CRITICAL TYPES

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Abstract. Economic models employ assumptions about agents' infinite hierarchies of belief. We might hope to achieve reasonable approximations by specifying only finitely many levels in the hierarchy. However, it is well known since Rubinstein (1989) that the behaviors of some fully specified hierarchies can be very different from the behavior of such finite approximations. Examples and earlier results in the literature suggest that these *critical types* are characterized by some strong assumptions on higher-order beliefs. We formalize this connection. We define a critical type to be any hierarchy at which the rationalizable correspondence exhibits a discontinuity. We show that critical types are precisely those types for which there is common belief in a certain class of event. All types from finite type spaces and almost all types in common prior type spaces are critical. On the other hand, we show that *regular types*, i.e. types which exhibit no discontinuities, are generic. In particular they form a residual set in the product topology. This second result strengthens a previous one due to Weinstein and Yildiz (2006) in two ways. First, while Weinstein and Yildiz (2006) considered a fixed game, our regular types have continuous behavior across all games. Second, our result applies to an arbitrary space of basic uncertainty and does not require the rich-fundamentals assumption employed by Weinstein and Yildiz (2006). Our proofs involve a novel characterization of the strategic topology first introduced by Dekel, Fudenberg, and Morris (2006a).

1. Introduction

Economic models are simplified approximations of complicated environments. We can have confidence in simplifying assumptions when we know that they deliver conclusions which also approximate the outcomes of richer models. In game-theoretic models, to completely describe the environment requires specifying agents' infinite hierarchies of belief. In practice, researchers typically pay close attention only to beliefs of finite order and the model is closed by imposing assumptions about higher-order beliefs made mostly for convenience. More typically, the model is specified in reduced form using types in a Harsanyi type space, so that assumptions about hierarchies are made only implicitly. However, it is well-known since the example of Rubinstein (1989) that the details of higher-order beliefs can matter for predicted outcomes. Some *critical types* give rise to hierarchies whose behavior is far from the behavior of approximating hierarchies which coincide on finitely (but arbitrarily) many

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levels. Formally, a critical type is a point of discontinuity in the solution mapping relative to the standard product topology on hierarchies of belief. We characterize critical types. Our characterization is based on new results concerning the structure of the universal type space. First, we observe that the universal type space admits a non-trivial partial order $\succeq$, where $u_i \succeq v_i$ iff in every game the set of rationalizable actions for hierarchy u_i includes the set of rationalizable actions for hierarchy v_i. Second, we present new results on the topological structure of the universal type space. We show that the metric topology introduced by Dekel, Fudenberg, and Morris (2006a) is in fact the weakest topology such that the ε-rationalizable correspondences are both upper and lower-hemicontinuous.[1] It follows, that a type u_i is a critical type if and only if the product topology is strictly weaker than the strategic topology at the point u_i.

Our characterization of critical types establishes the connection between continuity of behavior and common-belief in events, as defined by Monderer and Samet (1989). Say that a subset W_i of hierarchies for i is an upper-contour set if it includes all hierarchies that are larger under $\succeq$ than the hierarchies in W_i. Our first main theorem shows that u_i is a critical type if and only if there exists an upper contour set W_i that is a closed (in the product topology) proper subset of the universal type space for i such that for some $p > 0$, u_i exhibits common p-belief in the set W_i. We show that nearly all types ever employed in applied models satisfy this condition and are therefore critical types.[2] In particular, all types from finite type spaces and almost all types in common prior type spaces are critical.

For applied analysis, our results have the following implications. When we use common-priors, finite type spaces or other simple models, we are implicitly imposing some common-belief assumptions. Often these type spaces are used for convenience and these implicit assumptions are unintended consequences, indeed they are typically too subtle to even see or state explicitly. But they bring with them behavioral consequences that may just as well be unintended. Indeed, for any critical type there are types which coincide with these up to arbitrarily high orders of the belief hierarchy but do not satisfy any common-belief assumption. These types are usually harder to describe and therefore less convenient, but they may do just as well as models for the informational assumptions that the analyst has explicitly in mind. Our results show that there always exist games in which the behavior of

[1] Dekel, Fudenberg, and Morris (2006a) define the strategic topology by a metric and show that convergence of a sequence in this topology is equivalent to convergence of rationalizable behavior. Because convergent sequences do not uniquely define a topology, their analysis leaves open the question of whether there exists a weaker, non-metric topology which characterizes continuity of rationalizable behavior.

[2] We know of no example of a type space used in applications which does not consist entirely of critical types.

these types is very different than the behavior of the critical type. The conclusions we draw from simple type spaces must therefore be interpreted with caution.

Nevertheless, our second main result shows that critical types are rare. Indeed, their complement in the universal type space (which we call the set of *regular* types) is a residual set: generic in the product topology. This result strengthens the earlier result due to Weinstein and Yildiz (2006) in two ways. First, in Weinstein and Yildiz (2006) a specific game was held fixed so that the regular types they found were only guaranteed to exhibit continuous behavior in that given game. We find a generic set of regular types which by construction have continuous behavior across all games.[3] Second the results in Weinstein and Yildiz (2006) require an assumption that the space of payoff-relevant events is sufficiently rich that all actions are dominant actions for some types. Our results require no assumption about the space of payoff-relevant events apart from the standard technical assumption that it is a compact Polish space. On the other hand, Weinstein and Yildiz (2006) obtain generic *uniqueness* of rationalizable actions, a property that would not hold without some assumption of this type. We draw further comparisons with Weinstein and Yildiz (2006) in subsubsection 1.2.3.

