unobserved heterogeneity discussed in section 5.3.[38] As instruments, $\mathbf{I}$, we use the numbers of sawmills and logging firms in the county of each sale or neighboring counties (cf. Haile (2001)). This approach adds a second least-squares projection to construct $Z = \eta(\mathbf{I}) = E[N|\mathbf{I}]$. For the comparison of pseudo-value distributions, we split the sample into thirds (halves when we compare only 2- and 3-bidder auctions) based on the number of predicted bidders. Figures 4 and 5 show the resulting empirical distributions of pseudo-values compared in each test. For the scaled sales, the distributions are generally close and exhibit no clear ordering. For the lumpsum sales the distributions also appear to be fairly similar, although most comparisons suggest the stochastic ordering predicted by a CV model. The formal testing results are given in Table 7. THe means tests again fail to reject the PV model in any specification. In one of the scaled sale specifications ($n = 2 - -4$) the K-S test would reject at the 10 percent level (p-value .088). In two of the lumpsum sale specifications ($n = 2 - -4 and 2 - -7$) the K-S test would reject at the 10 percent level or better (p-values of .048 and .080).

As a specification check, we have examined the relationship between the estimated pseudo-values and the associated bids. Under the maintained assumptions of equilibrium bidding in the Milgrom-Weber model, this relation must be strictly monotone. While testing this restriction has been suggested by GPV and LPV, we are not aware of any formal testing approach that is directly applicable (cf. GPV). However, here this does not appear to be essential in our case; the importance of a formal test is in giving the appropriate allowance for deviations from strict monotonicity that would arise from sampling error. In most cases we have no deviations from strict monotonicity, so that no formal test could reject. In particular, we have examined the relation between bids and estimated pseudovalues in each subset of the data used to estimate the pseudovalues. For the case in which no instrumental variables are used (so that the samples are divided based on the value of n) we

[37] Haile (2001) provides some evidence using a different set of USFS auactions.

[38] We continue to assume the absence of a binding reserve price. See, e.g., Mead, Schniepp, and Watson (1981), Baldwin, Marshall, and Richard (1997), Haile (2001), Haile and Tamer (2003) for arguments that Forest Service reserve prices are nonbinding, explanations for why this might be the case, and supporting evidence.

find violations only in the case of lumpsum sales with $n = 6$, and even here only in the right tail. When instrumental variables are used, the sample is split based on the value of both n and the instrument, leading to smaller samples and greater sampling error. Nonetheless, even here there are only a few violations. For scaled sales, violations occur at no more than 2 points (i.e., 2 bids) per subsample, and the magnitudes of the violations are extremely small—on the order of 0.03 to 0.3 percent of the pseudovalues themselves. The handful of noticeable violations for lumpsum sales again occur only when auctions with $n = 6$ are examined.

While the failure to find evidence of common values in the scaled sales is consistent with *a priori* arguments for private values offered in the literature, the very limited evidence against the PV hypothesis for the lumpsum sales may be more surprising. Of course, the estimates published following the Forest Service cruise may be sufficiently precise that they leave little role for private information of a common value nature.[39] In fact, the cruises performed by the Forest Service for lumpsum sales are more thorough than those for scaled sales, a fact reflected in the name "tree measurement sale" given to such sales by the Forest Service. Hence, the intuitive argument for common values at the lumpsum sales might simply be misleading. It is, of course, a desire to avoid relying on intuition alone that led us to pursue a formal testing approach in the first place.

Nonetheless, we interpret the results with some caution. While we have allowed a rich class of models in our underlying framework, we have maintained the assumption of equilibrium competitive bidding in a static game, ruling out collusion and dynamic factors that might influence bidding decisions. While an examination of the monotonicity restriction these assumptions imply provides some comfort, a test of monotonicity cannot detect all violations of these assumptions. Even if these assumptions are satisfied, our econometric techniques for dealing with endogenous participation and auction heterogeneity have required additional assumptions and finite sample approximations that may cloud the analysis. Finally, while our tests are consistent, it could well be that the effect of the winner's curse in these auctions is sufficiently small that it is very difficult to detect with the moderate sample sizes available—an interpretation given some support by a graphical comparison of the estimated distributions. While this this would not rule out the presence of a common value element altogether, it would suggest that any CV elements are small relative to other sources of variation in the data.

[39] The fact that bidders conduct their own tract cruises does not contradict this, since the information obtained from such cruises could relate primarily to firm-specific (private value) factors.

8 Conclusion and Extensions

We have developed nonparametric tests for common values in first-price sealed-bid auctions. The tests are nonparametric, require observation only of bids, and are consistent against all common values alternatives within Milgrom and Weber's (1982) general framework. The tests perform well in Monte Carlo simulations and and can be adapted to incorporate auction-specific covariates and a range of models of endogenous participation. In addition to providing an approach for formal testing, comparing distributions of pseudo-values obtained from auctions with different numbers of bidders provides one natural way for quantifying the *magnitude* of any deviation from a private values model. An application to USFS timber auctions finds almost no significant evidence of common values, even for auctions in which *a priori* arguments might sugggest a CV model. Limitations of our tests are their reliance on a maintained assumption of equilibrium bidding, and the absence of dynamics in our underlying framework.

One further limitation of the approach as we have described it above is an assumption of symmetry. This is not necessary, however. It is possible to extend our methods to detect common value elements with asymmetric bidders (i.e., dropping the exchangeability assumption), as long as at least one bidder participates in auctions with different numbers of competitors. Two modifications of the basic approach above are required. The first is that we must focus on one bidder at a time rather than treating them symmetrically. In particular, consider, without loss of generality, bidder 1. A test for the presence of common values for bidder 1 can be based on (e.g.) the pseudo-value corresponding to $b^n_{1\tau}$, the empirical τ-th quantile of the bids submitted by bidder 1 in n-bidder auctions:

$$\hat{v}_{\tau,n,1} \equiv \left(b^n_{1\tau} + \frac{\hat{G}_1\left(b^n_{1\tau}\right)}{\hat{g}_1\left(b^n_{1\tau}\right)} \right)$$

where $\hat{G}_1(b)$ and $\hat{G}_{1j}(b)$ are nonparametric estimates analogous to those in equation (5) above, considering only the bids of bidder 1's opponents.

Under the PV hypothesis, the population analog, $v_{\tau,n,1}$, is constant across n for all $\tau \in (0,1)$. Under the CV alternative, in order to obtain the stochastic ordering[40]

$$v_{\tau,2,1} \geq v_{\tau,3,1} \geq \cdots \geq v_{\tau,\bar{n},1}.$$

Here we require the second modification: in considering auctions with $n = 2, 3, \ldots,$ we construct a sequence of sets of opponents faces by bidder 1, e.g., {bidder 2}, {bidder 2,

[40]We use a comparison of quantiles only for illustration. In practice, comparisons of means or of distribution functions are possible here as above.

bidder 3}, {bidder 2, bidder 3, bidder 4}, etc. This structure ensures that the severity of the winner's curse faced by bidder 1 is unambiguously greater in auctions with larger numbers of participants, even though opponents are not perfect substitutes for each other. While constructing such a sequence for n large would typically require a great deal of data, doing so for $n \in \{2,3\}$ (where the change in the severity of the winner's curse is typically largest) may be feasible in some applications.

Appendix

A Proof of Theorem 2

1. This is a standard result on the $\sqrt{T_n}$-convergence of sample to population quantiles (cf. van der Vaart (1999), Corollary 21.5).

