# The Harmonic Fisher Equation: Inflationary Bias and the Need for an Active Central Bank

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**Abstract**

We derive a harmonic Fisher equation, relating expected inflation to the nominal and real interest rates, in a stationary general equilibrium economy with real uncertainty. We find that if the central bank sets the nominal interest rate equal to the discount rate of the representative agent, then the long-run rate of inflation is positive (and the same) on almost every path, unless utility is logarithmic. By contrast the classical Fisher equation asserts that inflation should be zero. In fact, no constant interest rate will stabilize prices, even if the economy is stationary with bounded i.i.d. shocks. The central bank must actively manage interest rates if it wants to keep prices bounded forever. However, not even an active central bank can fix prices completely.

*Key Words:* Inflation, equilibrium, control, interest rate, central bank, Fisher, harmonic

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# 1 Introduction

Money is used to transact and to store value. Corresponding to these two rôles, there are two prices, the monetary price of goods and the monetary rate of interest. A central bank typically has one policy instrument, controlling the quantity of money. We investigate the consequences for price-inflation when the central bank holds the rate of interest constant, allowing the money stock to fluctuate endogenously depending on how much money agents borrow or deposit with it. We couch our analysis in terms of a representative agent model with a single good and independent, identically distributed endowments, where it is possible to provide explicit formulae for equilibrium.

According to the famous Fisher equation, the rate of inflation should depend on the monetary rate of interest and on the time-preference of the agents, and on nothing else. In a nonstochastic, stationary economy, this is precisely the case:

$$\frac{p_{n+1}}{p_n} = \frac{m_{n+1}}{m_n} = \beta(1+\rho), \tag{1.1}$$

where $\beta$ denotes the discount rate of the agents (its reciprocal is the rate of time-preference), $1+\rho$ denotes the gross monetary rate of interest, and $m_{n+1}/m_n$ denotes the rate of monetary growth. If the central bank sets the rate of interest (the same for borrowers and for depositors) equal to the rate of time-preference for agents, the equilibrium rate of inflation and of monetary growth will be zero.

We show that a Fisher-like equation still holds in a stochastic stationary world, but with the one-period rate of inflation replaced by the harmonic mean of the long-run rate of inflation or the one-period money growth rate:

$$\lim_{n\to\infty} E\left[\frac{1}{p_n^{1/n}}\right] = E\left[\frac{1}{m_{n+1}/m_n}\right] = \frac{1}{\beta(1+\rho)}. \tag{1.2}$$

In case $\beta(1+\rho) = 1$, the harmonic mean of the money growth rate is one. As long as there is any random variation in the money growth rate, its geometric mean must therefore be greater than one. With independent draws, the law of large numbers guarantees that the long run (geometric) growth rate of the money stock is positive on almost every path. Since the long run rate of inflation must be equal to the long run rate of growth of the money stock, this shows that when $\beta(1+\rho) = 1$, the slightest bit of monetary fluctuation creates an inflationary bias, irrespective of the utility function of the representative agent.[1]

Suppose that physical endowments $y_n$ are i.i.d. and that $yu'(y)$ is not constant. We show that no matter what fixed interest rate $\rho$ the central bank maintains, monetary stocks $m_n$ and prices $p_n$ must fluctuate unboundedly. Even if $y$ takes on just two values, prices $p_n$ will eventually become unboundedly large or small, or both. Only by active management, setting $\rho$ as a function of the real shock $y$, can the central bank ensure stable prices that stay bounded. Even such an active bank cannot maintain absolutely fixed price levels $p_n = p$ for all $n$.

On the other hand, we show that if agents *do not know their income* before they are called upon to commit themselves to expenditures, then the original Fisher equation is restored irrespectively of the agents' utility function,[2] and setting the rate of interest equal

[1] The sole exception occurs when there is no variation in the rate of growth of the money supply. We show that this can only happen if $yu'(y)$ is constant over all random endowments $y$. Thus logarithmic utility turns out to be the exception to our inflationary bias, rather than the archetypal example it often is in other contexts.

[2] In an actual economy, the control problem for a central bank is far more complicated than illustrated here or in the models of Lucas (1978, 1980, 1990). Only outside, or government (fiat), money is considered here. But in reality, the amount of credit in an economy is usually both larger and more volatile than the supply of government-money; thus, governmental control, of either the money supply or of the interest-rate, is far less effective than indicated in these models.

to the rate of time preference will result in an expected rate of inflation equal to zero.

Our model is in the spirit of the representative agent approach of Lucas (1978); we use dynamic programming methods in a microeconomic model of money, in the tradition of Shubik (1972), Shubik and Whitt (1973), Lucas (1980, 1990), Stokey and Lucas (1989) and Karatzas, Shubik and Sudderth (1994). The microeconomic tradition of analyzing policy and money in a market-clearing model is vast; see, for example, Phelps (1967, 1970, 1973), Kydland and Prescott (1977), Barro (1990), Chari et al. (1991), Mankiw (1992), Sargent (1987, 1999), Alvarez, Lucas and Weber (2001), and Dubey and Geanakoplos (2001). To our knowledge, however, the questions addressed in this paper seem to be treated here for the first time. The model of Lucas (1990), for instance, is extremely close to ours, but analyzes the case where the central bank behavior is random and output is fixed; agents in this model may or may not know the interest rate when they make their savings consumption choice. Models of Weil ( ), Campbell ( ) and others examine the real interest rate, that is the interest rate on bonds that pay one unit of good in each period. Surprisingly, much less attention has been paid to the nominal interest rate.

## 1.1 Preview

The derivation of the harmonic Fisher equation (1.2) will be undertaken in a completely specified general equilibrium model with a representative agent. It may be instructive to briefly see how to derive the harmonic Fisher equation in a reduced form model based on two premises, stationarity and money neutrality. First, we suppose that prices at time $n$ are proportional to the stock of money at time $n$:

$$p_n = \bar{p}(y_n)m_n, \tag{1.3}$$

where $y_n$ is the stochastic endowment of the perishable good at time $n$. Secondly, we suppose that the money stock at time $n+1$ is proportional to the stock at time $n$, and depends only on the random variable at time $n$:

$$m_{n+1} = \tau(y_n)m_n. \tag{1.4}$$

In equilibrium, the agent is indifferent between spending a dollar on consumption and depositing it in the bank with interest to consume next period:

$$\frac{u'(y_n)}{p_n} = \beta(1+\rho)E_n\left[\frac{u'(y_{n+1})}{p_{n+1}}\right],$$

where $E_n[\cdot] = E[\cdot|\mathcal{F}_n]$ denotes conditional expectation with respect to the information $\mathcal{F}_n$ available to agents at time $n$. Substituting for $p_n$, $p_{n+1}$ and $m_{n+1}$, gives

$$\frac{u'(y_n)}{\bar{p}(y_n)m_n} = \beta(1+\rho)E_n\left[\frac{u'(y_{n+1})}{\bar{p}(y_{n+1})m_{n+1}}\right] = \beta(1+\rho)E_n\left[\frac{u'(y_{n+1})}{\bar{p}(y_{n+1})\tau(y_n)m_n}\right] = \frac{\beta(1+\rho)}{\tau(y_n)m_n}E_n\left[\frac{u'(y_{n+1})}{\bar{p}(y_{n+1})}\right].$$

Let $z(y) \triangleq u'(y)/\bar{p}(y)$. Cancelling $m_n$ from both sides, and bringing $\tau(y_n)$ to the left, gives

$$\tau(y_n) = \beta(1+\rho)\frac{1}{z(y_n)}E_n[z(y_{n+1})],$$

Inverting both sides gives

$$\frac{1}{\tau(y_n)} = \frac{1}{\beta(1+\rho)}\frac{z(y_n)}{E_n[z(y_{n+1})]}.$$

Assuming that the $y_n$ are independent of $\mathcal{F}_s$ for $s < n$ and taking expectations gives

$$E\left[\frac{1}{\tau(y)}\right] = E_s\left[\frac{1}{\tau(y_n)}\right] = \frac{1}{\beta(1+\rho)}\frac{E_s[z(y_n)]}{E_n[z(y_{n+1})]} = \frac{1}{\beta(1+\rho)}, \tag{1.5}$$

because $y_n$ and $y_{n+1}$ have the same distribution.

# 2 Equilibrium

## 2.1 The Model

We consider a representative agent model extending over days or time periods $n = 1, 2, ....$ On each day the agent receives a random endowment $Y_n(\omega)$ of a single perishable commodity, where $Y_n$ is a random variable on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\omega$ is an element of $\Omega$. The random variables $Y_1, Y_2, ...$ corresponding to the successive random endowments of the agent are assumed to be independent with a common distribution $\lambda$. We often use $Y$ with no subscript to denote a generic random variable with this distribution. We further assume that the support $\mathcal{Y}$ of the endowment variables is bounded away from 0 and $\infty$.

The total payoff to the agent in state $\omega$ from consumption $(x_1(\omega), x_2(\omega), ...)$ is

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n(\omega)),$$

where $\beta \in (0, 1)$ is the discount factor and the utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is concave, strictly increasing, and differentiable on $(0, \infty)$.