Our results on the special nature of commonly used type spaces extend an earlier insight of Morris (2002). In that paper, it was shown that all types in finite type spaces and types in continuum type spaces with bounded densities exhibit discontinuous rationalizable behavior in a specific, infinite, "higher-order expectations" game. Our contribution here is to characterize exactly what makes a type critical, namely common-belief in certain events. Indeed, the crucial property identified in Morris (2002), namely that the higher-order expectations of these types converge to a cycle, is a special case of our characterization. Our characterization reveals the internal source of the discontinuity and also allows us to go further and identify additional classes of critical types, for example common prior type spaces attach probability one to critical types.

1.1. **Overview of Results.**

In this subsection we briefly preview some of our main results. The exposition here is informal and precise definitions are deferred to the main body of the paper.

Throughout, an underlying space Ω of basic uncertainty is held fixed. The elements of Ω are called states of nature. We consider the universal type space $U_i(\Delta\Omega)$ introduced in

[3]We follow Ely and Peski (2006) and Dekel, Fudenberg, and Morris (2006a) in characterizing types in terms of their behavior in all games. That is, we fix the space of hierarchies of beliefs over fundamental uncertainty and allow the game to vary. In addition to being convenient and instructive, it is essential for applications such as mechanism design where the game is not fixed but instead chosen as a response to the players' information.

Ely and Peski (2006).[4] The elements of $U_i(\Delta\Omega)$ represent players' hierarchies of beliefs as originally modeled by Mertens and Zamir (1985), but suitably enriched in order to capture the determinants of interim rationalizable behavior. The space $U_i(\Delta\Omega)$ has a natural product topology which captures convergence of higher-order beliefs.

A two-player game G of incomplete information is defined by, for each i, a set of actions A_i, and a payoff function $g_i : A_1 \times A_2 \times \Omega \to \mathbf{R}$ depending on the pair of actions chosen and the state. For any game G and any type $u_i \in U_i(\Delta\Omega)$, we can identify the set of interim rationalizable actions for type u_i, denoted $R_i(u_i|G)$. Similarly, given $\varepsilon > 0$, the set of interim ε-rationalizable actions is denoted $R_i(u_i|G, \varepsilon)$.[5]

We can partially-order the types in $U_i(\Delta\Omega)$ according to their rationalizable behaviors. Take two hierarchies u_i and u_i'. We say that $u_i \succeq u_i'$ if u_i has a weakly larger set of ε-rationalizable actions than u_i' in all games, for all $\varepsilon \geq 0$, i.e. $R_i(u_i'|G, \varepsilon) \subseteq R_i(u_i|G, \varepsilon)$. A set $V \subseteq U_i(\Delta\Omega)$ is called an *upper-contour set* if V includes all hierarchies that are larger than those in V under the relation $\succeq$. Formally $V = \cup_{v_i \in V}\{u_i : u_i \succeq v_i\}$.

Monderer and Samet (1989) introduced the concept of common p-belief. A hierarchy u_i exhibits common p-belief in a subset $V \subseteq U_i(\Delta\Omega)$ if $u_i \in V$, and u_i assigns probability at least p to the set of hierarchies for $-i$ who assign probability at least p to V, etc. A formal definition is given in Section 4.2 below. We let $C_i^p(V)$ denote the set of i's hierarchies for which it is common p-belief that i's hierarchy is in V. Our characterization of critical types is based on the following two lemmas.

Lemma 1. *Let $V \subset U_i(\Delta\Omega)$ (strict inclusion.) For any $p > 0$, the complement of $C_i^p(V)$ is dense in the product topology. If V is closed in the product topology, then the complement of $C_i^p(V)$ is open.*

Thus, given any type u_i, there is a sequence of types which do not have common p-belief in V but whose higher-order beliefs converge to those of u_i. In fact, our proof of Lemma 1 shows that for every k, there is a hierarchy in the complement of $C_i^p(V)$ which is *identical* to u_i up to order k.

[4]We showed in Ely and Peski (2006) that the space $U_i(\Delta\Omega)$ was the minimal type space for characterizing interim rationalizability. All of the results in this paper have counterparts for the alternative concept of interim correlated rationalizability introduced by Dekel, Fudenberg, and Morris (2007).

[5]To be precise, the solution concept of interim rationalizability is not defined directly on hierarchies, but rather on Harsanyi type spaces which are implicit models of hierarchies. But any two types with the same hierarchies in $U_i(\Delta\Omega)$ will have the same rationalizable actions in all games, regardless of the type space to which they belong. We may thus think of the rationalizable correspondence as depending only on hierarchies. See Ely and Peski (2006) and also Theorem 1 below.