2. For simplicity we introduce the notation $G_n \equiv G_n(b;b)$, $g_n \equiv g_n(b;b)$, $\hat{G}_n \equiv \hat{G}_n(b;b) = \frac{1}{nT_nh}\sum_{t=1}^{T_n}\sum_{i=1}^{n} 1\left(b_{it}^{*} < b\right) K\left(\frac{b-b_{it}}{h}\right)$ and $\hat{g}_n \equiv \hat{g}_n(b;b) = \frac{1}{nT_nh^2}\sum_{t=1}^{T_n}\sum_{i=1}^{n} K\left(\frac{b-b_{it}}{h}\right) K\left(\frac{b-b_{it}^{*}}{h}\right)$. Then we can use a standard first-order Taylor expansion to write

$$\hat{v}\left(s^{-1}(b), s^{-1}(b), n\right) - v\left(s^{-1}(b), s^{-1}(b), n\right) = \frac{\hat{G}_n}{\hat{g}_n} - \frac{G_n}{g_n}$$

$$= \frac{\hat{G}_n - G_n}{g_n} - \frac{G_n}{g_n^2}\left(\hat{g}_n - g_n\right) + o\left(\hat{G}_n - G_n\right) + o\left(\hat{g}_n - g_n\right)$$

$$= \frac{\hat{G}_n - E\hat{G}_n}{g_n} + \frac{E\hat{G}_n - G_n}{g_n} - \frac{G_n}{g_n^2}\left(\hat{g}_n - E\hat{g}_n\right) - \frac{G_n}{g_n^2}\left(E\hat{g}_n - g_n\right) + o\left(\hat{G}_n - G_n\right) + o\left(\hat{g}_n - g_n\right).$$

Standard bias calculation for kernel estimation shows that by Assumption 5, both

$$|E\hat{G}_n - G_n| \le |\int \left(G_n(b; b+uh) - G_n(b;b)\right) K(u)\, du| \le Ch^R \int |u|^R K(u)\, du = o\left(\frac{1}{\sqrt{Th^2}}\right)$$

and

$$|E\hat{g}_n - g_n| \le \left|\int\int \left(g_n(b+uh; b+vh) - g_n(b;b)\right) K(u) K(v)\, dudv\right| \le Ch^R = o\left(\frac{1}{\sqrt{Th^2}}\right).$$

Next it will be shown that

$$\sqrt{T_nh^2}\left(\hat{g}_n - E\hat{g}_n\right) \xrightarrow{d} N\left(0, \frac{1}{n}\left(\int\int K^2(x) K^2(y)\, dxdy\right) g_n(b;b)\right).$$

For this purpose it suffices to show that

$$\lim_{T_n\to\infty} Var\left(\sqrt{T_nh^2}\left(\hat{g}_n(b;b) - E\hat{g}_n(b;b)\right)\right) = \frac{1}{n}\left(\int\int K^2(x) K^2(y)\, dxdy\right) g_n(b;b).$$

This is verified by the following calculation:

$$
\begin{aligned}
&Var\left(\frac{1}{\sqrt{T_n h^2}\cdot n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\left[K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right)\right]\right)\\
=&T_n\left(\frac{1}{T_n n^2 h^2}Var\left(\sum_{i=1}^{n}\left[K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right)\right]\right)\right)\\
=&\frac{1}{nh^2}\left\{Var\left[K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right)\right]\right.\\
&\left.+\left(n-1\right)Cov\left[K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right),K\left(\frac{b_{jt}-b}{h}\right)K\left(\frac{b_{jt}^{*}-b}{h}\right)\right]\quad j\neq i.\right\}
\end{aligned}
$$

It is a standard result that

$$E\left(K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right)\right)=O\left(h^2\right)$$

and it can be verified that

$$E\left[K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right)K\left(\frac{b_{jt}-b}{h}\right)K\left(\frac{b_{jt}^{*}-b}{h}\right)\right]=O\left(h^4\right).$$

Therefore we can write

$$
\begin{aligned}
&Var\left(\sqrt{T_n h^2}\left(\hat{g}_n\left(b;b\right)-E\hat{g}_n\left(b;b\right)\right)\right)=\frac{1}{n\,h^2}E\left[K^2\left(\frac{b_{it}-b}{h}\right)K^2\left(\frac{b_{it}^{*}-b}{h}\right)\right]+O\left(h^4\right)\\
&=\frac{1}{n}\int\int\frac{1}{h^2}K^2\left(\frac{u-b}{h}\right)K^2\left(\frac{v-b}{h}\right)g_n\left(u,v\right)dudv+O\left(h^4\right)\\
&=\frac{1}{n}\left(\int\int K^2\left(x\right)K^2\left(y\right)dxdy\right)g_n\left(b;b\right)+o\left(1\right)
\end{aligned}
$$

where the last equality uses the substitutions $x=\left(u-b\right)/h$ and $y=\left(v-b\right)/h$. Finally the same type of variance calculation shows that

$$Var\left(\sqrt{T_n h^2}\left(\hat{G}_n-E\hat{G}_n\right)\right)\longrightarrow 0.$$

Hence the proof for part 2 is complete.

3. Since the sample quantiles of the bid distribution converge at rate $\sqrt{T_n}$ to the population quantile, which is faster than the convergence rate for the pseudo-values, for large T_n the sampling error in the τth quantile of the bid distribution does not affect the large sample properties of the estimated quantiles of the pseudo-value distribution. For each $\tau\in\{\tau_1,\ldots,\tau_l\}$:

$$\left(\hat{v}\left(s_n^{-1}\left(\hat{b}_{\tau_l,n}\right),s_n^{-1}\left(\hat{b}_{\tau_l,n}\right),n\right)-\hat{v}\left(x_\tau,x_\tau,n\right)\right)=O_p\left(\frac{1}{\sqrt{T_n}}\right)=o_p\left(\frac{1}{T_n h^2}\right). \tag{28}$$

A formal proof would proceed using uniform convergence of the kernel estimates of $G_n\left(b;b\right)$ and $g_n\left(b;b\right)$ and their derivatives, and stochastic equicontinuity arguments (see for example

Andrews (1994) and Pollard (1984)). We omit these rather tedious technical details and proceed by noting that (28) implies that the limiting distribution of

$$\sqrt{T_n h^2}\left(\hat{\xi}\left(\hat{b}_{\tau_l,n};n\right)-v\left(F_x^{-1}\left(\tau\right),F_x^{-1}\left(\tau\right),n\right)\right) \qquad \tau=\{\tau_1,\ldots,\tau_L\}$$

is the same as the limiting distribution of the vector

$$\sqrt{Th^2}\left(\hat{\xi}\left(s_n\left(x_\tau\right);n\right)-v\left(x_\tau,x_\tau,n\right)\right) \qquad \tau=\{\tau_1,\ldots,\tau_L\}$$

In part 2 we showed that each element of this vector is asymptotically normal with limit variance given by the diagonal element of Ω. It remains to show that the off-diagonal elements of Ω are 0. For this purpose it suffices to show, using the standard result that nonparametric kernel estimates at two distinct points (here, two quantiles $b_\tau \equiv s(x_\tau)$ and $b_{\tau'} \equiv s(x_{\tau'})$) are asymptotically independent; i.e.,

$$\lim_{T_n\to\infty} Cov\left(\sqrt{T_n h^2}\left(\hat{\xi}\left(b_\tau;n\right)-v\left(x_\tau,x_\tau,n\right)\right),\sqrt{T_n h^2}\left(\hat{\xi}\left(b_{\tau'};n\right)-v\left(x_{\tau'},x_{\tau'},n\right)\right)\right)=0.$$

Using the bias calculation and convergence rates derived in part 2, it suffices for this purpose to show that

$$\lim_{T_n\to\infty} Cov\left(\sqrt{T_n h^2}\left(\hat{g}_n\left(b_\tau;b_\tau\right)-Eg_n\left(b_\tau;b_\tau\right)\right),\sqrt{T_n h^2}\left(\hat{g}_n\left(b_{\tau'};b_{\tau'}\right)-Eg_n\left(b_{\tau'};b_{\tau'}\right)\right)\right)=0$$

To show this, first observe that the left-hand side can be written

$$\begin{aligned}
&Cov\left[\frac{1}{\sqrt{T_n h^2}n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b_{it}^*-b_\tau}{h}\right),\frac{1}{\sqrt{T_n h^2}n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}K\left(\frac{b_{it}-b_{\tau'}}{h}\right)K\left(\frac{b_{it}^*-b_{\tau'}}{h}\right)\right]\\
=&\frac{1}{n^2h^2}Cov\left[\sum_{i=1}^{n}K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b_{it}^*-b_\tau}{h}\right),\sum_{i=1}^{n}K\left(\frac{b_{it}-b_{\tau'}}{h}\right)K\left(\frac{b_{it}^*-b_{\tau'}}{h}\right)\right].
\end{aligned}$$