The agent in the economy must sell his entire endowment $Y_n(\omega)$ for money in each period $n$ at price $p_n(\omega)$, thereby receiving $p_n(\omega)Y_n(\omega)$. He consumes only by purchasing commodities with the money he already has on hand (cash in advance). The money prices of the commodity are random variables $(p_1(\omega), p_2(\omega), ...)$. The agent regards himself as so small as to be unable to affect these prices by his actions.

In period 1 the agent begins with a quantity of fiat money $m_1$ called his liquid wealth. A government bank stands ready to loan or borrow money at a given interest rate $\rho \geq 0$. The agent can lend an amount up to $m_1$ or borrow up to a limit $L_1(\omega)$ that we shall specify. Having chosen to borrow or lend $\tilde{m}_1(\omega)$ with $-m_1 \leq \tilde{m}_1(\omega) \leq L_1(\omega)$, the agent spends $b_1(\omega) = m_1 + \tilde{m}_1(\omega)$ on commodities and consumes $x_1(\omega) = b_1(\omega)/p_1(\omega)$. At every subsequent period $n = 2, 3, ...$, the agent begins with liquid wealth

$$m_n(\omega) = (1+\rho)(m_{n-1}(\omega) - b_{n-1}(\omega)) + p_{n-1}(\omega)Y_{n-1}(\omega) \tag{2.1}$$

and by choosing to borrow or lend $\tilde{m}_n(\omega)$, $-m_n(\omega) \leq m_n(\omega) \leq L_n(\omega)$, the agent spends $b_n(\omega) = m_n(\omega) + \tilde{m}_n(\omega)$ and consumes $x_n(\omega) = b_n(\omega)/p_n(\omega)$. For notational simplicity we shall avoid using $\tilde{m}_n$ and $x_n$, since they are determined at every stage by $b_n$ and $p_n$.

## 2.2 Equilibrium

At the beginning of period $n$, the agent knows the values of $m_1, m_2, ..., m_{n-1}, m_n$; $p_1, p_2, ..., p_{n-1}$, $p_n$; and $Y_1, Y_2, ..., Y_{n-1}, Y_n$. (Notice that the agent is assumed to know $p_n$ and his endowment $Y_n$ for period $n$ at the *beginning* of the period. Eventually we shall consider a model in which the agent does not know either $p_n$ or $Y_n$ when he chooses his bid in the period.) The agent must choose his bid $b_n$ in period $n$ to be a function of these variables, or, equivalently, to be measurable with respect to the sigma-field $\mathcal{F}_n$ generated by them.

The agent's budget set is

$$\begin{aligned} B(m_1) &= \{b_n(\omega) : 0 \le b_n(\omega) \le m_n(\omega) + L_n(\omega) \text{ for almost all } \omega \in \Omega \\ &\quad \text{and } b_n \text{ is } \mathcal{F}_n\text{-measurable for all } n \ge 1\}, \end{aligned} \tag{2.2}$$

where $m_n$ is determined by $(m_1, p_1(\omega), ..., p_{n-1}(\omega); b_1(\omega), ..., b_{n-1}(\omega); Y_1(\omega), ..., Y_{n-1}(\omega))$ as in (2.1) above.

The economy $(\beta, u, Y, m_1)$ is in *equilibrium* at $p \equiv \{p_n(\omega) : n \ge 1, \omega \in \Omega\}$ when the representative agent is optimizing in his budget set while consuming $Y_n(\omega)$. Letting

$$b_n(\omega) \equiv p_n(\omega)Y_n(\omega) \text{ for all } n \ge 1,\ \omega \in \Omega$$

$b \equiv \{b_n(\omega) : n \ge 1,\, \omega \in \Omega\}$ satisfies

$$b \in \underset{d \in B(m_1)}{\arg\max}\, E\left[\sum_{n=1}^{\infty} \beta^{n-1} u(d_n(\omega)/p_n(\omega))\right]. \tag{2.3}$$

We distinguish two cases for the borrowing limits $L_n(\omega)$. Setting

$$L_n(\omega) = \frac{p_n(\omega)Y_n(\omega)}{1+\rho}, \tag{2.4}$$

we get an economy with a bank that permits the agent to borrow up to the amount he is sure to receive in income. By setting $L_n(\omega) = 0$ and $\rho = 0$, we effectively obtain an economy without a bank.

In the following two sections we construct equilibria for these two cases. Afterwards we shall consider a model in which the bank chooses an interest rate $\rho_n(\omega)$ in period $n$ that need not be constant. We shall also construct equilibria with and without a bank when the agent does not know his endowment $Y_n(\omega)$ or the price $p_n(\omega)$ when he is called upon to borrow and bid at time $n$.

We are primarily concerned with the behavior of the prices $p_n(\omega)$. For the models with a bank we will give conditions on the interest rate $\rho$ that result in inflation, deflation, or neither. Consumption in our models is trivial with the agent consuming his endowment $Y_n(\omega)$ in each period $n$.

## 2.3 Neutral, Stationary Equilibrium (NSE)

We shall focus our attention on *stationary equilibria* in which prices $p_n(\omega)$, expenditures (bids) $b_n(\omega)$, and the liquidity constraints $L_n(\omega)$ at time $n$ can all be expressed in terms of functions $p : \mathbb{R}_{++} \times \mathcal{Y} \to \mathbb{R}$ and $b : \mathbb{R}_{++} \times \mathcal{Y} \to \mathbb{R}$ that depend only on the liquid wealth $m_n(\omega)$ and the endowment $Y_n(\omega)$ in the period:

$$\begin{aligned} p_n(\omega) &= p(m_n(\omega), Y_n(\omega)) \\ b_n(\omega) &= b(m_n(w), Y_n(\omega)) \\ L_n(\omega) &= \mathcal{L}(Y_n(\omega), p_n(\omega)) \end{aligned} \tag{2.5}$$

where $\mathcal{L} : \mathcal{Y} \times \mathbb{R}_{++} \to \mathbb{R}$,

$$\mathcal{L}(y, p) = \begin{cases} \dfrac{p \cdot y}{1+\rho} & \text{if banking} \\ 0 & \text{if not} \end{cases} \tag{2.6}$$

In fact we shall construct *stationary* equilibria in which fiat money is *neutral* (doubling the money doubles prices and bids without changing consumption):

$$\begin{aligned} p_n(\omega) &= m_n(\omega)\bar{p}(Y_n(\omega)) \\ b_n(\omega) &= m_n(\omega)\bar{b}(Y_n(\omega)) \end{aligned} \tag{2.7}$$

where $\bar{p} : \mathcal{Y} \to \mathbb{R}_{++}$ and $\bar{b} : \mathcal{Y} \to \mathbb{R}_{++}$. Clearly $\bar{b}(y) = \bar{p}(y)y$, for all $y \in \mathcal{Y}$. It is evident from the monotonicity of $u$ that there can be no equilibrium with $p_n(\omega) = 0$ with positive probability, for some $n \geq 1$. Hence in neutral stationary equilibrium we must have $\bar{b}(y) > 0$ and $\bar{p}(y) > 0$ for all $y \in \mathcal{Y}$, and $m_n(\omega) > 0$ almost surely, for all $n \geq 1$. We can think of $\bar{p}(y)$ as the reciprocal of real liquid wealth in the economy $1/\bar{p}(y) = m/p(m, y)$.

## 2.4 Dynamic Programming

In neutral, stationary equilibrium the *macro* variables $(m, y, p)$ follow a stationary Markov process defined by the price function $p(m, y)$. Let $\Gamma$ be the graph of the function $p$, $\Gamma = \{(m, y, p) \in \mathbb{R}_{++} \times \mathcal{Y} \times \mathbb{R}_{++} : p(m, y) = p\}$. Then the law of motion of the macro variables is given by transition probabilities $\Gamma \to \Delta(\Gamma)$ indicated below for each $(m, y, p) \in \Gamma$:

$$\begin{aligned} m' &= (m - py)(1 + \rho) + py \\ &= m(1 + \rho) - \rho py \\ &= m(1 + \rho - \rho\bar{b}(y)) \\ y' &\in \mathcal{Y}, \text{ distributed as } Y \\ p' &= p(m', y'). \end{aligned} \tag{2.8}$$

Note that $m'$ is determined without any uncertainty by $(m, y, p)$, whereas $y'$ and $p'$ are stochastic. Note also that in order for $m' > 0$, we must have $\bar{b}(y) \equiv p(m, y)y/m < (1+\rho)/\rho$ for all $y \in \mathcal{Y}$.

Facing this dynamic system, the representative agent with liquid wealth $s$ must choose an expenditure rule at each macro state $(m, p, y)$ to maximize his utility. In stationary equilibrium, when $s = m$, the expenditure rule $(s, m, y, p) \to py$ must be optimal.