Lemma 2. *Let $W \subset U_i(\Delta\Omega)$ (strict inclusion) be an upper contour set which is closed in the product topology. For $p > 0$ small enough, for any $\varepsilon > 0$ there exists $V \subset U_i(\Delta\Omega)$ (strict inclusion), a game G, and an action a_i such that $W \subseteq V$ and*

(1) *If $u_i \in C_i^p(W)$ then a_i is interim-rationalizable for u_i.*
(2) *If $u_i \notin C_i^{p-2\varepsilon(1-p)}(V)$ then a_i is not interim-ε-rationalizable for u_i.*

This lemma says that we can always find a game with an action whose rationalizability hinges on whether there is common p-belief in certain events. There is a simple intuition behind the Lemma. We construct a coordination game with a pair of actions (a_i, a_{-i}) such that player i plays a only if her hierarchy of beliefs belongs to V and she believes with probability at least p that the opponent plays a_{-i} On the other hand, player $-i$ plays a_{-i} only if she believes with probability at least p that player i plays a_i

Putting these lemmas together provides a sufficient condition for a type to be critical. Suppose that type t_i has a hierarchy u_i which exhibits common p-belief in some product-closed, upper-contour, proper subset of $U(\Delta\Omega)$. Then by the first lemma, for any set V, there is a sequence of hierarchies which do not have common $p/2$-belief in V and whose higher-order beliefs converge to those of u_i. By the second lemma, there exists a game G and an action a_i such that a_i is rationalizable for t_i but not ε-rationalizable for any type along the sequence, where $\varepsilon = \frac{p}{4(1-p)}$. Thus there is a discontinuity[6] in rationalizable behavior at the type t_i, i.e. t_i is a critical type.

In fact we show the converse: critical types are exactly the types whose hierarchies satisfy common p-belief of this form for some $p > 0$. The necessity part is a consequence of the following lemma:

Lemma 3. *For any game G, any player i, there is a product-closed, upper contour, proper subset $W_i \subset U_i(\Delta\Omega)$ and $p^* > 0$, such that for any $p \leq p^*$, for any type with hierarchy $u_i^* \notin C_i^p(W_i)$, there is an open neighborhood $V \ni u_i^*$ such that if action a_i is interim rationalizable for any type with a hierarchy u_i^*, then, it is $6p$-interim rationalizable for any type with a hierarchy $u_i \in V$.*

The proof of Lemma 3 shows that there exists a set of actions $A^* \subseteq A_i$ and an open, lower contour set of hierarchies $U = U_i(\Delta\Omega) \backslash W$, such that $a_i \in A^*$ is a rationalizable action for any type with a hierarchy in U. Suppose that $p > 0$ is small and find any type with hierarchy $u_i^* \notin C_i^p(U_i(\Delta\Omega) \backslash U)$. Such a type believes with a probability at least $1-p$ that player $-i$ believes with a probability at least $1-p$ that hierarchy of beliefs of i belongs to set U, or that i believes with a probability at least $1-p$ that ... (at most, finitely many times) that player i's hierarchy of beliefs belongs to U. We use the fact that U is open to show that any

[6]Precisely, there is a failure of lower hemi-continuity.

action that is rationalizable for a type with hierarchy u_i^* is also $6p$-rationalizable for types with hierarchies sufficiently close to u_i^*. (Note that set A^* plays a similar role to the dominant action in Weinstein and Yildiz (2006). Because we can find A^* with appropriate properties for any game G, we do not need their richness assumption.)

Suppose now that u_i does not satisfy any non-trivial common p-belief statement:

$$u_i \notin \bigcup_{p>0} \bigcup_{\substack{W \subset U_i(\Delta\Omega), \\ W \text{ is closed and upper contour}}} C_i^p(W).$$

It follows from Lemma 3 that u_i is not critical.

Finally, we show that *regular* types, i.e. those that are not critical, are generic. For any $p > 0$ and closed proper subset W, the set of types which do not have common p-belief in W is open and dense. By the previous results, the set of regular types is the intersection of all such sets. We show there is a countable collection of closed upper-contour sets W^k such that the regular types consist of all hierarchies which do not have common p belief in W^k for all k and for all rational p. Thus, the set of regular types is a countable intersection of open and dense sets, hence residual in the product topology.

1.2. **Examples.** In this subsection, we present some examples to illustrate critical and regular types and other aspects of the results.

1.2.1. *Partial Order.* An important ingredient in our characterization is a partial order on types. We say that $u_i \succeq u_i'$ if u_i has a weakly larger set of ε-rationalizable actions than u_i' in all games, for all $\varepsilon \geq 0$, i.e. $R_i(u_i'|G,\varepsilon) \subseteq R_i(u_i|G,\varepsilon)$. To see that this is indeed a non-trivial order, recall the example from Ely and Peski (2006), reproduced below.

	-1	$+1$
-1	0	1/4
$+1$	1/4	0

$\omega = -1$

	-1	$+1$
-1	1/4	0
$+1$	0	1/4

$\omega = +1$

FIGURE 1. A type space

The figure illustrates a type space over a space of basic uncertainty containing two elements, $\omega \in \{-1, +1\}$. There are two players, each with two types, also labeled $\{-1, +1\}$. The type space has a common prior and the tables show the probabilities of various type-profile/state combinations. We can compare this type to a simpler type space in which each player has exactly one type, labeled $*$ and the common-prior attaches equal probability to the two states.