Using the fact that for each i,

$$E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b_{it}^*-b_\tau}{h}\right)\right]=O\left(h^2\right)$$

and for each $i\neq j$,

$$E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b_{it}^*-b_\tau}{h}\right)K\left(\frac{b_{jt}-b_{\tau'}}{h}\right)K\left(\frac{b_{jt}^*-b_{\tau'}}{h}\right)\right]=O\left(h^4\right)$$

we can further rewrite the covariance function as

$$\begin{aligned}
&\frac{1}{n^2h^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b_{it}^*-b_\tau}{h}\right)K\left(\frac{b_{jt}-b_{\tau'}}{h}\right)K\left(\frac{b_{jt}^*-b_{\tau'}}{h}\right)\right]+O\left(h^2\right)\\
=&\frac{1}{n^2h^2}\sum_{i=1}^{n}E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b_{it}^*-b_\tau}{h}\right)K\left(\frac{b_{it}-b_{\tau'}}{h}\right)K\left(\frac{b_{it}^*-b_{\tau'}}{h}\right)\right]+O\left(h^2\right)\\
=&\frac{1}{n}\int\int K\left(x\right)K\left(y\right)K\left(x+\frac{b_\tau-b_{\tau'}}{h}\right)K\left(y+\frac{b_\tau-b_{\tau'}}{h}\right)g_n\left(b_\tau+xh,b_\tau+yh\right)dxdy+O\left(h^2\right)\to 0.
\end{aligned}$$

B Proof of Theorem 3

Assumption 5 directly implies the following uniform rates of convergence for $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ (see Horowitz (1998) and Guerre, Perrigne, and Vuong (2000)).

$$\sup_{b\in\mathbb{R}}\left|\tilde{G}_n(b;b)\right| \equiv \sup_{b\in\mathbb{R}}\left|\hat{G}_n(b;b)-G_n(b;b)\right| = O_p\left(\sqrt{\frac{\log T}{Th}}\right)+O\left(h^R\right)$$
$$\sup_{b\in R}\left|\tilde{g}_n(b;b)\right| \equiv \sup_{b\in R}\left|\hat{g}_n(b;b)-g_n(b;b)\right| = O_p\left(\sqrt{\frac{\log T}{Th^2}}\right)+O\left(h^R\right).$$

Since part (i) is an immediate consequence of part (ii), we proceed to prove part (ii) directly. Letting $\xi(b;n)=v\left(s^{-1}(b),s^{-1}(b),n\right)$, we can decompose the left hand side of part (ii) as

$$\begin{aligned}&\sqrt{T_nh}\left(\hat{\mu}_{n,\tau}-E\left[\xi(b;n)\,1\left(b_{\tau,n}\le b\le b_{1-\tau,n}\right)\right]\right)\\ =&\sqrt{T_nh}\left(\frac{1}{T_nn}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\left(b_{it}+\frac{\hat{G}_n(b_{it};b_{it})}{\hat{g}_n(b_{it};b_{it})}\right)1\left(\hat{b}_{\tau,n}\le b_{it}\le\hat{b}_{1-\tau,n}\right)-E\left[\xi(b;n)\,1\left(b_{\tau,n}\le b\le b_{1-\tau,n}\right)\right]\right)\\ =&\hat{\mu}^1_{n,\tau}+\hat{\mu}^2_{n,\tau}+\hat{\mu}^3_{n,\tau}+\hat{\mu}^4_{n,\tau}\end{aligned}$$

where

$$\begin{aligned}\hat{\mu}^1_{n,\tau}=&\sqrt{T_nh}\left(\frac{1}{T_nn}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\left(\frac{\hat{G}_n(b_{it};b_{it})}{\hat{g}_n(b_{it};b_{it})}-\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right)\left(1\left(\hat{b}_{\tau,n}\le b_{it}\le\hat{b}_{1-\tau,n}\right)-1\left(b_{\tau,n}\le b_{it}\le b_{1-\tau,n}\right)\right)\right)\\ \hat{\mu}^2_{n,\tau}=&\sqrt{T_nh}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\left[\frac{\hat{G}_n(b_{it};b_{it})}{\hat{g}_n(b_{it};b_{it})}-\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right]1\left(b_{\tau,n}\le b_{it}\le b_{1-\tau,n}\right)\\ \hat{\mu}^3_{n,\tau}=&\sqrt{T_nh}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\left(b_{it}+\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right)\left(1\left(\hat{b}_{\tau,n}\le b_{it}\le\hat{b}_{1-\tau,n}\right)-1\left(b_{\tau,n}\le b_{it}\le b_{1-\tau,n}\right)\right)\\ =&\sqrt{T_nh}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\xi(b_{it};n)\left(1\left(\hat{b}_{\tau,n}\le b_{it}\le\hat{b}_{1-\tau,n}\right)-1\left(b_{\tau,n}\le b_{it}\le b_{1-\tau,n}\right)\right)\\ \hat{\mu}^4_{n,\tau}=&\sqrt{T_nh}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\left(\left(b_{it}+\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right)1\left(b_{\tau,n}\le b_{it}\le b_{1-\tau,n}\right)-E\left[\xi(b;n)\,1\left(b_{\tau,n}\le b\le b_{1-\tau,n}\right)\right]\right)\\ =&\sqrt{T_nh}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\left(\xi(b_{it};n)\,1\left(b_{\tau,n}\le b_{it}\le b_{1-\tau,n}\right)-E\left[\xi(b;n)\,1\left(b_{\tau,n}\le b\le b_{1-\tau,n}\right)\right]\right).\end{aligned}$$

We consider the properties of each of these terms in turn. For $\hat{\mu}^4_{n,\tau}$, a standard application of the law of large numbers gives

$$\hat{\mu}^4_{n,\tau}=\sqrt{h}\;O_p(1)=o_p(1).$$

The function in the summand of $\hat{\mu}^3_{n,\tau}$ satisfies stochastic equicontinuity conditions (a type I function of Andrews (1994)). Hence using the parametric convergence rates of $\hat{b}_\tau$ and $\hat{b}_{1-\tau}$,

$$\begin{aligned}\hat{\mu}^3_{n,\tau}=&\sqrt{T_nh}\left(E_b\xi(b;n)\,1\left(\hat{b}_{\tau,n}\le b\le\hat{b}_{1-\tau,n}\right)-E_b\left[\xi(b;n)\,1\left(b_{\tau,n}\le b\le b_{1-\tau,n}\right)\right]\right)+o_p(1)\\ =&C\sqrt{T_nh}\left(O\left(\hat{b}_{\tau,n}-b_{\tau,n}\right)+O\left(\hat{b}_{1-\tau,n}-b_{1-\tau,n}\right)\right)+o_p(1)=\sqrt{T_nh}O_p\left(\frac{1}{\sqrt{T_n}}\right)+o_p(1)=o_p(1).\end{aligned}$$

Similarly, the function in the summand of $\hat{\mu}^1_{n,\tau}$ also satisfies stochastic equicontinuity conditions (product of type I and type III functions in Andrews (1994)), and hence

$$\begin{aligned}\hat{\mu}^1_{n,\tau} =& \sqrt{T_n h} E_b \left(\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)} - \frac{G_n(b;b)}{g_n(b;b)} \right) \left(1\left(\hat{b}_\tau \leq b \leq \hat{b}_{1-\tau}\right) - 1\left(b_{\tau,n} \leq b \leq b_{1-\tau,n}\right)\right) + o_p(1) \\ =& O_p\left(\sup_{b\in[b_\tau-\epsilon, b_{1-\tau}+\epsilon]} \left| \frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)} - \frac{G_n(b;b)}{g_n(b;b)} \right| \right) \sqrt{T_n h} \left(O\left(\hat{b}_{\tau,n} - b_{\tau,n}\right) + O\left(\hat{b}_{1-\tau,n} - b_{1-\tau,n}\right)\right) + o_p(1) \\ =& o_p(1) \sqrt{T_n h} O_p\left(\frac{1}{\sqrt{T_n}}\right) + o_p(1) = o_p(1).\end{aligned}$$

Combining the above results of rates of convergence, we have thus far shown that

$$\sqrt{T_n h}\left(\hat{\mu}_{n,\tau} - E\left[\xi(b;n) 1\left(b_{\tau,n} \leq b \leq b_{1-\tau,n}\right)\right]\right) = \hat{\mu}^2_{n,\tau} + o_p(1).$$