To verify this it suffices that there is a valuation function $V : \mathbb{R}_+ \times \Gamma \to \mathbb{R}$ satisfying

$$V(s, m, y, p) = \max_{0 \leq b \leq s + \mathcal{L}(y,p)} \left[ u\left(\frac{b}{p}\right) + \beta E_Y V((s - b)(1 + \rho) + py, m', Y, p') \right] \tag{2.9}$$

such that for all $(m, y, p) \in \Gamma$ and $s = m$,

$$p \cdot y \in \underset{0 \leq b \leq m + \mathcal{L}(y,p)}{\arg\max} \left[ u\left(\frac{b}{p}\right) + \beta E_Y V((m - b)(1 + \rho) + py, m', Y, p') \right] \tag{2.10}$$

i.e.,

$$V(m, m, y, p) = u(y) + \beta E_Y[V(m', m', Y, p')]. \tag{2.11}$$

# 3 Constructing NS-Equilibrium without a Bank

Without a bank $\mathcal{L}(y, p) = 0$ and the money supply will not change, so the representative agent will begin each period with the same liquid wealth $m$. We can see that $m' = m$ by taking $\rho = 0$ in the law of motion (2.8).

The representative agent always has the choice of spending his money at time $n$ or saving it to spend in the future. In equilibrium he is always spending $p_n(\omega)Y_n(\omega) > 0$ in every period. Hence for all $(m, y, p) \in \Gamma$, in NSE

$$\frac{u'(y)}{p} \geq \beta E\left[\frac{u'(Y)}{p'}\right] \tag{3.1}$$

(with equality if $py < m$) or equivalently

$$\frac{u'(y)}{\bar{p}(y)} \geq \beta E\left[\frac{u'(Y)}{\bar{p}(Y)}\right] \text{ for all } y \in \mathcal{Y} \tag{3.2}$$

(with equality if $\bar{p}(y)y < 1$).

We must find prices such that (3.1) is satisfied.

One natural (but generally insufficient) guess is that the agent will spend all his money every period, $b(m, y) = m$, $\bar{b}(y) = 1$, $p(m, y) = m/y$, $\bar{p}(m, y) = 1/y$. Plugging this candidate equilibrium into (3.2) gives

$$yu'(y) \geq \beta E[Yu'(Y)] \tag{3.3}$$

for all $y \in \mathcal{Y}$.

This indeed is an equilibrium if $yu'(y)$ is a constant (as it will be for $u(y) = \log(y)$), or more generally, if (3.3) holds for all $y \in \mathcal{Y}$. But if there is enough variation in $yu'(y)$ over $y \in \mathcal{Y}$, then (3.3) will fail for some $y$ and $(p, b)$ is not an equilibrium.

Another plausible (but insufficient) guess is to set price always proportional to marginal utility, $\bar{p}(y) = \frac{1}{a}u'(y)$ for some constant $a > 0$, for all $y \in \mathcal{Y}$. But plugging this into (3.2) gives a strict inequality for all $y \in \mathcal{Y}$, implying that $\bar{p}(y) = 1/y$ for all $y \in \mathcal{Y}$. Hence $1/y = u'(y)/a$ or $yu'(y) = a$ for all $y \in \mathcal{Y}$, which can only happen if $u$ is logarithmic or there is no endowment uncertainty.

The solution to (3.2) must be that price is proportional to marginal utility when the representative agent is not spending all his money, and equal to $m/y$ when he is. Since the RHS of (3.2) does not depend on $y$, let us call it $a \equiv \beta E[u'(Y)/\bar{p}(Y)]$. Then equilibrium requires that

$$p(m, y) = \begin{cases} m/y & \text{for } yu'(y) \geq a \\ \dfrac{u'(y)}{a}m & \text{for } yu'(y) < a \end{cases} \tag{3.4}$$

The rub is that $\bar{p}(y)$ then depends on $a$, $\bar{p}_a(y) = 1/y$ or $u'(y)/a$. Plugging this into the definition of $a$, we get

$$a = \beta E[\max\{a, Yu'(Y)\}]. \tag{3.5}$$

It is evident that at $a = 0$, LHS (3.5) $<$ RHS(3.5), since $Yu'(Y) > 0$ for all $Y \in \mathcal{Y}$, while for a very large LHS(3.5) $>$ RHS(3.5) since $\beta < 1$ and $E[Yu'(Y)] < \infty$. Hence there must be an $a \in (0, \infty)$ solving (3.5). Indeed, it is evident that near any $a$ solving (3.5), the RHS increases less quickly in $a$ than the LHS, so there is a unique $a$ solving (3.5).

**Theorem 3.1:** *Defining $a$ as in* (3.5), *and $p(m, y)$ as in* (3.4), *and $b(m, y) = yp(m, y)$, gives a neutral stationary equilibrium for the economy without a bank. This is the unique neutral stationary equilibrium.*

**Proof:** This candidate equilibrium satisfies (3.2) by construction. We leave to the appendix the proof that $b$ is fully optimal given prices defined by (3.4). ■

In the equilibrium just constructed liquid wealth remains constant at $m$, and prices fluctuate independently and identically across periods according to

$$\begin{aligned} p_n(\omega) &= p(m, Y_n(\omega)) \\ &= \min\left\{\frac{1}{Y_n(\omega)}, \frac{u'(Y_n(\omega))}{a}\right\} m \end{aligned} \tag{3.6}$$

No matter what the discount rate $\beta$, there is no inflation. If the shocks $Y$ are bounded away from 0 and $\infty$, so are the prices.

# 4 Constructing NS-Equilibrium with a Bank

With a bank the money supply is endogenous, and the representative agent may well begin each period with different liquid wealth. We suppose that a bank stands ready to accept deposits or give loans at an exogenous interest rate $\rho > 0$. We shall construct a neutral stationary equilibrium. From the law of motion (2.8) we have

$$m' = \tau(y)m \tag{4.1}$$

where

$$\tau(y) \equiv 1 + \rho - \rho\bar{b}(y) \text{ for all } y \geq 0. \tag{4.2}$$

As we noted earlier, neutrality implies that the growth rate $\tau$ of liquid wealth between periods $n-1$ and $n$ depends only on $y_{n-1}$, and therefore that $m_n$ is $\mathcal{F}_{n-1}$-measurable.

Note that as long as $\bar{b}(y) \leq (1+\rho)/\rho$, the agent's borrowing constraint (2.5) is satisfied, and as long as $\bar{b}(y) < (1+\rho)/\rho$, his future liquid money stays strictly positive, which is necessary in neutral, stationary equilibria.

In neutral, stationary equilibrium, if there is one, the agent could always have borrowed (deposited) a little more or less. Hence we must have

$$\frac{u'(y)}{p} = (1+\rho)\beta E\left[\frac{u'(Y)}{p'}\right]. \tag{4.3}$$

Plugging in our formulas for $p$ and $p'$, and recalling $m\bar{b}(y) = py$ gives

$$\begin{aligned} \frac{yu'(y)}{m\bar{b}(y)} &= (1+\rho)\beta E\left[\frac{Yu'(Y)}{m'\bar{b}(Y)}\right] \\ &\equiv \frac{(1+\rho)\beta}{\tau(y)m} E\left[\frac{Yu'(Y)}{\bar{b}(Y)}\right]. \end{aligned} \tag{4.4}$$

Cancelling $m$ from both sides, we have for all $y \in \mathcal{Y}$

$$\frac{yu'(y)}{\bar{b}(y)} = \frac{(1+\rho)\beta}{\tau(y)} E\left[\frac{Yu'(Y)}{\bar{b}(Y)}\right]. \tag{4.5}$$

$\rho > 0$ can now be seen to be necessary to the existence of neutral, stationary equilibrium with a bank. If $\rho = 0$, then by (4.2), $\tau(y) = 1$ for all $y$, giving $yu'(y)/\bar{b}(y) = \beta E[Yu'(Y)/\bar{b}(Y)]$ for all $y \in Y$, by (4.5). Taking expectations on both sides contradicts $\beta < 1$.

**Lemma 4.1:** *There is a unique function $\bar{b} : \mathcal{Y} \to \mathbb{R}_+$ that simultaneously satisfies* (4.2) *and the optimality condition* (4.5), *and satisfies $0 < \bar{b}(y) < (1+\rho)/\rho$ for all $y \in \mathcal{Y}$:*

$$\bar{b}(y) = \frac{1+\rho}{\rho}\left[\frac{(1-\beta)yu'(y)}{(1-\beta)yu'(y) + \beta E[Yu'(Y)]}\right] \tag{4.6}$$

**Proof:** Substituting the expression for $\tau(y)$ in terms of $\bar{b}(y)$ given by (4.2) into (4.5), and then solving for $\bar{b}(y)$, uniquely gives (4.6), as we show by some tedious algebra in the appendix. ■

**Theorem 4.2:** *Defining $\bar{b}(y)$ as in (4.6), and letting $p(m,y) = m\bar{b}(y)/y$ and $b(m,y) = m\bar{b}(y)$ gives the unique neutral, stationary equilibrium for the economy with a bank.*

The proof is given in the appendix.