Let us first compare, for player 1 say, type $*$ with any of the types from Figure 1, say $+1$. There is a close connection between their best-reply correspondences. Recall that a game is an action set and a payoff function mapping action profiles and states into utilities. For any game, take any action a played by type $*$ of player 2, and consider the set of best-replies for type $*$ of player 1. This is exactly the set of best-replies for type $+1$, to the strategy of player 2 which plays a irrespective of type. The same argument applies to ε-best-replies. Thus, any action which can be a best-reply for $*$ is also a best-reply for type $+1$. It follows that (the hierarchy represented by) $+1$ is weakly larger than (the hierarchy represented by) $*$. Indeed the ordering is strict. As demonstrated by the example in Ely and Peski (2006), there are games in which the set of rationalizable actions for $+1$ strictly includes the set of rationalizable actions for $*$. On the other hand, the types $+1$ and -1 have the same Best-response correspondences and therefore their hierarchies are equivalent under the ordering.

This example illustrates a partial characterization of the ordering $\succeq$ that can be stated directly in terms of belief-hierarchies. First $u_i \sim u_i'$ if and only if $u_i = u_i'$. That is, two types have the same rationalizable actions in all games if and only if they have the same Δ-hierarchies. This was the main result in Ely and Peski (2006) and the result is restated below in Theorem 1. Second, $u_i \succeq u_i'$ only if u_i and u_i' represent the same Mertens-Zamir hierarchies of belief. This follows from a result in Dekel, Fudenberg, and Morris (2006a), namely that for any two types with distinct Mertens-Zamir hierarchies there is a game in which they have mutually disjoint rationalizable sets.

1.2.2. *Finite Types.* We now turn to some examples illustrating our characterization of critical types. Critical types are pervasive. In most models of incomplete information used in applied analysis, simple type spaces are used for convenience. For example, finite type spaces are often adopted for analytical simplicity. All types in finite type spaces are critical types. We can derive this result from our characterization as follows. Let T be a finite type space. First, consider the finite subset W of the universal type space corresponding to the hierarchies represented by the types in T_i. The set W is common-knowledge for all of the hierarchies in W. Next, for each of the hierarchies u_i in W, consider the corresponding upper-contour set $\{v_i : v_i \succeq u_i\}$. This is a closed, proper subset of the universal type space. The fact that it is closed follows from the result that rationalizable behavior is upper hemi-continuous in the product topology (see Dekel, Fudenberg, and Morris (2006a).) Because it contains only hierarchies that represent the same Mertens-Zamir hierarchy, it is a proper subset. It follows that the set $V := \cup_{u_i \in W}\{v_i : v_i \succeq u_i\}$ is an upper-contour proper subset that is closed (as the union of finitely many closed sets) and includes W. Since W is common-knowledge among all the hierarchies in W, it follows that V is also common-knowledge among these hierarchies.

We have found a closed, upper-contour proper subset that is common-knowledge, hence all of the types in T_i are critical.

Rubinstein first illustrated the special nature of finite types in the example of the e-mail game. While the example is now familiar, we will revisit it here as it will be a useful starting point for discussing critical types in more complicated cases.

Two players play a game of incomplete information in which there are two payoff-relevant states $\omega \in \{-1, +1\}$. We first consider the simplest type space in which it is common-knowledge that each player has exactly one type and with probability 1 the state of the world is $+1$. A game is now described by a set of actions for each player and a payoff function giving the players' utilities as a function of the action profile and the state. The Rubinstein game has two actions A and B for each player and the payoffs are given in the following tables.

	A	B
A	3,3	0,0
B	0,0	2,2

$\omega = +1$

	A	B
A	0,3	0,0
B	1,0	1,2

$\omega = -1$

FIGURE 2. The E-mail game

When it is common-knowledge that the state is $+1$, the game is effectively one of complete information and both actions are rationalizable for each player. In particular, the action profile (A, A) is rationalizable and indeed constitutes a (Bayesian) Nash equilibrium. Next we consider a type space in which there is some non-trivial incomplete information. Each player i has a countable set of types $T_i = \{t_i^k\}_{k=0}^{\infty}$. There is a common prior $\rho \in \Delta(\{-1, +1\} \times T_1 \times T_2)$ from which the state and the type profile are drawn. The prior ρ is defined by $\rho(-1, t_1^0, t_2^0) = 1/2$ and for some β less than but close to 1,

$$\rho(+1, t_1^k, t_2^l) = \begin{cases} \frac{(1-\beta)\beta^{k+l}}{2} & \text{if } k = l \text{ or } k = l + 1 \\ 0 & \text{otherwise} \end{cases}$$

For each player, we can view the sequence of types (t_i^k) as approximating the complete-information type from the first type space. Indeed, for every k, the first k orders of belief for the type t_i^k are identical to the first k orders of belief of the complete-information type, namely mutual knowledge of order k (1 knows that 2 knows that ..., k times) that the state is $+1$. Formally, this sequence of hierarchies converges in the product topology to the hierarchy representing complete information.