The term $\hat{\mu}^2_{n,\tau}$ can be further decomposed using a second order Taylor expansion:

$$\hat{\mu}^2_{n,\tau} = \hat{\mu}^5_{n,\tau} + \hat{\mu}^6_{n,\tau} + \hat{\mu}^7_{n,\tau}$$

where

$$\begin{aligned}\hat{\mu}^5_{n,\tau} =& \sqrt{T_n h} \frac{1}{nT_n} \sum_{t=1}^{T_n} \sum_{i=1}^{n} \frac{1}{g_n(b_{it};b_{it})} \left(\hat{G}_n(b_{it};b_{it}) - G_n(b_{it};b_{it})\right) 1\left(b_{\tau,n} \leq b_{it} \leq b_{1-\tau,n}\right) \\ \hat{\mu}^6_{n,\tau} =& -\sqrt{T_n h} \frac{1}{nT_n} \sum_{t=1}^{T_n} \sum_{i=1}^{n} \frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2} \left(\hat{g}_n(b_{it};b_{it}) - g_n(b_{it};b_{it})\right) 1\left(b_{\tau,n} \leq b_{it} \leq b_{1-\tau,n}\right) \\ \hat{\mu}^7_{n,\tau} =& \sqrt{T_n h} \frac{1}{nT_n} \sum_{t=1}^{T_n} \sum_{i=1}^{n} h^1_n(b_{it}) \left(\hat{G}_n(b_{it};b_{it}) - G_n(b_{it};b_{it})\right)^2 1\left(b_{\tau,n} \leq b_{it} \leq b_{1-\tau,n}\right) \\ &+ \sqrt{T_n h} \frac{1}{nT_n} \sum_{t=1}^{T_n} \sum_{i=1}^{n} h^2_n(b_{it}) \left(\hat{g}_n(b_{it};b_{it}) - g_n(b_{it};b_{it})\right)^2 1\left(b_{\tau,n} \leq b_{it} \leq b_{1-\tau,n}\right).\end{aligned}$$

Here $h^1_n(\cdot)$ and $h^2_n(\cdot)$ are the second derivatives with respect to $G_n(\cdot)$ and $g_n(\cdot)$ evaluated at some mean values between $\hat{G}_n(\cdot)$ and $G_n(\cdot)$ and between $\hat{g}_n(\cdot)$ and $g_n(\cdot)$. We first bound $\hat{\mu}^7_{n,\tau}$ using the uniform convergence rates of $\hat{G}_n(\cdot)$ and $\hat{g}_n(\cdot)$:

$$\begin{aligned}\left|\mu^7_{n,\tau}\right| \leq& C\sqrt{T_n h}\left(O_p\left(\frac{\log T}{T_n h} + h^{2R}\right) + O_p\left(\frac{\log T}{T_n h^2} + h^{2R}\right)\right) \\ =& O_p\left(\frac{\log T}{\sqrt{T_n h}} + \frac{\log T}{\sqrt{T_n h^3}} + \sqrt{T_n h^{1+4R}}\right) = o_p(1).\end{aligned}$$

Now consider

$$\begin{aligned}
\hat{\mu}_{n,\tau}^{6} &= -\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\frac{G_n\left(b_{it};b_{it}\right)}{g_n\left(b_{it};b_{it}\right)^2}\left(\hat{g}_n\left(b_{it};b_{it}\right)-E\left[\hat{g}_n\left(b_{it};b_{it}\right)\right)1\left(b_{\tau,n}\leq b_{it}\leq b_{1-\tau,n}\right)\right] \\
&\quad -\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\frac{G_n\left(b_{it};b_{it}\right)}{g_n\left(b_{it};b_{it}\right)^2}\left(E\left[\hat{g}_n\left(b_{it};b_{it}\right)\right]-g_n\left(b_{it};b_{it}\right)\right)1\left(b_{\tau,n}\leq b_{it}\leq b_{1-\tau,n}\right) \\
&= -\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\frac{G_n\left(b_{it};b_{it}\right)}{g_n\left(b_{it};b_{it}\right)^2}\left(\hat{g}_n\left(b_{it};b_{it}\right)-E\left[\hat{g}_n\left(b_{it};b_{it}\right)\right)1\left(b_{\tau,n}\leq b_{it}\leq b_{1-\tau,n}\right)\right]+o_p\left(1\right) \\
&\equiv \hat{\mu}_{n,\tau}^{8}+o_p\left(1\right)
\end{aligned}$$

because by assumption the bias in the second term on the right-hand side of the first line is of the order

$$\sqrt{T_n h}O\left(h^R\right)=O\left(\sqrt{T_n h^{1+2R}}\right)=o\left(1\right).$$

Next we show that

$$\hat{\mu}_{n,\tau}^{8}\xrightarrow{d}N\left(0,\Omega=\left[\int\left(\int K\left(v\right)K\left(u+v\right)dv\right)^2du\right]\left[\frac{1}{n}\int_{F_b^{-1}(\tau)}^{F_b^{-1}(1-\tau)}\frac{G_n\left(b;b\right)^2}{g_n\left(b;b\right)^3}g_n\left(b\right)^2db\right]\right).$$

This follows from a limit variance calculation for U-statistics. We can write

$$\hat{\mu}_{n,\tau}^{8}=\sqrt{T_n h}\frac{1}{n^2T_n^2}\sum_{t=1}^{T_n}\sum_{s=1}^{T_n}m\left(w_t,w_s\right)$$

where

$$\begin{aligned}
m\left(w_t,w_s\right)=&\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{G_n\left(b_{it};b_{it}\right)}{g_n^2\left(b_{it};b_{it}\right)}\left[\frac{1}{h^2}K\left(\frac{b_{sj}-b_{ti}}{h}\right)K\left(\frac{b_{sj}^{*}-b_{ti}}{h}\right)\right. \\
&\left.-E\frac{1}{h^2}K\left(\frac{b_{sj}-b_{ti}}{h}\right)K\left(\frac{b_{sj}^{*}-b_{ti}}{h}\right)\right]1\left(b_{\tau,n}\leq b_{it}\leq b_{1-\tau,n}\right).
\end{aligned}$$

Using lemma 8.4 of Newey and McFadden (1994), we can verify that

$$\begin{aligned}
\sqrt{T_n h}\frac{E|m\left(w_t,w_t\right)|}{T_n}&=O_p\left(\sqrt{T_n h}\frac{1}{T_n h}\right)=O_p\left(\frac{1}{\sqrt{T_n h}}\right)=o_p\left(1\right),\quad\text{and} \\
\sqrt{T_n h}\frac{\sqrt{Em\left(w_t,w_s\right)^2}}{T_n}&=O_p\left(\sqrt{T_n h}\frac{1}{T_n\sqrt{h^3}}\right)=O_p\left(\frac{1}{\sqrt{T_n h^2}}\right)=o_p\left(1\right).
\end{aligned}$$

It then follows from Lemma 8.4 of Newey and McFadden (1994) that

$$\hat{\mu}_{n,\tau}^{8}=\sqrt{T_n h}\frac{1}{n^2T_n}\sum_{t=1}^{T_n}\left[E\left(m\left(w_t,w_s\right)|w_t\right)+E\left(m\left(w_t,w_s\right)|w_s\right)\right]+o_p\left(1\right).$$

The first term is asymptotically negligible, since

$$\sqrt{T_n h}\frac{1}{n^2 T_n}\sum_{t=1}^{T_n} E\left(m\left(w_t, w_s\right)|w_t\right)$$
$$=\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T_n}\left[g_n\left(b_{it}; b_{it}\right)1\left(b_{\tau,n}\leq b_{it}\leq b_{1-\tau,n}\right)-E\left[g_n\left(b_{it}; b_{it}\right)1\left(b_{\tau,n}\leq b_{it}\leq b_{1-\tau,n}\right)\right]+O\left(h^R\right)\right]$$
$$=\sqrt{T_n h}O_p\left(\frac{1}{\sqrt{T_n}}\right)+O\left(\sqrt{T_n h^{1+2R}}\right)=o_p\left(1\right).$$