Having described the unique neutral, stationary equilibrium, we now proceed to study the growth rate of liquid wealth and price. The "natural" choice for the central bank interest rate is $1+\rho = 1/\beta$. When there is no uncertainty, the standard Fisher equation shows that this natural $\rho$ generates zero inflation.

**Theorem 4.3:** *The expected gross rate of future money growth has harmonic mean $\beta(1+\rho)$:*

$$E\left[\frac{1}{\tau(Y)}\right] = \frac{1}{(1+\rho)\beta} = E_n\left[\frac{m_k}{m_{k+1}}\right](\omega), \tag{4.7}$$

*for all $k > n$ and almost all $\omega \in \Omega$. In particular, if $(1+\rho)^{-1} = \beta$, then the harmonic mean of gross future money growth is one.*

**Proof:** The proof of this theorem was given in the Introduction. (Notice that (1.5) corresponds to the conclusion of the theorem. The argument for (1.5) in Section 1 was based on (1.3) and (1.4), which correspond to (4.2) and (4.1), respectively.)

Here is an alternative argument based on the formula

$$\tau(y) = \beta(1+\rho)\frac{E[Yu'(Y)]}{(1-\beta)yu'(y) + \beta E[Yu'(Y)]}. \tag{4.8}$$

This formula follows from (4.2) and (4.6) by trivial algebra. Invert both sides of (4.8) to get a formula for $1/\tau(y)$. Then integrate with respect to the distribution of $Y$ to obtain (4.7). ■

**Corollary 4.4:** *If $yu'(y)$ is constant for all $y \in \mathcal{Y}$, then in the unique neutral, stationary equilibrium with a bank,*

$$m_n(\omega) = m_1[\beta(1+\rho)]^{n-1}$$

*for all $n$ and almost all $\omega \in \Omega$, and*

$$p_n(\omega) = \frac{\bar{b}(Y_n(\omega))}{Y_n(\omega)} m_1[\beta(1+\rho)]^{n-1}.$$

*In particular, provided that $\beta(1+\rho) = 1$, there is no inflationary trend if $y$ is constant or if $u(y) = \log y$ for all $y \in \mathcal{Y}$. In that case, prices $p_n(\omega)$ are* i.i.d. *and bounded away from 0 and $\infty$.*

When $y$ is almost surely constant (so $\mathcal{Y}$ consists of a single point), the classical Fisher theorem obtains. Surprisingly, the same formula holds even with real uncertainty, provided that utility is logarithmic. But the next two corollaries show that the logarithmic case is very misleading. In every other case, there will be inflation with probability one if the central bank set $\rho = (1/\beta) - 1$.

**Corollary 4.5:** *In the natural case where the central bank sets $\rho = (1/\beta) - 1$, the liquid wealth $m_n \to \infty$ almost surely, provided $Yu'(Y)$ is not almost surely constant.*

**Proof:** By repeated application of (4.1),

$$m_n(\omega) = m_1 \prod_{k=1}^{n-1} \tau(Y_k(\omega)). \tag{4.9}$$

Thus

$$\log m_n(\omega) = \log m_1 + \sum_{k=1}^{m-1} \log \tau(Y_k(\omega)).$$

So

$$\frac{\log m_n(\omega) - \log m_1}{n-1} = \frac{1}{n-1} \sum_{k=1}^{n-1} \log \tau(Y_k(\omega)).$$

By the law of large numbers, $(\log m_n - \log m_1)/(n-1)$ converges almost surely to $E[\log \tau(Y)]$, which is greater than 0 if $E[1/\tau(Y)] = 1$, since the geometric mean is greater than the harmonic mean whenever there is nontrivial randomness. Hence $\log m_n \to \infty$ almost surely. ■

The price process $p_n$ is more complex than the liquid wealth process $m_n$, since $p_n$ depends on $Y_n$ while $m_n$ is independent of $Y_n$. Knowing $Y_n(\omega)$, the rate of inflation between periods $n$ and $n+1$ will not be $\tau(Y_n(\omega))$. But if $Y$ is bounded away from 0 and $\infty$, we can prove that prices grow at the same rate as liquid money *in the long run.* In particular, the harmonic mean of price inflation between periods $n$ and $n+r$ converges to $(1+\rho)\beta$ as $r \to \infty$, almost surely.

**Corollary 4.6:** *Assume $Y$ is bounded away from zero and infinity, and that $Yu'(Y)$ is not almost surely constant. Then*

$$\frac{p_{n+r}(\omega)}{p_n(\omega)} = \frac{\bar{b}(Y_{n+r}(\omega))}{Y_{n+r}(\omega)} \left( \frac{Y_n(\omega)}{\bar{b}(Y_n(\omega))} \right) \prod_{k=n}^{n+r-1} (\tau(Y_k(\omega))$$

*and thus, for almost all $\omega \in \Omega$,*

$$\frac{\log[p_{n+r}(\omega)] - \log p_n(\omega)}{r-1} \to E \log(\tau(Y)) > (1+\rho)\beta$$

*and*

$$E\left[ \left( \frac{p_n}{p_{n+r}} \right)^{1/r} \right] \to E\left[ \frac{1}{\tau(Y)} \right].$$

In particular, when $\beta(1+\rho) = 1$,

$$p_n(\omega) \to \infty \text{ almost surely.}$$

**Proof:** From (2.6) and (4.9),

$$\begin{aligned} p_n(\omega) &= \bar{p}(Y_n(\omega)) m_n(\omega) \\ &= \frac{\bar{b}(Y_n(\omega))}{Y_n(\omega)} m_n(\omega) \\ &= \frac{\bar{b}(Y_n(\omega))}{Y_n(\omega)} m_1 \prod_{k=1}^{n-1} \tau(Y_k(\omega)). \end{aligned}$$

Replacing $n$ with $n+r$, we get the formula for $\log[p_{n+r}(\omega)/p_n(\omega)]$. The strong law of large numbers and the boundedness of $\bar{b}(Y)/Y$ gives the second result. Since $Y$ is bounded, $(\bar{b}(Y(\omega))/Y(\omega))^{1/r} \to 1$. Noting that the $\tau(Y_n)$ are independent gives the third result. ■

**Corollary 4.7:** *Under the same hypotheses as Corollary* 4.6, *there exists* $0 < \rho^* < 1/\beta - 1$ *such that for* $\rho > \rho^*$, $p_n \to \infty$ *almost surely, and for* $\rho < \rho^*$, $p_n \to 0$ *almost surely. Furthermore, when* $\rho = \rho^*$, $\limsup_{n\to\infty} p_n(\omega) = \infty$ *almost surely* and $\liminf_{n\to\infty} p_n(\omega) = 0$ *almost surely. Thus no constant interest rate* $\rho$ *can keep prices bounded away from zero and infinity.*

**Proof:** Clearly $E[\log_\rho \tau(Y)]$, given by (4.8), is continuous and strictly increasing in $\rho$. When $\rho = 0$, $\tau_\rho(y) < 1$ (thus $\log \tau_\rho(y) < 0$) for all $y$, and when $p = 1/\beta - 1$, $E[\log(\tau_\rho(y))] > 0$, is shown in Corollary 4.5. Hence there is a unique $\rho^*$ with

$$E[\log \tau_{\rho^*}(Y))] = 0.$$

By Corollary 4.6, $p_n \to 0$ or $\infty$ if $\rho < \rho^*$ or $\rho > \rho^*$, respectively. But if $\rho = \rho^*$, then by Corollary 4.5, $\log m_n$ is a random walk without drift, and so $\limsup m_n(\omega) = \infty$ and $\liminf m_n(\omega) = 0$ almost surely. By Corollary 4.6, the same must be true of $p_n$. ■

It is interesting to note that when $\rho$ is low enough that $E[\tau_\rho(y)] = 1$, then $E[\log(\tau_\rho(y))] < 0$, so $p_n(\omega) \to 0$ almost surely. But $\mathrm{Var}(p_n) \to \infty$, so with smaller and smaller probability a price path might shoot to a higher and higher level before eventually falling to 0.[3]

# 5 A High-Information Model with an Active Bank

Suppose that we are in exactly the same situation as the one studied in Section 4, and under exactly the same assumptions, except that now the bank sets an interest rate $\rho(y) \in [0, \infty)$ in each period based on the observed value $y$ of the endowment variable $Y$ in the period, with positive probability $\rho(y) > 0$. As in the previous section we assume that the agent's expenditure

$$b(y, m) = \bar{b}(y)m,$$

is proportional to his liquid wealth $m$, when the observed endowment value is $y$. The old calculation (4.1) shows that

$$m_{n+1}(\omega) = \tau(Y_n(\omega))m_n(\omega)$$

where now the rate of growth of liquid wealth is

$$\tau(y) \triangleq 1 + \rho(y) - \rho(y)\bar{b}(y). \tag{5.1}$$

[3] To see that $\mathrm{Var}(m_n) \to \infty$ when $E[\tau_\rho(y) = 1]$, observe that

$$\begin{aligned}\mathrm{Var}(m_n) &= E(m_n^2) - (E(m_n))^2 \\ &= E(m_n^2) - m_1^2\end{aligned}$$

and

$$Em_n^2 = m_1^2 \prod_{k=1}^{n-1} E(\tau(Y_n)^2) = m_1^2[E(\tau(Y)^2)]^{n-1}$$

where

$$E(\tau(Y)^2) > [E\tau(Y)]^2 = 1.$$

We shall construct an equilibrium for this model, which generalizes that of Theorem 4.2. Then we shall consider the question of whether the bank can select the interest rates $\rho(y)$ in such a way that prices are bounded away from zero and infinity for all $n$. (In the next section, we shall consider the more difficult problem for the bank of holding prices constant. We shall conclude that this is typically impossible to achieve in our models.)