Rubinstein showed that B is the unique rationalizable action for every type of both players. In fact the action A is not even *approximately* rationalizable for any type. Formally, for $\varepsilon > 0$, an action is an ε-best-reply to some strategy for the opponent, if it earns an expected payoff within ε of the payoff to a best-reply. An action is ε-rationalizable if it survives an iterative procedure of elimination of actions which are not ε-best-replies to the surviving strategies. Action A is not ε-rationalizable for any type of either player for any $\varepsilon < 1/2$. Here is the discontinuity. The minimum ε such that A is ε-rationalizable for the approximating types does not converge to zero, which is the corresponding value for the complete-information type.

1.2.3. *Common Prior Types.* In the Rubinstein example, the "limit" type is a critical type because it exhibits common-knowledge in a single payoff-relevant state. On the other hand, the types in the email information structure do not have common-knowledge in any proper subset of payoff-relevant states. They would therefore seem to be immune to the type of construction we applied to the complete-information type. Nevertheless, we can apply our characterization and show that all of the e-mail types are critical types as well. This is because our characterization of critical types is in terms of common belief in subsets of hierarchies, not states. Common belief in a proper subset of hierarchies is a strictly weaker requirement than common-belief in a subset of payoff-relevant states.

Indeed, we show below in Theorem 5 that in any type space with a common prior, almost all types have common belief in a proper subset of hierarchies, and indeed almost all types are critical types. (Formally, critical types have probability 1 under the common prior.)

We can illustrate this result using the email types from the Rubinstein example. We will show that for each of these types, rationalizable behavior is "discontinuous" as we approximate their hierarchies to higher and higher orders. Indeed, for this example we are able to construct a single game to exhibit the discontinuity for all of the email types. The game is presented below.

	L	$\varnothing$	R
U	$2, 2+\omega$	$0, 5/2$	$2, 2-\omega$
D	$1, 0$	$2, 5/2$	$1, 0$

FIGURE 3. A game showing that the Rubinstein types are critical

This game, together with the Rubinstein information structure, has a Bayesian Nash equilibrium in pure strategies as follows. All types of player 1 play U. All types of player 2 play L with the exception of type 0 who plays R. This is clearly a best-reply for player 1, whose payoffs do not depend on the state. For player 2, the best reply to U is either L or

R depending on the probabilities attached to the two states. Type 0 who, for β close to 1 is almost certain of state -1, optimally chooses R, and all other types, who are certain of state 1 optimally choose L.

We will now show how to construct for each player i, each type t_i^k and for every n, a hierarchy which coincides exactly with the hierarchy represented by t_i^k up to order n. The actions L and R will not be rationalizable for any of the hierarchies we construct for player 2 and the action U will not be rationalizable for any of the hierarchies for 1. To begin with we consider any hierarchy u^0 for player 2 whose first-order belief assigns equal probability to the two states. For any such hierarchy, the action $\varnothing$ is (interim) strongly dominant and hence the unique rationalizable action. Now consider any of the Rubinstein types t_i^k and any n. Consider a hierarchy which coincides with that of t_i^k up to order n and for all orders $m > n$, exhibits mutual knowledge of order $m - n$ that player 2 has hierarchy u^0. It is straightforward but notationally demanding to show the existence of such a hierarchy, either implicitly through a type space or by explicit construction. The formal construction appears below in the proof of Lemma 1.[7]

When player 1 knows that player 2 has hierarchy u^0, player 1 knows that player 2 will play $\varnothing$. The unique best-reply for player 1, regardless of the state, is to play D. Thus, D is the unique rationalizable action for any such hierarchy for 1. When player 2 knows that player 1 knows that 2 has hierarchy u^0, player 2 knows that 1 will play D. Neither action L nor R is a best-reply to D, regardless of the state. Thus, neither L nor R can be rationalizable for such a hierarchy for 1. We can continue the argument to show that for none of the hierarchies we have constructed will U, L or R be rationalizable. In fact, for no ε smaller than 1/2 will any of these actions be even ε-rationalizable.

The construction in this example exploited the fact that for all of the Rubinstein types, it is common-knowledge that the first-order beliefs of both players belong to a proper subset, indeed a finite subset, of the set of all first-order beliefs. However, a similar construction can be applied to types which have only common p-belief for some $p < 1$ in a proper subset of first-order beliefs. Indeed, even types for which there is no common belief in any proper subset of first-order beliefs can be critical types. Our characterization shows that any type which has common p-belief for some $p > 0$ in some proper subset of *hierarchies* (that is closed and upper-contour) is a critical type.

The result that common priors attach probability 1 to critical types may at first seem at odds with the result from Weinstein and Yildiz (2006) that common prior types generically exhibit robust rationalizable behavior. There are two differences between our statement

[7]The difficulty in explicitly constructing the hierarchy is in ensuring that it satisfies the requirement of *coherency* which is a necessary condition for a hierarchy to be derivable from some type space.

and theirs. First, Weinstein and Yildiz (2006) consider a fixed game and characterize the types with robust behavior for that particular game, whereas we are interested in types with robust behavior across all games. This first difference is less important than it appears. For example, suppose that the space of basic uncertainty Ω is finite, and we consider finite action spaces. The set of games is isomorphic to the set of payoff functions over actions and Ω and there is a countable dense set Γ of these. For each of the games G in Γ, there is an open and dense set of common-prior types with robust behavior in G. Thus, as a consequence of the results in Weinstein and Yildiz (2006), the set of types whose behavior is robust in *all* of the games in Γ is, in a topological sense, generic among common-prior types: a countable intersection of open and dense subsets. On the other hand, our results imply that this set is, for practical purposes, vanishingly small. Indeed, in any common-prior type space, the common prior itself attaches probability zero to this set. In applications, common priors are modeled using type *spaces* and not individual types, the negative result would seem to carry the more important message for applied work.