It remains only to verify by straightforward though somewhat tedious calculation that

$$Var\left(\sqrt{T_n h}\frac{1}{n^2 T_n}\sum_{t=1}^{T_n} E\left(m\left(w_t, w_s\right)|w_s\right)\right)$$
$$=hVar\left(\frac{1}{n}\sum_{j=1}^{n}\int_{b_\tau}^{b_{1-\tau}}\frac{G_n\left(b;b\right)}{g_n^2\left(b;b\right)}\frac{1}{h^2}K\left(\frac{b_{sj}-b}{h}\right)K\left(\frac{b_{sj}^*-b}{h}\right)g_n\left(b\right)db\right)$$
$$=h\frac{1}{n}Var\left(\int_{b_\tau}^{b_{1-\tau}}\frac{G_n\left(b;b\right)}{g_n^2\left(b;b\right)}\frac{1}{h^2}K\left(\frac{b_{sj}-b}{h}\right)K\left(\frac{b_{sj}^*-b}{h}\right)g_n\left(b\right)db\right)+o\left(1\right)$$
$$=h\frac{1}{n}E\left(\int_{b_\tau}^{b_{1-\tau}}\frac{G_n\left(b;b\right)}{g_n^2\left(b;b\right)}\frac{1}{h^2}K\left(\frac{b_{sj}-b}{h}\right)K\left(\frac{b_{sj}^*-b}{h}\right)g_n\left(b\right)db\right)^2+o\left(1\right)$$
$$\longrightarrow\Omega\equiv\left[\int\left(\int K\left(v\right)K\left(u+v\right)dv\right)^2 du\right]\left[\frac{1}{n}\int_{G_n^{-1}(\tau)}^{G_n^{-1}(1-\tau)}\frac{G_n^2\left(b;b\right)}{g_n^3\left(b;b\right)}g_n^2\left(b\right)db\right].$$

Finally, we note that if we apply the calculations performed for $\hat{\mu}_{n,\tau}^6$ to $\hat{\mu}_{n,\tau}^5$, we see that

$$E\left[\hat{\mu}_{n,\tau}^5\right]=o\left(1\right)\qquad\text{and } Var\left(\hat{\mu}_{n,\tau}^5\right)=o\left(1\right)$$

which then implies that $\hat{\mu}_{n,\tau}^5\xrightarrow{p}0$. The proof is now completed by putting these terms together.□

References

Anderson, G. (1996): "Nonparametric Tests of Stochastic Dominance in Income Distributions," *Econometrica*, 64, 1183–1194.

Andrews, D. (1994): "Empirical Process Methods in Econometrics," in *Handbook of Econometrics, Vol. 4*, ed. by R. Engle, and D. McFadden. North Holland.

——— (1998): "Hypothesis Testing With a Restricted Parameter Space," *Journal of Econometrics*, 86, 155–199.

Athey, S., and P. Haile (2002): "Identification of Standard Auction Models," *Econometrica*, 70, 2107–2140.

Athey, S., and J. Levin (2001): "Information and Competition in U.S. Forest Serve Timber Auctions," *Journal of Political Economy*.

Ayer, M., D. Brunk, G. Ewing, W. Reid, and E. Silverman (1955): "An Empirical Distribution Function for Sampling with Incomplete Information," *Annals of Mathematical Statistics*, 26, 641–647.

Bajari, P., and A. Hortacsu (2003): "Winner's Curse, Reserve Prices, and Endogenous Entry: Empirical Insights from eBay Auctions," *RAND Journal of Economics*, 34, 329–355.

Baldwin, L. (1995): "Risk Aversion in Forest Service Timber Auctions," working paper, RAND Corporation.

Baldwin, L., R. Marshall, and J.-F. Richard (1997): "Bidder Collusion at Forest Service Timber Sales," *Journal of Political Economy*, 105, 657–699.

Barrett, G., and S. Donald (2003): "Consistent Tests for Stochastic Dominance," *Econometrica*, 71, 71–104.

Bartholomew, D. (1959): "A Test of Homogeneity for Ordered Alternatives," *Biometrika*, 46, 36–48.

Bikhchandani, S., P. Haile, and J. Riley (2002): "Symmetric Separating Equilibria in English Auctions," *Games and Economic Behavior*, 38, 19–27.

Bikhchandani, S., and C. Huang (1989): "Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets," *Review of Financial Studies*, 2, 311–339.

Chatterjee, K., and T. Harrison (1988): "The Value of Information in Competitive Bidding," *European Journal of Operational Research*, 36, 322–333.

Chen, X., and X. Shen (1998): "Sieve Extremum Estimates for Weakly Dependent Data," *Econometrica*, 66, 289–314.

Chernozhukov, V. (2002): "Inference on Quantile Regression Process, An Alternative," mimeo., MIT.

Cummins, J. (1994): "Investment Under Uncertainty: Estimates from Panel Data on Pacific Northwest Forest Products Firms," working paper, Columbia University.

Davidson, R., and J.-Y. Duclos (2000): "Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality," *Econometrica*, 68, 1435–1464.

Elyakime, B., J. Laffont, P. Loisel, and Q. Vuong (1994): "First-Price Sealed-Bid Auctions with Secret Reserve Prices," *Annales d'Economie et Statistiques*, 34, 115–141.

Engelbrecht-Wiggans, R., J. List, and D. Lucking-Reiley (1999): "Demand Reduction in Multi-unit Auctions with Varying Numbers of Bidders: Theory and Field Experiments," working paper, Vanderbilt University.

Gilley, O., and G. Karels (1981): "The Competitive Effect in Bonus Bidding: New Evidence," *Bell Journal of Economics*, 12, 637–648.

Guerre, E., I. Perrigne, and Q. Vuong (2000): "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 68, 525–74.

Haile, P. (1999): "Auctions with Resale," mimeo, University of Wisconsin-Madison.

——— (2001): "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales," *American Economic Review*, pp. 399–427.

Haile, P., and E. Tamer (2003): "Inference with an Incomplete Model of English Auctions," *Journal of Political Economy*, 111, 1–51.

Hajek, J., Z. Sidak, and P. Sen (1999): *Theory of Rank Tests*. Academic Press.

Hansen, R. (1985): "Empirical Testing of Auction Theory," *American Economic Review, Papers and Proceedings*, 75, 156–159.

——— (1986): "Sealed-Bid Versus Open Auctions: The Evidence," *Economic Inquiry*, 24, 125–143.

Hendricks, K., J. Pinkse, and R. Porter (2003): "Empirical Implications of Equilibrium Bidding in First-Price, Symmetric, Common-Value Auctions," *Review of Economic Studies*, 70, 115–146.

Hendricks, K., and R. Porter (1988): "An Empirical Study of an Auction with Asymmetric Information," *American Economic Review*, pp. 865–883.

Horowitz, J. (1998): "Bootstrap Methods for Median Regression Models," *Econometrica*, 66, 1327–1352.

Jofre-Bonet, M., and M. Pesendorfer (forthcoming): "Bidding Behavior in a Repeated Procurement Auction," *Econometrica*.

Johnson, R. (1979): "Oral Auction Versus Sealed Bids: An Empirical Investigation," *Natural Resources Journal*, 19, 315–335.

Klemperer, P. (1999): "Auction Theory: A Guide to the Literature," *Journal of Economic Surveys*, 13, 227–286.

Krasnokutskaya, E. (2003): "Identification and Estimation of Auction Models under Unobserved Auction Heterogeneity," working paper, Yale University.

Krishna, V. (2002): *Auction Theory*. Academic Press.

Laffont, J. J., and Q. Vuong (1993): "Structural Econometric Analysis of Descending Auctions," *European Economic Review*, 37, 329–341.

——— (1996): "Structural Analysis of Auction Data," *American Economic Review, Papers and Proceedings*, 86, 414–420.

Lavergne, P., and Q. Vuong (1996): "Nonparametric Selection of Regressors: the Nonnested Case," *Econometrica*, pp. 207–219.

Lederer, P. (1994): "Predicting the Winner's Curse," *Decision Sciences*, 25, 79–101.

Leffler, K., R. Rucker, and I. Munn (1994): "Transaction Costs and the Collection of Information: Presale Measurement on Private Timber Sales," working paper, University of Washington.

Levin, D., and J. Smith (1994): "Equilbrium in Auctions with Entry," *American Economic Review*, 84, 585–599.

Lewbel, A. (1998): "Semiparametric Latent Variable Model Estimation with Endogenous or Mismeasured Regressors," *Econometrica*, 66, 105–122.