The optimality condition (4.5) is replaced in this section by

$$\frac{yu'(y)}{\bar{b}(y)} = \frac{\beta(1+\rho(y))}{1+\rho(y)-\rho(y)\bar{b}(y)} \cdot E\left[\frac{Yu'(Y)}{\bar{b}(Y)}\right], \quad \forall\, y \in \mathcal{Y}. \tag{5.2}$$

**Theorem 5.1:** *Given the interest rate function $\rho(\cdot)$, with $\rho(y) > 0$ with positive probability, there is a unique function $\bar{b}(\cdot)$ such that equation (5.2) holds, namely*

$$\frac{1}{\bar{b}(y)} = \frac{\rho(y)}{1+\rho(y)} + \frac{\beta}{1-\beta} \cdot \frac{1}{yu'(y)} \cdot E\left(\frac{\rho(Y)}{1+\rho(Y)} \cdot Yu'(Y)\right). \tag{5.3}$$

Defining $b(m,y) = m\bar{b}(y)$ and $p(m,y) = m\bar{b}(y)/y$ gives the unique neutral, stationary equilibrium.

**Proof:** The proof is given in the appendix, and is similar to the proof of Lemma 4.1 and Theorem 4.2. ■

## 5.1 Stabilizing Prices

Is there some policy $\rho(y)$ that stabilizes prices, keeping them bounded away from 0 and $\infty$ forever? If $\tau(y)$ is not identically 1 (almost surely), then the arguments of Corollaries 4.6 and 4.7 show that equilibrium prices cannot remain bounded away from 0 and $\infty$ forever. But in the next theorem we show that by taking $\rho(y) = 0$ or $\rho(y)$ just high enough that $\bar{b}(y) = 1$, the active central bank can stabilize prices. It does so effectively by putting itself out of business, reducing the equilibrium to the no banking equilibrium of Section 3.

**Theorem 5.2:** *Suppose $Y$ is bounded away from 0 and $\infty$ almost surely. Then there is indeed an active bank policy $\rho(\cdot)$ that stabilizes prices $p_n$: $\exists K > 0$ such that in the unique neutral stationary equilibrium corresponding to $\rho$,*

$$0 < \frac{1}{K} < p_n(\omega) < K < \infty \textit{for all } n \textit{and almost all } \omega \in \Omega.$$

**Proof:** Consider the equilibrium without a bank given in Section 3. Define

$$\rho(y) = \begin{cases} \frac{yu'(y)}{a} - 1 & \text{if } yu'(y) \geq a \\ 0 & \text{if } yu'(y) < a \end{cases}$$

where $a \equiv \beta E[u'(Y)/\bar{p}(Y)]$. Recalling that $\bar{b}(y) = 1$ if $yu'(y) = a$, this gives $\tau(y) = 1$ for all $y$. Recalling that $1/\bar{p}(y) = Y/\bar{b}(Y)$, we see that the optimality condition (5.2) is also satisfied. By the argument of Theorem 5.2, this is an equilibrium. (The reader can check that with positive probability, $yu'(y) > a$, hence with positive probability $\rho > 0$, and then the explicit equilibrium constructed here corresponds to the equilibrium given by Theorem 5.1.) By (3.6) these prices are indeed bounded away from zero and infinity.■

## 5.2 Fixed Prices

Another possible goal for an active bank might be to hold prices *exactly* constant, rather than holding prices within finite bounds as in Theorem 5.2. However, if the endowment variable $Y$ is not itself constant, *it is typically impossible for the bank to actively adjust interest rates so as to hold prices constant*. To avoid unenlightening technicalities, we shall give a proof only for the special case where the endowment variable $Y$ takes three values $a$, $b$, and $c$ with positive probabilities where $0 < a < b < c$. We assume this special structure for the rest of the subsection.

Suppose that we want the price $p_n$ in every period $n$ to be the same, say $p_n \equiv 1$. Thus, for each value $y \in \mathcal{Y} \equiv \{a, b, c\}$, if $Y_n = y$, we require

$$p_n = b_n/y = 1.$$

$$b_n = b(y) = y \text{ for all } y \in \{a, b, c\}.$$

The optimality condition (5.2) takes the form:

$$\frac{u'(y)}{1} = \beta(1 + \rho(y)) \cdot E\left[\frac{u'(Y)}{1}\right], \quad y \in \{a, b, c\},$$

or equivalently

$$1 + \rho(y) = \frac{u'(y)}{\beta \cdot E[u'(Y)]}, \; y \in \{a, b, c\}.$$

Suppose $u'(c) < u'(b) < u'(a)$. If $\rho(c) \geq 0$, then $\rho(a) > \rho(b) > 0$.

Next, we look at the behavior of the liquid wealth of the agent. The law of motion gives

$$m_{n+1} = (1 + \rho(Y_n)) \cdot (m_n - Y_n) + Y_n.$$

An easy proof by induction shows that, if $Y_1 = Y_2 = \cdots = Y_n = y$, then,

$$m_n = (1 + \rho(y))^{n-1} \cdot (m_1 - y) + y.$$

Now consider possible values for the initial money-supply $m_1$. If $m_1 < a$ and $Y_1 = Y_2 = \cdots = Y_n = a$, then for very large $n$ we have

$$m_n = (1 + \rho(a))^n \cdot (m_1 - a) + a < 0,$$

a contradiction. On the other hand, if $m_1 > a$, and $Y_1 = Y_1 = \cdots = Y_n = a$, then $m_n = (1 + \rho(a))^n(m_1 - a) + n > c$ for sufficiently large $n$. But then following the rule $b(y) = y$, the agent will never spend more than $c < m_k$ for all $k \geq n$, which cannot be optimal, another contradiction. Thus there is no equilibrium unless $m_1 = a$. Applying the same argument, there is no equilibrium unless $m_1 = b$, a contradiction.

# 6 Equilibrium with Low Information

Suppose now that the agent does not know $Y_n(\omega)$ or $p_n(\omega)$ when he visits the bank or decides on his expenditure $b_n(\omega)$ at time $n$. We shall show that the classical Fisher equation is restored.

Formally, the only change we need to make in the model is to leave $Y_n$ and $p_n$ out of the set of random variables generating the $\sigma$-field $\mathcal{F}_n$. Hence the agent's bid must be

$$b(m, y) = b(m) = m\bar{b} \tag{6.1}$$

Equilibrium prices $p(m,y)$ will now satisfy

$$p(m,y) = b(m)/y = m\bar{b}/y \tag{6.2}$$

Since $\bar{b}$ is now a constant, so is the rate of growth of liquid money

$$\tau = 1 + \rho - \rho\bar{b} \tag{6.3}$$

## 6.1 Low Information Equilibrium without a Bank

First we turn to the case without a bank. The first-order condition (3.2) now becomes

$$E\left[\frac{Yu'(Y)}{\bar{b}}\right] \geq \beta E\left[\frac{Yu'(Y)}{\bar{b}}\right] \tag{6.4}$$

with equality if $\bar{b} < 1$.It follows immediately that $\bar{b} = 1$ in neutral, stationary equilibrium. The next theorem implies that $b(m,y) = m$ and $p(m,y) = m/y$ is indeed an equilibrium.

**Theorem 6.1:** *In the low information model without a bank, there is a unique neutral, stationary equilibrium $p$, in which*

$$p(m,y) = m/y.$$

**Proof:** We only need to verify that $b(m,y) \equiv m$ is optimal for the agent, given the prices above. The proof is short so we give it here, rather than deferring it to the appendix.