1.3. **Upper Contour Sets.** Because our characterization refers to upper contour sets, it is desirable to have an alternative characterization that is stated completely in terms of the primitive description of a type, i.e. its hierarchy of beliefs. We are able to provide a sufficient condition in these terms by exploiting a result due to Dekel, Fudenberg, and Morris (2006a). They showed that any two types with distinct Mertens-Zamir belief hierarchies have non-nested sets of rationalizable actions.[8] This implies that any set of hierarchies which reduce to the same Mertens-Zamir hierarchy is an upper contour set. Moreover, the mapping which associates hierarchies in $U(\Delta\Omega)$ with their associated Mertens-Zamir hierarchy is continuous and hence any set in $U(\Delta\Omega)$ which maps to a closed (in the product topology) subset of Mertens-Zamir hierarchies is a closed upper-contour set. Thus, any type in $U(\Delta\Omega)$ which exhibits common p-belief in some closed proper subset of Mertens-Zamir hierarchies is a critical type.

This sufficient condition in fact is satisfied by all of the examples above. It is not, however, a necessary condition for a critical type. In Appendix A, following our construction of a regular type, we show how to construct a critical type which does not satisfy this condition.

1.3.1. *Regular Types.* While critical types are pervasive in applications, they are in a formal sense very rare: they form a residual subset of the universal type space relative to the natural product topology on higher-order beliefs. The *regular* types, those with no discontinuities in behavior, are the typical ones. Nevertheless, they are in a certain sense elusive: actually describing a regular type is a serious challenge in its own right. It is thus not surprising that

[8] We quote their result below (12.) To be precise, their result applies to an alternative version of interim rationalizability, but the same proof works for the version of rationalizability we are using here.

they do not appear in applied analysis. Indeed, without the simplifying tools of either finite or common-prior type spaces to implicitly describe hierarchies, we are not well-equipped to describe them at all. In Appendix A we provide a non-constructive description of a regular hierarchy via a type space. Here we give an informal sketch.

The universal type space has a countable, dense subset Q. We consider the set of all finite truncations of the hierarchies in Q. Now we construct a single hierarchy by "stacking" these finite truncations. To see how this is done, take the first k-order belief truncation of a hierarchy u, and the first l-order belief truncation of a hierarchy v. Now construct a $k+l$-order finite hierarchy. We first copy the initial k-orders of belief from u. Then, the orders $k+m$ ranging between $k+1$ and $k+l$ are taken to be k-order mutual certainty of the m-th order belief of v. We continue in this way, interweaving all of the finite truncations of all of the hierarchies in Q. The resulting type u^* is regular. This follows from two observations. First, any set W which is common belief for u^* must include Q. This is because every hierarchy in Q is believed with probability 1 at some level of the hierarchy. Next, since Q is dense, there is no closed, proper subset which includes Q, thus by our characterization u^* is regular.

2. Model

If X is a measurable space, then ΔX refers to the space of Borel probability measures on X. When X is a topological space, we treat X as a measurable space equipped with the Borel σ-algebra. If $f : X \to Y$ is a mapping between two measurable spaces, then we write $\Delta f : \Delta X \to \Delta Y$ for the induced mapping between the corresponding spaces of measures.

We consider two-player games with incomplete information. We fix throughout a space of basic uncertainty Ω. In a game with incomplete information, payoffs depend on action choices as well as the realization of Ω. We assume that Ω is a compact Polish space with at least two elements[9].

The players' uncertainty is modeled by a Harsanyi type space over Ω. A type space over Ω, denoted $T = (T_i, \mu_i)_{i=1,2}$ consists of a pair of measurable spaces T_i and two belief mappings $\mu_i : T_i \to \Delta\left(\Omega \times T_{-i}\right)$. The probability measure $\mu_i(t_i) \in \Delta(\Omega \times T_{-i})$ indicates the belief of type t_i about the basic uncertainty and the type of the opponent. Throughout, we use the notation $C_\Omega \mu_i(t_i)(\cdot) : T_{-i} \to \Delta(\Omega)$ to represent a version of conditional probability over Ω as a function of the opponent's type. We assume that there exist jointly measurable functions

[9]This ensures that there is non-trivial incomplete information and there exists more than one (in fact infinitely many) hierarchies of belief.

$\beta_i : T_i \times T_{-i} \to \Delta\Omega$, such that

$$\beta_i (t_i, t_{-i}) = C_\Omega \mu (t_i) (t_{-i})\,[10].$$

Let $\mathcal{T}(\Omega)$ be the collection of all type spaces over Ω.

A *game form* (or simply game) over Ω is a tuple $G = (A_i, g_i)_{i=1,2}$, where for each i, A_i is a finite set of actions and $g_i : A_i \times A_{-i} \times \Omega \to \mathbf{R}$ is a continuous payoff function.