Li, T. (2002): "Econometrics of First Price Auctions with Entry and Binding Reservation Prices," working paper, Indiana University.

Li, T., I. Perrigne, and Q. Vuong (2000): "Conditionally Independent Private Information in OCS Wildcat Auctions," *Journal of Econometrics*, 98, 129–161.

——— (2002): "Structural Estimation of the Affiliated Private Value Auction Model," *RAND Journal of Economics*, 33, 171–193.

Linton, O., E. Massoumi, and Y. Whang (2002): "Consistent Testing for Stochastic Dominance: A Subsampling Approach," mimeo., LSE.

McAfee, P., and J. McMillan (1987): *Incentives in Government Contracting*. University of Toronto Press.

McFadden, D. (1989): "Testing for Stochastic Dominance," in *Studies in the Economics of Uncertainty*, ed. by T. B. Fomby, and T. K. Seo. Springer-Verlag.

Mead, W., M. Schniepp, and R. Watson (1981): "The Effectiveness of Competition and Appraisals in the Auction Markets for National Forest Timber in the Pacific Northwest," Washington: U.S. Department of Agriculture, Forest Service.

Milgrom, P. (1981): "Good News and Bad News: Representation Theorems and Applications," *The Bell Journal of Economics*, 13, 380–391.

Milgrom, P., and R. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

National Resources Management Corporation (1997): "A Nationwide Study Comparing Tree Measurement and Scaled Sale Methods for Selling United States Forest Service Timber," Report to the U.S. Forest Service, Department of Agriculture.

Newey, W. (1994): "Kernel Estimation of Partial Means and a General Variance Estimator," *Econometric Theory*, 10, 233–253.

Newey, W., and D. McFadden (1994): "Large Sample Estimation and Hypothesis Testing," in *Handbook of Econometrics, Vol. 4*, ed. by R. Engle, and D. McFadden. North Holland.

Paarsch, H. (1991): "Empirical Models of Auctions and an Application to British Columbian Timber," University of Western Ontario, Department of Economics Technical Report 91-19.

——— (1992): "Deciding Between the Common and Private Value Paradigms in Empirical Models of Auctions," *Journal of Econometrics*, 51, 191–215.

——— (1997): "Deriving an Estimate of the Optimal Reserve Price: An Application to British Columbian Timber Sales," *Journal of Econometrics*, 78, 333–357.

Pinkse, J., and G. Tan (2002): "The Affiliation Effect in First-Price Auctions," mimeo, Penn State University.

Pollard, D. (1984): *Convergence of Stochastic Processes*. Springer Verlag.

Samuelson, W. F. (1985): "Competitive Bidding with Entry Costs," *Economics Letters*, 17, 53–57.

Sareen, S. (1999): "Posterior Odds Comparison of a Symmetric Low-Price, Sealed-Bid Auction Within the Common Value and the Independent Private Value Paradigms," *Journal of Applied Econometrics*, 14, 651–676.

Schneyerov, A. (2002): "Applying Auction Theory to Municipal Bond Auctions: Market Power and the Winner's Curse," mimeo., University of British Columbia.

van der Vaart, A. (1999): *Asymptotic Statistics*. Cambridge University Press.

Wolak, F. (1989): "Testing Inequality Constraints in Linear Econometric Models," *Journal of Econometrics*, 41, 205–235.

——— (1991): "The Local Nature of Hypothesis Tests Involving Inequality Constraints in Nonlinear Models," *Econometrica*, 59, 981–995.

Table 1: Monte Carlo Results
200 replications of each experiment.

	PV1		CV1		PV2		CV2	
Range of n:	2–4	2–5	2–4	2–5	2–4	3–5	3–5	3–6
T_n	200	200	200	200	200	200	200	200
share of p-values $< 10\%$	0.21	0.39	1.00	1.00	0.12	0.27	0.94	0.99
share of p-values $< 5\%$	0.11	0.29	1.00	1.00	0.05	0.18	0.91	0.99

Table 2: Monte Carlo Results
Bootstrap Estimation of Σ
200 replications of each experiment.

	PV1		CV1		PV2		CV2	
Range of n:	2–4	2–5	2–4	3–6	2–4	3–5	3–5	3–6
T_n	200	200	200	200	200	200	200	200
share of p-values $< 10\%$	0.14	0.18	1.00	1.00	0.13	0.21	0.80	0.91
share of p-values $< 5\%$	0.10	0.12	1.00	1.00	0.04	0.11	0.70	0.83

Table 3: Monte Carlo Results
K-S Test using Subsampled critical values.[a]
200 replications of each experiment.

	PV1			CV1		
Range of n:	2–3	2–4	2–5	2–3	2–4	2–5
T^b	200	200	200	200	200	200
B^c	50	50	50	50	50	50
S^d	200	200	200	200	200	200
%(reject at 5%)	0.01	0.01	0.01	0.24	0.56	0.58
%(reject at 10%)	0.14	0.07	0.04	0.68	0.97	0.97

	PV2			CV2		
Range of n:	2–3	2–4	2–5	2–3	2–4	2–5
T	200	200	200	200	200	200
B	50	50	50	50	50	50
S	200	200	200	200	200	200
%(reject at 5%)	0.02	0.01	0.00	0.17	0.31	0.40
%(reject at 10%)	0.07	0.06	0.06	0.51	0.65	0.82

[a] The bandwidth sequence is $h_T = O(T^{-\frac{1}{4}})$.
[b] Number of auctions.
[c] Number of auctions in each subsampled dataset.
[d] Number of subsamples taken.

Table 4: Data Configuration
USFS Timber Auctions

	Scaled Sales		Lumpsum Sales	
	number of auctions	number of bids	number of auctions	number of bids
$n = 2$	63	126	54	108
$n = 3$	39	117	40	120
$n = 4$	42	168	33	132
$n = 5$	33	165	16	80
$n = 6$	23	138	18	108
$n = 7$	14	98	11	77
$n = 8$	4	32	6	48
$n = 9$	9	81	7	63
$n = 10$	11	110	3	30
$n = 11$	1	11	0	0
$n = 12$	4	48	3	36
TOTAL	243	1094	191	802

Table 5: Summary Statistics
USFS Timber Auctions

	Scaled Sales		Lumpsum Sales	
	mean	std dev	mean	std dev
number of bidders	4.50	2.47	4.20	2.30
winning bid	80.50	51.49	77.53	46.57
appraised value	36.12	32.56	36.10	29.08
estimated volume	609.89	640.50	390.04	555.86
est. manuf cost	141.51	45.79	153.46	43.08
est. harvest cost	120.57	29.55	118.36	24.92
est. selling value	312.04	75.85	335.74	96.88
species concentration	0.5267	0.5003	0.5497	0.4988
6-month inventory	334161	120445	389821	139625
contract term	7.31	3.27	6.39	3.63
acres	697.78	2925.45	266.82	615.28
region 5 dummy	0.8519		0.6806	

Table 6: Test Results
Without Instrumental Variables

Scaled Sales

range of n	2-3	2-4	2-5	2-6	2-7
bid regression R^2	.730	.668	.753	.712	.702
means	27.06	31.25	31.07	29.45	26.14
	28.25	33.46	33.51	32.78	30.14
		41.35	40.32	37.79	35.34
			37.23	34.87	30.31
				39.29	35.79
					49.76
p-values:					
means test	.500	.677	.740	.800	.827
K-S test	.296	.714	.166	.700	.576

Lumpsum Sales

range of n	2-3	2-4	2-5	2-6	2-7
bid regression R^2	.752	.736	.627	.574	.566
means	23.73	22.02	3.52	8.09	9.67
	23.85	24.60	5.85	10.62	12.09
		17.97	0.90	6.48	8.15
			15.12	17.32	18.68
				17.35	18.67
					10.71
p-values:					
means test	.502	.432	.730	.813	.764
K-S test	.432	.226	.330	.630	.124

Table 7: Test Results
With Instrumental Variables

Scaled Sales

range of n	2-3	2-4	2-5	2-6	2-7
IV regression R^2	.172	.134	.178	.190	.215
bid regression R^2	.740	.671	.758	.722	.724
means	24.34	37.13	34.53	32.92	34.27
	23.74	46.55	37.45	29.35	29.66
		36.77	41.09	32.75	37.74
p-values:					
means test	.486	.543	.662	.613	.646
K-S test	.362	.088	.766	.496	.214