Consider more generally a single agent with initial wealth $s \geq 0$, but who will receive $m$ as revenue each period The agent can bid any amount $b \in [0,s]$, receive $u(b/p(Y))$ in utility, and then move to $s - b + Yp(Y) = s - b + m$ at the next stage. Let $V(s)$ be the optimal reward for this agent. Then the function $V(\cdot)$ satisfies the Bellman equation

$$V(s) = \sup_{0\leq b\leq s}\left[E\left(u\left(\frac{b}{p(Y)}\right)\right) + \beta V(s-b+m)\right] = \sup_{0\leq b\leq s}[\tilde{u}(b) + \beta V(s-b+m)], \tag{6.5}$$

where

$$\tilde{u}(b) \triangleq E\left[u\left(\frac{b}{p(Y)}\right)\right] = E\left[u\left(\frac{bY}{m}\right)\right], \qquad b \in [0,\infty)$$

can be regarded as another concave utility function. Standard arguments show that $V(\cdot)$ inherits from $u(\cdot)$ the properties of continuity, concavity and strict increase. Consequently,

$$\psi(b;s) \triangleq \tilde{u}(b) + \beta\, V(s-b+m),\ 0 \leq b \leq s,$$

has a point of maximum, namely

$$c(s) \in \arg\max \psi(\cdot\,; s).$$

We need to show that $c(m) = m$. Of course, $c(m) \leq m$, by the rules of the game. Suppose, by way of contradiction, that $c(m) < m$.

Now

$$V(m) = \tilde{u}(c(m)) + \beta V(2m - c(m)),$$

and clearly,

$$V(c(m)) \geq \tilde{u}(c(m)) + \beta \cdot V(c(m) - c(m) + m) = \tilde{u}(c(m)) + \beta V(m),$$

from (6.5). Subtracting the expression for $V(c(m))$ from $V(m)$ gives the first inequality below, and the strict increase of $V(\cdot)$ and the assumption $c(m) < m$ then imply

$$V(m) - V(c(m)) \leq \beta[V(2m - c(m)) - V(m)] < V(2m - c(m)) - V(m),$$

contradicting the concavity of $V(\cdot)$. ■

In the equilibrium of Theorem 6.1, the money supply stays fixed and the successive prices are $m_1/Y_1, m_1/Y_2, \ldots$. Although they fluctuate randomly, these prices have the same distribution, and are bounded away from 0 and $\infty$. There is no inflation or deflation in this economy.

### 6.2 A Low-Information Model with a Bank

Again, agents must bid without knowledge of their endowment in each period. However, they are now permitted to borrow or make deposits in an outside bank. The bank charges borrowers and pays depositors at a fixed rate of interest $\rho \in (0, \infty)$.

The first-order condition (4.5) now becomes

$$E\left[\frac{Yu'(Y)}{\bar{b}}\right] = \beta(1+\rho)E\left[\frac{Yu'(Y)}{\tau\bar{b}}\right]. \tag{6.6}$$

This can only be satisfied if

$$\tau = \beta(1+\rho) \tag{6.7}$$

restoring the old Fisher equation. Combining (6.3) and (6.7) gives

$$\bar{b} = (1-\beta)(1+\rho)/\rho \tag{6.8}$$

**Theorem 6.2:** *The low information economy with a bank has an equilibrium $(p, b)$ at which*

$$\begin{aligned} b(m,y) &= \frac{(1-\beta)(1+\rho)}{\rho} m \\ p(m,y) &= \frac{(1-\beta)(1+\rho)}{\rho} \frac{m}{y}. \end{aligned}$$

**Proof:** See appendix.

## 7 Extensions

### 7.1 Production

We considered a simple representative agent economy in which equilibrium consumption was always the same, $x_n(\omega) = Y_n(\omega)$, independent of central bank policy. It would be much more interesting to consider a model with heterogeneous agents, or with a representative agent with production, in which the central bank had to balance twin goals of price stabilization and efficient consumption.

### 7.2 Heterogeneous Agents

The harmonic Fisher equation was derived from the premise that

$$m_{n+1} = f(y_n)m_n, \tag{7.1}$$

where $y_n$ represented the real variables at time $n$. In a heterogeneous agent economy, we could take $y_n$ to represent the real shocks to all the agents. But equation (7.1) would not hold in general because the distribution of liquid wealth would also matter. It would be very interesting to work out whether there is an analogue nevertheless for the Fisher equation.

## 7.3 Differential Information

We present in this subsection a simple example involving two types of agents, who differ only in the information they receive about their respective endowments. As the example illustrates, differences in information may result in differences in wealth and consumption.

**Example 8.1:** Assume that there is no bank, that every agent is risk-neutral with utility function $u(x) \equiv x$, and that the endowment variable $Y$ takes on the values 1 and 5 with probability 1/2 each. Let the discount factor be $\beta = 1/2$ and let the supply of money held by the agents be $m = 1$. Finally, suppose that half of the agents, called type 1, have low information in that they have no knowledge of the endowment variable $Y$ before bidding in each period; and that the other half of the agents, called type 2, have high information in that they do know $Y$ before bidding.

Then there is an equilibrium with two wealth states: In the first, type 1 agents have wealth $s = 1$, and type 2 agents have the same wealth $\tilde{s} = 1$; in the second type 1 agents have wealth $s = 3/5$ and type 2 agents have wealth $\tilde{s} = 7/5$. It can be shown that, in equilibrium, an optimal strategy for type 1 agents is always to bid their entire wealth, and an optimal strategy for type 2 agents is to bid all if $Y = 5$, but to bid 1/5 if $Y = 1$ and $\tilde{s} = 1$ and to bid 3/5 if $Y = 5$ and $\tilde{s} = 7/5$. The price depends on the value of $Y$. For example, if $s = \tilde{s} = 1$ and $Y = 1$, then the total bid is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{5} = \frac{3}{5}$$

and the price is

$$p_1 = \frac{3/5}{1} = 3/5.$$

The law of motion gives the new wealth values for the two types as

$$s_1 = 1 - 1 + \frac{3}{5} \cdot 1 = \frac{3}{5},\ \tilde{s}_1 = 1 - \frac{1}{5} + \frac{3}{5} \cdot 1 = \frac{7}{5}.$$

If $s = \tilde{s} = 1$ and $Y = 5$, then the price is

$$p_2 = \frac{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1}{5} = \frac{1}{5},$$

and the new wealth values are

$$s_1 = \tilde{s}_1 = 1 - 1 + \frac{1}{5} \cdot 5 = 1.$$

Similar calculations show that for $s = 3/5$, $\tilde{s} = 7/5$, and $Y = 1$, the price is $p_1 = 3/5$ and the next wealth values are $s_1 = 3/5$, $\tilde{s}_1 = 7/5$; while for $s = 3/5$, $\tilde{s} = 7/5$, $Y = 5$, the price is $p_2 = 1/5$ and $s_1 = \tilde{s}_1 = 1$. If the economy is equally likely to start in either of the two wealth states (1,1) and (3/5,7/5), then another easy calculation shows that the average daily utility earned by type 1 agents is 8/3 and that earned by the better-informed type 2 agents is 10/3.

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# A Appendix: The Proofs

Here we supply the proofs that were omitted above.

## A.1 The Proof of Theorem 3.1

We must show that, if prices are given by (3.4) and (3.5), then it is optimal for an agent with wealth $m$ and endowment $y$ to bid $b(m,y) = yp(m,y)$ in any given period. To show that this is so, consider more generally an agent with initial wealth $s \geq 0$ and endowment $y$. The agent can bid any amount $b \in [0,s]$, and begin the next period with wealth $s - b + yp(m,y) = s - b + b(m,y)$. By assumption, the agent is so small that his actions do not affect the price or the total wealth $m$, which remains constant in this economy with no bank. So, for simplicity of notation, we write $p(y)$ for $p(m,y)$ and $b(y)$ for $b(m,y)$.

Thus the agent with wealth $s$ and endowment $y$ faces a dynamic programming problem with optimal reward function $V(s,y)$ satisfying the Bellman equation:

$$V(s,y) = \sup_{0 \leq b \leq s} \left[ u\left(\frac{b}{p(y)}\right) + \beta \cdot E[V(s - b + b(y), Y)] \right], \quad s \geq 0,\ y \in \mathcal{Y}.$$

Notice that this dynamic programming problem has state space $[0,\infty) \times \mathcal{Y}$, action sets $\mathcal{A}(s,y) = [0,s]$, law of motion

$$(s,y) \to (s - b + b(y), Y)$$

under action $b$, and daily reward $r((s,y),b) = u(b/p(y))$. It suffices to show that the optimal bid $b$ at state $(m,y)$ is $b(y)$, for every $y \in \mathcal{Y}$.

To prove that this is so, we introduce *another* dynamic programming problem with the same states $(s,y)$ and the same law of motion, but with larger action sets $\tilde{\mathcal{A}}(s,y) = [-m,s]$ and with daily reward for taking action $b$ at $(s,y)$ equal to

$$\tilde{r}((s,y),b) = u_y(b) \equiv A_y + \lambda_y b, \quad -m \leq b < \infty,$$

where

$$\lambda_y \equiv \frac{u'(y)}{p(y)} = \frac{1}{m} \cdot \max\{a, yu'(y)\}, \qquad A_y \equiv u(y) - \lambda_y b(y). \tag{A.1}$$

Notice that

$$\begin{aligned} u_y(b(y)) &= u(y) = u\left(\frac{b(y)}{p(y)}\right), \\ u_y'(b(y)) &= \lambda_y = \frac{1}{p(y)}\ u'(y) = \frac{1}{p(y)}\ u'\left(\frac{b(y)}{p(y)}\right). \end{aligned}$$

Thus, the affine function $u_y(\cdot)$ is the tangent line to the graph of the concave function $b \mapsto u(b/p(y))$ at the point $b = b(y)$. In particular, $u_y(b) \geq u(b/p(y))$ for all $b \in [0,s]$. Consequently, the expected return from any strategy $\pi$, which is available in both problems, will be at least as large in the modified problem as it was in the original problem.