2.1. **Interim Rationalizability.** We base our analysis on the concept of interim rationalizability. An early definition was given in Morris and Skiadis (2000) for games with finitely many types. Our definition of interim rationalizability generalizes to games with infinitely many types.[11] An alternative concept, interim correlated rationalizability has been proposed by Dekel, Fudenberg, and Morris (2007). We discuss in Section 6 how to extend the results in this paper to interim correlated rationalizability.

Fix a type space $T \in \mathcal{T}(\Omega)$, and a game $G = (A_i, g_i)$. An assessment is a pair of subsets $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i \subseteq T_i \times A_i$. Alternatively an assessment can be defined by the pair of correspondences $\alpha_i : T_i \rightrightarrows A_i$, with $\alpha_i(t_i) := \{a_i : (t_i, a_i) \in \alpha_i\}$. The image $\alpha_i(t_i)$ is interpreted as the set of actions that player i of type t_i could conceivably play.

A *behavioral strategy* for player i is a measurable function $\sigma_i : T_i \to \Delta A_i$. The expected payoff to type t_i of player i from choosing action a_i when the opponent's strategy is σ_{-i} is given by

$$\pi_i(a_i, \sigma_{-i}|t_i) = \int_{T_{-i}\times\Omega} \int_{A_{-i}} g_i(a_i, a_{-i}, \omega) d\sigma_{-i}(t_{-i}) d\mu_i(t_i). \tag{2.1}$$

The strategy σ_i is a selection from the assessment α if for each i, $\sigma_i(t_i) \in \Delta\alpha_i(t_i)$ for all $t_i \in T_i$. Let $\Sigma_i(\alpha_i)$ be the set of all strategies for i that are selections from α.

For any $\varepsilon \geq 0$, an action a_i is an interim $\varepsilon-$best-response for t_i against σ_{-i} if $\pi_i(a_i, \sigma_{-i}|t_i) \geq \pi_i(a'_i, \sigma_{-i}, |t_i) - \varepsilon$ for all $a'_i \in A_{-i}$. Let $B_i(\sigma_{-i}|t_i, \varepsilon)$ denote the set of all interim $\varepsilon-$best-responses for t_i to σ_{-i}. If α an assessment, then $B_i(\alpha_{-i}|t_i; \varepsilon)$ is the set of all $\varepsilon-$best-responses to strategies in $\Sigma_{-i}(\alpha_{-i})$.

An assessment α has the $\varepsilon-$*best-response property* if every action attributed to player i is a interim ε-best-reply to some selection from α_{-i}, i.e.,

$$\alpha_i \subseteq \{(t_i, a_i) : a_i \in B_i (\alpha_{-i}|t_i; \varepsilon)\}$$

If the above is satisfied with equality, then we say that α has the ε fixed-point property.

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[10] Thus, we consider type spaces with strongly measurable beliefs as in Ely and Peski (2006). It has recently been shown by Shmaya (2007) that this is without loss of generality when Ω and the type spaces are Polish.

[11] See Ely and Peski (2006) for additional details.

Definition 1. *Given a type space T, and a game G, the interim $\varepsilon-$rationalizable correspondence is the maximal assessment with the ε-fixed-point property, denoted $R(\cdot|G,\varepsilon)$. We say that a_i is interim $\varepsilon-$rationalizable for type t_i if $a_i \in R_i(t_i|G,\varepsilon)$.*

Ely and Peski (2006) show that the interim $0-$rationalizable correspondence exists and can be understood as a measurable function from T_i to 2^{A_i}. The same is true of the $\varepsilon-$rationalizable correspondence. The next Lemma makes a convenient connection between ε-rationalizability and ε'-rationalizability:

Lemma 4. *For each game $G = (A_j, g_i)$ and each $\varepsilon \geq 0$, there is a game $G' = \left(A'_j, g'_j\right)$, where $A'_j = A_i \times A_{-i}$, such that for any $t_i \in T_i$, any $\varepsilon' \geq 0$*

$$a_i \in R_i\left(t_i|G,\varepsilon+\varepsilon'\right) \text{ if and only if } \left(a_i, a_{-i}\right) \in R_i\left(t_i|G',\varepsilon'\right).$$

Proof. Define payoffs

$$g'_i\left(\left(a^i_i, a^{-i}_i\right), \left(a^{-i}_{-i}, a^i_{-i}\right), \omega\right) := g_i\left(a^i_i, a^{-i}_{-i}, \omega\right) + \varepsilon \cdot \mathbf{1}\left\{a^i_i = a^i_{-i}\right\}.$$

Then, for any player i, her payoffs do not depend on the choice of action a^{-i}_i. They depend on the opponent's choice of action a^i_{-i} - and in particular, player i may get ε payoff additionally if the opponent's choice is equal to i's choice $a^i_i = a^i_{-i}$. □

We will also consider the *strict* ε-rationalizable correspondence, denoted

$$R^\circ\left(\cdot|G,\varepsilon\right) = \bigcup_{\varepsilon'<\varepsilon} R\left(\cdot|G,\varepsilon'\right). \tag{2.2}$$

Thus, an action is strictly ε-rationalizable if it is ε'-rationalizable for some $\varepsilon' < \varepsilon$. In particular, no action is strict 0-rationalizable. [12] We conclude this subsection with the following lemma whose proof is in Appendix C.