Lumpsum Sales

range of n	2-3	2-4	2-5	2-6	2-7
IV regression R^2	.387	.258	.291	.292	.321
bid regression R^2	.754	.745	.642	.639	.633
means	28.98	35.43	7.15	33.15	28.33
	24.11	24.84	3.48	29.84	24.94
		24.18	10.29	28.39	22.82
p-values:					
means test	.374	.321	.665	.585	.541
K-S test	.270	.048	.628	.172	.080

Figure 1. Empirical Distributions of Pseudo-values
From One Monte Carlo Sample

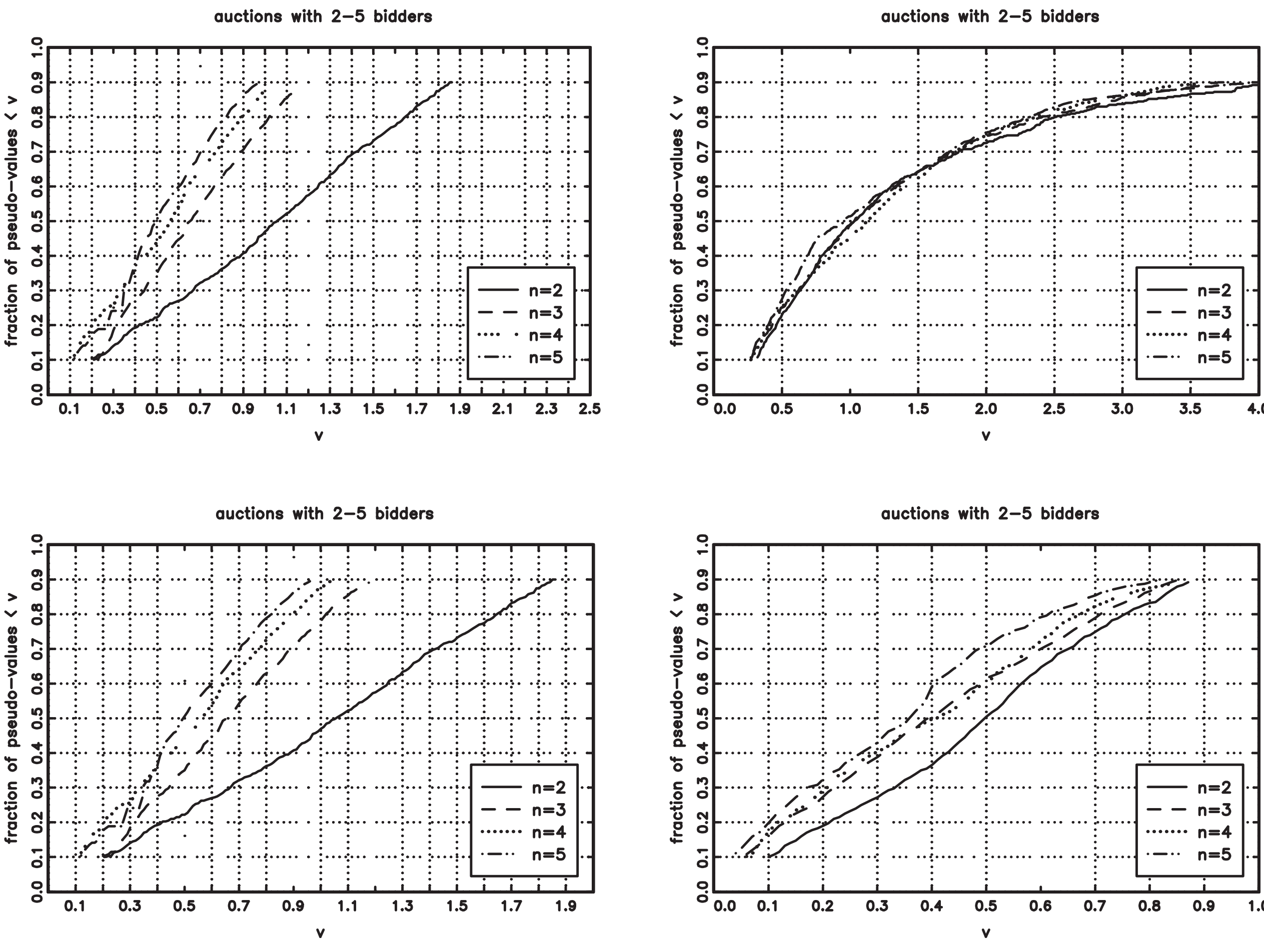