Let $\pi^*$ be the strategy that, at each state $(s,y)$, uses action

$$b^*(s,y) \equiv \left\{ \begin{array}{ll} s & ;\text{ if } \lambda_y > a/m \\ b(y) + s - m & ;\text{ if } \lambda_y \leq a/m \end{array} \right\} \in \mathcal{A}(s,y)\,. \tag{A.2}$$

Notice that, for every $y \in \mathcal{Y}$, we have

$$b^*(m,y) \equiv \left\{ \begin{array}{ll} m & ;\text{ if } \lambda_y < a/m \\ b(y) & ;\text{ if } \lambda_y \leq a/m \end{array} \right\} = b(y),$$

and that under the law of motion

$$(m,y) \to (m - b^*(m,y) + b(y), Y) = (m,y).$$

Thus, for an initial state $(m,y)$, the return from $\pi^*$ is the same in both problems; namely,

$$u(y) + \frac{\beta}{1-\beta}\ E[u(Y)].$$

It now suffices to show that the strategy $\pi^*$ is optimal in the modified problem, for it must then be optimal at states $(m,y)$ in the original problem as well.

Let $W(s,y)$ be the optimal reward function in the modified problem. Then $W$ satisfies the Bellman equation

$$W(s,y) = (TW)(s,y),$$

where $T$ is the operator

$$(T\Phi)(s,y) \equiv \sup_{-m \le b \le s} [u_y(b) + \beta \cdot E\Phi(s - b + b(y), Y)], \tag{A.3}$$

whose domain is the collection of functions $\Phi : [0,\infty) \times \mathcal{Y} \to \mathbb{R}$ for which the right-hand side of (A.3) is well-defined.

Define $Q(s,y)$ to be the expected return in the modified problem from the strategy $\pi^*$ at the initial state $(s,y)$, and let

$$v(y) \equiv Q(m,y) = u(y) + \frac{\beta}{1-\beta}\ E[u(Y)]. \tag{A.4}$$

Clearly $Q(s,y) \le W(s,y)$, and $E[v(Y)] = (1-\beta)^{-1}E[u(Y)]$, so

$$v(y) = u(y) + \beta \cdot E[v(Y)]. \tag{A.5}$$

**Lemma A.1:** *For every initial state* $(s,y)$*, we have:* (i) $Q(s,y) = v(y) + \lambda_y(s-m)$*, and* (ii) $(TQ)(s,y) = Q(s,y)$.

**Proof:** From (A.2), we have

$$s - b^*(s,y) + b(y) = m.$$

Hence,

$$\begin{aligned} Q(s,y) &= u_y(b^*(s,y)) + \beta \cdot E[Q(m,Y)] \\ &= A_y + \lambda_y b^*(s,y) + \beta \cdot E[v(Y)] \\ &= A_y + \lambda_y b(y) + \lambda_y(s-m) + \beta \cdot E[v(Y)] \\ &= u(y) + \lambda_y(s-m) + \beta \cdot E[v(Y)] \\ &= v(y) + \lambda_y(s-m), \end{aligned} \tag{A.6}$$

thanks to (A.4), (A.5), and (i) is verified. To verify (ii), let

$$\begin{aligned} \psi_y(b) &\equiv u_y(b) + \beta \cdot E[Q(s-b+b(y),Y)] \\ &= A_y + \lambda_y b + \beta \cdot E[v(Y) + \lambda_Y(s - b + b(y) - m)]. \end{aligned} \tag{A.7}$$

The coefficient of $b$ in this expression is

$$\lambda_y - \beta \cdot E[\lambda_Y] \ = \ \frac{u'(y)}{p(y)} - \beta \cdot E\left[\frac{u'(Y)}{p(Y)}\right].$$

By (3.2) and (3.4), this coefficient is positive for $\lambda_y > a$, and the maximum of $\psi_y(\cdot)$ on the interval $[-m, s]$ is then attained at $b^*(s,y) = s$ ; whereas for $\lambda_y = a$, the coefficient is zero and in this case every point of the interval, including $b^*(s,y)$, attains the maximum. In either case, we have:

$$(TQ)(s,y) = \max_{0 \le b \le s} \psi_y(b) = \psi_y(b^*(s,y)) = u_y(b^*(s,y)) + \beta \cdot E[Q(m,Y)] = Q(s,y)\,,$$

■

**Lemma A.2:** *There is a real number $k$ such that $u_y(b) \ge -k > -\infty$ for all $y \in \mathcal{Y}$ and $b \ge -m$.*

**Proof:** This is a simple calculation based on (A.1) and (3.4):

$$\begin{aligned} u_y(b) &= A_y + b\lambda_y \ge A_y - m\lambda_y \\ &= u(y) - [m - b(y)]\frac{u'(y)}{p(y)} \\ &\ge u(y) - 2m\frac{u'(y)}{p(y)} \\ &= \left\{ \begin{array}{ll} u(y) - 2yu'(y) & \text{for } yu'(y) \ge a, \\ u(y) - 2m & \text{for } yu'(y) < a. \end{array} \right\}. \end{aligned}$$

By assumption, $y$ is bounded away from zero and infinity. Hence, $u(y)$ and $-yu'(y)$ are bounded away from negative infinity. ■

To complete the proof of Theorem 3.1, first notice that, by adding the constant $k$ from Lemma A.2 to the daily reward $\tilde{r}((s,y),b) = u_y(b)$, we obtain an equivalent problem with positive daily rewards. Indeed, by adding $k$ to the daily reward, we merely add $k/(1-\beta)$ to the total discounted reward. For a dynamic programming problem with positive daily rewards, a theorem of Blackwell (1966) states that the optimal reward function $W$ is the least nonnegative fixed point of the operator $T$ of (A.3). By Lemma A.1, $Q$, is such a fixed point and, being the expected reward from $\pi^*$, $Q \le W$. Hence, $Q = W$ and $\pi^*$ is optimal. The proof of Theorem 3.1 is now complete.

## A.2 The Proof of Lemma 4.1

Set

$$\theta \equiv E\left[\frac{Yu'(Y)}{Y}\right]$$

and rewrite (4.5) as

$$\frac{yu'(y)}{\bar{b}(y)} = \frac{(1+\rho)\beta\theta}{\tau(y)} = \frac{(1+\rho)\beta\theta}{1+\rho-\beta\bar{b}(y)}$$

or, equivalently,

$$\frac{yu'(y)}{\bar{b}(y)} = \frac{\rho}{1+\rho} \cdot yu'(y) + \beta\theta. \tag{A.8}$$

Take the expected value to get

$$\theta = \frac{\rho}{1+\rho} E[Yu(Y)] + \beta\theta$$

or

$$\theta = \frac{\rho}{(1-\beta)(1+\rho)} E[Yu'(Y)].$$

Substituting for $\theta$ in (A.8) and dividing by $yu'(y)$, we have

$$\frac{1}{\bar{b}(y)} = \frac{\rho}{1+\rho}\left[1 + \frac{\beta}{1-\beta} \cdot \frac{E[Yu'(Y)]}{yu'(y)}\right] = \frac{\rho}{1+\rho}\left[\frac{(1-\beta)yu'(y) + \beta E[Yu'(Y)]}{(1-\beta yu'(y)}\right].$$

Take the reciprocal to obtain (4.5).

We have shown that bids $\bar{b}(y)$ that satisfy (4.5) must be given by formula (4.6). The argument reverses to show that, if we define $\bar{b}$ by (4.6), then (4.5) holds. Thus (4.6) gives the unique solution to the functional equation (4.5).

## A.3 The Proof of Theorem 4.2

The proof is similar to that of Theorem 3.1. Let $\bar{b}(y)$ be defined by (4.6) and suppose prices satisfy $p(m,y) = m\bar{b}(y)/y$. We must show that $b(m,y) = m\bar{b}(y)$ is the optimal bid for an agent with wealth $m$ and endowment $y$. For the proof we consider the more general situation of an agent with wealth $s \geq 0$ and endowment $y \in \mathcal{Y}$. The agent can bid any amount $b \in [0, s + b(m,y)/(1+\rho)]$ borrowing or lending the difference between $s$ and $b$ according as $s$ is smaller or larger than $b$. The agent receives $u(b/p(m,y))$ in utility and begins the next period with wealth $(1+\rho)(s-b)+yp(m,y) = (1+\rho)(s-b)+b(m,y)$. The total wealth in the economy becomes $\tau(y)m$ as in (4.1) and (4.2). Thus the agent faces a dynamic programming problem with optimal reward function $V(s,y,m)$, which satisfies the Bellman equation

$$V(s,y,m) = \sup_{0 \leq b \leq s + b(m,y)/(1+\rho)} \left[u\left(\frac{b}{p(m,y)}\right) + \beta EV((1+\rho)(s-b) + b(m,y), Y, \tau(y)m)\right] \tag{A.9}$$

This dynamic programming problem has state space $[0,\infty) \times \mathcal{Y} \times [0,\infty)$, action sets $\mathcal{A}(s,y,m) = [0, s + b(m,y)/(1+\rho)]$, law of motion

$$(s,y,m) \to ((1+\rho)(s-b) + b(m,y), Y, \tau(y)m)$$

under action $b$, and daily reward function $r((s,y,m),b) = u(b/p(m,y))$. What must be shown is that an optimal bid $b$ at states of the form $(m,y,m)$ is $b(m,y)$.

As in the proof of Theorem 3.1, we introduce a modified dynamic programming problem with the same states $(s,y,m)$ and the same law of motion, but with larger action sets $\widetilde{\mathcal{A}}(s,y,m) = [-m, s + b(m,y)/(1+\rho)$ and with daily reward

$$\tilde{r}((s,y,m),b) = u_{y,m}(b) = A_{y,m} + \lambda_{y,m} b,$$

where

$$\lambda_{y,m} \equiv \frac{u'(y)}{p(m,y)} = \frac{yu'(y)}{m\bar{b}(y)}, \quad A_{y,m} = u(y) - \lambda_{y,m} b(m,y). \tag{A.10}$$

The affine function $u_{y,m}(\cdot)$ is tangent to the concave function $b \mapsto u(b/p(m,y))$ at the point $b = b(m,y)$. Thus $u_{y,m}(y) \geq u(b/p(m,y))$ for all $b$; so the return from any strategy available in both problems is at least as large in the modified problem as in the original problem.

Let $\pi^*$ be the strategy for the modified problem that, at each state $(s,y,m)$, uses the action

$$b^*(s,y,m) = b(m,y) + (s-m). \tag{A.11}$$

Since

$$b^*(m,y,m) = b(m,y),$$

and, under the law of motion,

$$(m,y,m) \to ((1+\rho)(m-b(m,y)) + b(m,y), Y, \tau(y)m) = (\tau(y)m, Y, \tau(y)m), \qquad \text{(A.12)}$$

the strategy $\pi^*$ chooses the same actions and has the same expected return in both problems for an initial state of the form $(m,y,m)$. Thus, if $\pi^*$ is optimal in the modified problem, then it must be optimal in the original problem as well for initial states $(m,y,m)$.

So it suffices to show $\pi^*$ is optimal in the modified problem. To do so, let $W(s,y,m)$ be the optimal reward and let $Q(s,y,m)$ be the expected return from $\pi^*$ for an initial state $(s,y,m)$. The Bellman equation can be written as

$$W(s,y,m) = (TW)(s,y,m),$$

where

$$(T\Phi)(s,y,m) \equiv \sup_{-m \le b \le s+b(m,y)/(1+\rho)} [u_{y,m}(b) + \beta E[\Phi((1+\rho)(s-b)+b(m,y), Y, \tau(y)m)]$$

is an operator acting on function $\Phi : [0,\infty) \times \mathcal{Y} \times [0,\infty) \to \mathbb{R}$ for which the right-hand side of the equation above is well defined.

By analogy with (A.4), we also define

$$v(y) \equiv Q(m,y,m) = u(y) + \frac{\beta}{1-\beta} E[u(Y)] \qquad \text{(A.13)}$$

and observe that

$$v(y) = u(y) + \beta E v(Y). \qquad \text{(A.14)}$$

**Lemma A.3:** *For every initial state $(s,y,m)$, we have*
(i) $Q(s,y,m) = v(y) + \lambda_{y,m}(s-m)$,
(ii) $(TQ)(s,y,m) = Q(s,y,m)$.

**Proof:** (i) By (A.11),

$$(1+\rho)(s - b^*(s,y,m)) + b(m,y) = (1+\rho)(m - b(m,y)) + b(m,y)) = \tau(y)m.$$

Hence, by definition of $Q$, (A.12), (A.13), and (A.14),

$$\begin{aligned} Q(s,y,m) &= u_{y,m}(b^*(s,y,m)) + \beta E Q(\tau(y)m, Y, \tau(y)m) \\ &= A_{y,m} + \lambda_{y,m} b^*(s,y,m) + \beta E v(Y) \\ &= u(y) - \lambda_{y,m} b(m,y) + \lambda_{y,m}(b(m,y) + s - m) + \beta E v(Y) \\ &= v(y) + \lambda_{y,m}(s-m). \end{aligned}$$

(ii) Define

$$\psi(b) \equiv u_{y,m}(b) + \beta E[Q((1+\rho)(s-b) + b(m,y), Y, \tau(y)m)].$$

Part (i) implies that

$$\psi(b) = A_{y,m} + \lambda_{y,m} b + \beta E[v(Y) + \lambda_{Y,\tau(y)m}((1+\rho)(s-b) + b(m,y) - \tau(y)m)].$$

The coefficient of $b$ on the right-hand side is

$$\begin{aligned}\lambda_{y,m} - \beta(1+\rho)E[\lambda_{Y,\tau(y)m}] &= \frac{u'(y)}{p(m,y)} - \frac{\beta(1+\rho)}{\tau(y)}E\left[\frac{u'(Y)}{p(m,Y)}\right] \\ &= \frac{1}{m}\left[\frac{yu'(y)}{\bar{b}(y)} - \frac{\beta(1+\rho)}{\tau(y)}E\left[\frac{Yu'(Y)}{\bar{b}(Y)}\right]\right] \\ &= 0,\end{aligned}$$

by (4.5). Thus $\psi$ is a constant function and (ii) is a trivial consequence. ■

**Lemma A.4:** *There is a real number $k$ such that $u_{y,m}(b) \geq -k > -\infty$ for all $y \in \mathcal{Y}$, $m \in [0, \infty)$, and $b \geq -m$.*

**Proof:** Calculate as follows:

$$\begin{aligned}u_{y,m}(b) &= A_{y,m} + \lambda_{y,m}b \geq A_{y,m} - \lambda_{y,m}m \\ &= u(y) - \lambda_{y,m}[m + b(m,y)] = u(y) - m[1+\bar{b}(y)]\frac{yu'(y)}{m\bar{b}(y)} \\ &\geq u(y) - \frac{yu'(y)}{\bar{b}(y)}\left[1 + \frac{1+\rho}{\rho}\right] \\ &= u(y) - \frac{\beta(1+2\rho)}{1-\beta} \cdot \frac{E[Yu'(Y)]}{\tau(y)} \\ &= u(y) - \frac{1+2\rho}{1+\rho}\left[yu'(y) + \frac{\beta}{1-\beta}E[Yu'(Y)]\right].\end{aligned}$$

The inequality is by (4.6) and the last two equalities by (4.5) and (4.8), respectively. The desired result follows because $u(Y)$ and $Yu'(Y)$ are each bounded away from negative infinity. ■

The proof of Theorem 4.2 can now be completed by the same argument that was used to complete the proof of Theorem 3.1 following Lemma A.2.

## A.4 The Proof of Theorem 5.1

The proof of Theorem 5.1 is quite similar to the combined proofs of Lemma 4.1 and Theorem 4.2. So we omit it.

## A.5 The Proof of Theorem 6.2

Consider the situation of an agent with wealth $s \geq 0$ at the beginning of a period when the total wealth in the economy is $m > 0$. The agent can bid any amount $b \in [0, s + \bar{b}/(1+\rho)]$ where $\bar{b}$ is defined by (6.8). The agent does not know the value of his endowment $Y$ or the price $p(m,Y)$ prior to choosing $b$, but he can calculate the expected utility $E[u(b/p(m,Y))]$ that he will receive from consumption. He also knows that he will begin the next period with wealth $(1+\rho)(s-b) + Yp(m,Y) = (1+\rho)(s-b) + \bar{b}m$. Thus the agent faces a dynamic programming problem with optimal reward function $V(s,m)$ satisfying:

$$V(s,m) = \sup_{0 \leq b \leq s + \bar{b}m/(1+\rho)} \left[Eu\left(\frac{b}{p(m,Y)}\right) + \beta V((1+\rho)(s-b) + \bar{b}m, \tau m)\right]. \qquad \text{(A.15)}$$

It suffices to show that the optimal bid at states $(m,m)$ is $b = \bar{b}m$.

This dynamic programming problem corresponds to a special case of the one studied above in Section A.3. To obtain the special case, replace the utility function of (A.9) by

$$\tilde{u}(x) \equiv Eu(xY)$$

and then replace $Y$ by the constant variable $\tilde{Y} \equiv 1$. The Bellman equation (A.15) is then equivalent to (A.9). In particular, the optimal bid at states $(m, 1, m)$ is $b(m, 1) = \bar{b}m$, where the $\bar{b}$ of (4.6) is the same as that of (6.8) because $\tilde{Y}$ is constant.