Figure 2. Empirical Distributions of Pseudo-values

Scaled Sales

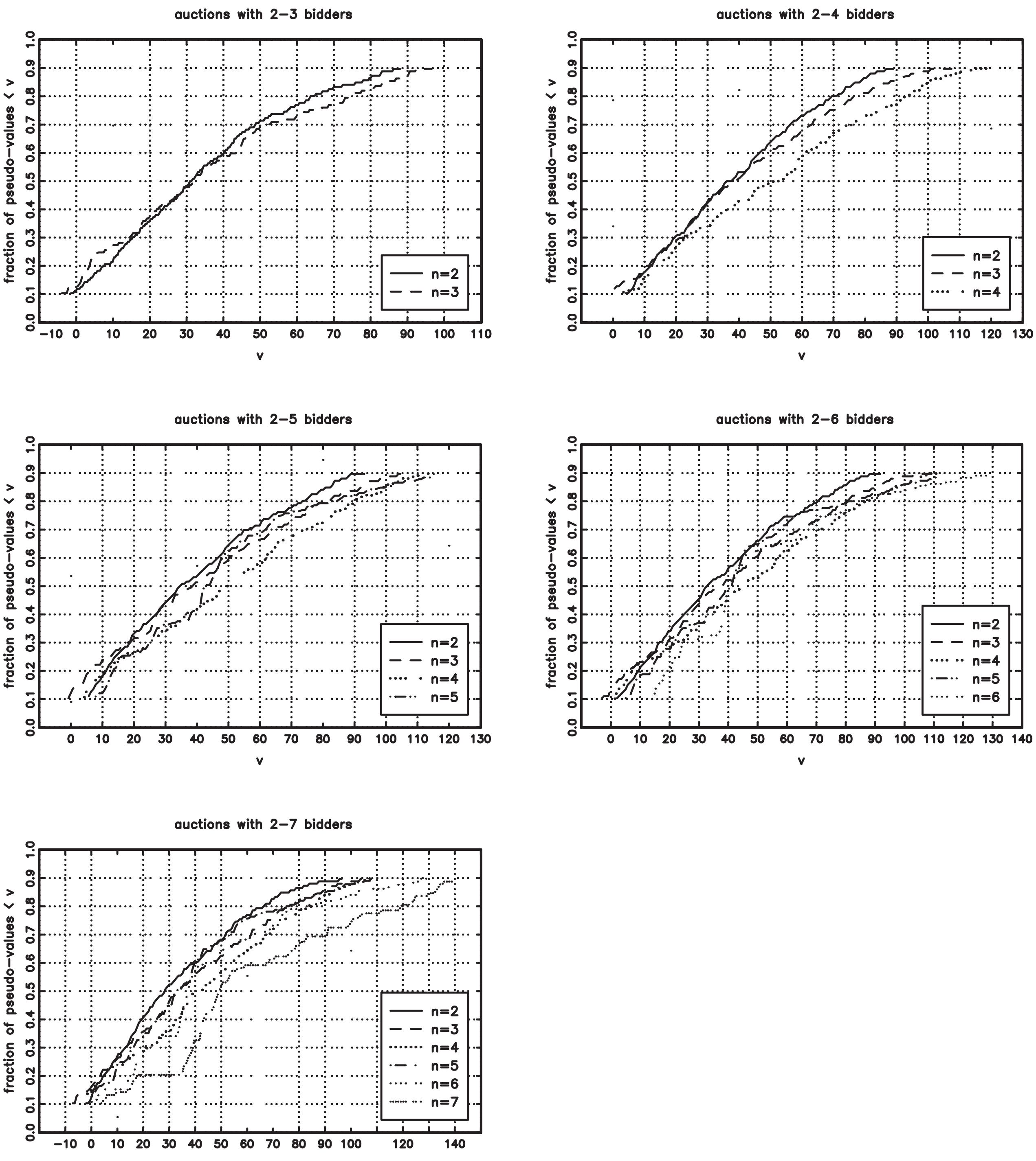

Figure 3. Empirical Distributions of Pseudo-values

Lumpsum Sales

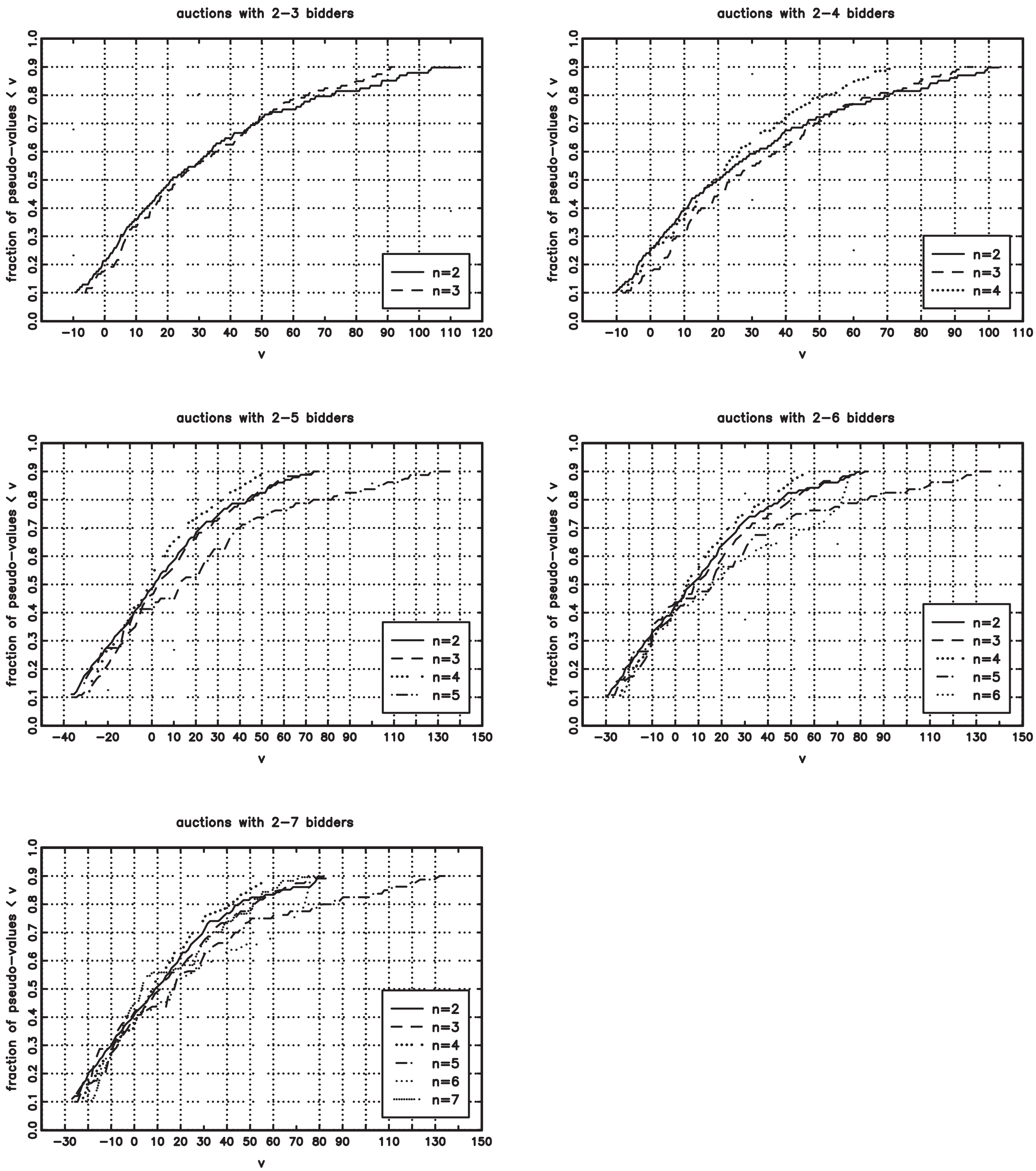

Figure 4. Empirical Distributions of Pseudo-values

Scaled Sales, Using Instrumental Variables

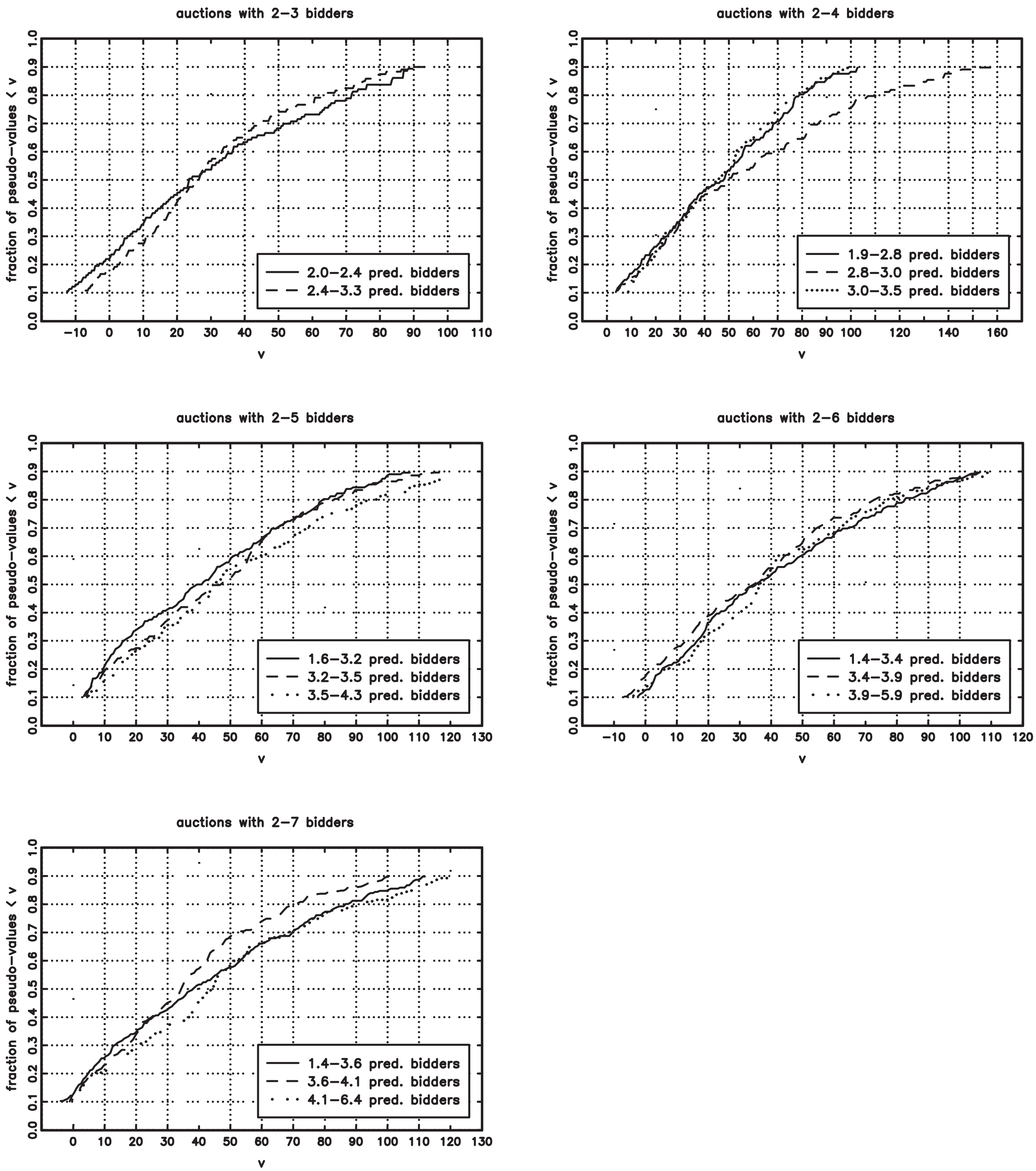

Figure 5. Empirical Distributions of Pseudo-values

Lumpsum Sales, Using Instrumental Variables