Lemma 5. *For any $\varepsilon \geq 0$, any game G,*

$$R_i\left(\cdot|G,\varepsilon\right) = \bigcap_{\varepsilon'>\varepsilon} R_i\left(\cdot|G,\varepsilon'\right).$$

2.2. **The Universal Type Space.** A type space is an implicit description of a player's higher-order beliefs. Our characterization of critical types will be in terms of their hierarchies of beliefs, explicitly described. This ensures that our classification is not dependent on any particular choice of type space.

Mertens and Zamir (1985) consider the space $U(\Omega) = \left(U_i(\Omega), \mu_i\right)_i$, where the belief mapping $\mu_i : U_i(\Omega) \to \Delta\left(\Omega \times U_{-i}(\Omega)\right)$ makes $U(\Omega)$ a type space over Ω. $U_i(\Omega)$ can be defined the set of all coherent hierarchies from infinite product $\prod_{k=1}^{\infty} U^k_i(\Delta\Omega)$ of sequences

[12] Note that this is distinct from iterated elimination of actions which are not ε-strict best responses.

of finite hierarchies. Ely and Peski (2006) consider the universal type space over $X = \Delta\Omega$, called the space of Δ-hierarchies. The higher-order beliefs of a given type implicitly described within a given type space associates this type with a unique element of $U_i(\Delta\Omega)$. Each hierarchy $u_i \in U_i(\Delta\Omega)$ is uniquely associated with a belief which is a probability measure in $\Delta(\Delta\Omega \times U_{-i}(\Delta\Omega))$. We use the symbol u_i interchangeably to refer to the hierarchy or the belief. It turns out that rationalizable behavior of any type is determined by this type's Δ-hierarchy:

Theorem 1 (Ely and Peski (2006))**.** *For any type space $T = (T_i, \mu_i)$, there exist measurable mappings $\phi_i^T : T_i \to U_i(\Delta\Omega)$, such that for any player i, any types $t_i, t_i' \in T_i$, any game G over Ω, any $\varepsilon > 0$*

$$R_i(t_i|G,\varepsilon) = R_i(t_i'|G,\varepsilon) \text{ iff } \phi_i^T(t_i) = \phi_i^T(t_i').$$

The mappings ϕ_i^T preserve beliefs. In particular, $\phi_i^T(t_i) = u_i$ implies that for any measurable $F \subseteq U_{-i}(\Delta\Omega)$,

$$\phi_i^T(F) = \mu_i(t_i)\left[\left(\phi_i^T\right)^{-1}(F)\right].$$

Also, there exists type space over Ω, denoted $L = \left(L_i, \mu_i^L\right)$, such that μ_i^L is continuous in the weak-topology on $\Delta(\Omega \times L_{-i})$, ϕ_i^L is continuous and onto, the inverse correspondence $(\phi_i^L)^{-1} : U_i(\Delta\Omega) \rightrightarrows L_i$ is continuous and the correspondence $R(\cdot|G,\varepsilon) : L_i \rightrightarrows A_i$ is upper hemi-continuous.*

Proof. Ely and Peski (2006) show the Theorem when $\varepsilon = 0$. For $\varepsilon > 0$. apply Lemma 4. ☐

Theorem 1 allows us to consider R_i as a correspondence defined directly on $U_i(\Delta\Omega)$, i.e. $R_i(u_i|G,\varepsilon)$ is the set of ε-rationalizable actions for any type t_i whose Δ-hierarchy is u_i, independent of the type space to which t_i belongs. It follows from the last part of the theorem that this correspondence is upper hemi-continuous.

2.3. **Structure of the Universal Type Space.** The space $U_i(\Delta\Omega)$ has some natural topological and order structure.

Definition 2. *The product topology on $U_i(\Delta\Omega)$ is the Tychonoff topology inherited from the infinite product $\prod_{k=1}^{\infty} U_i^k(\Delta\Omega)$.*

By standard results, this topology is separable and metrizable. Throughout the paper, we write $u_i^n \to u_i$ to denote convergence in the product topology.

We can partially order the hierarchies in $U_i(\Delta\Omega)$ according to the partial ordering of their rationalizable actions across games. Two hierarchies are ordered if the first has a smaller set of rationalizable actions in every game.

Definition 3. *For any $u_i, v_i \in U_i(\Delta\Omega)$, we write $v_i \preceq u_i$ iff for any game G and any $\varepsilon \geq 0$, $R(v_i|G,\varepsilon) \subseteq R(u_i|G,\varepsilon)$.*

Definition 4. *Set $A \subseteq U_i(\Delta\Omega)$ is a* lower contour *set if*

$$A = \{v_i : \text{there is } u_i \in A, \text{ s.t. } v_i \preceq u_i\}.$$

Set $A \subseteq U_i(\Delta\Omega)$ is an upper contour *set if*

$$A = \{v_i : \text{there is } u_i \in A, \text{ s.t. } u_i \preceq v_i\}.$$

Hence, set A is lower contour if it contains all hierarchies that are smaller with respect to the relation "$\preceq$" from all hierarchies in set A. Observe that if A is a lower contour set, then $U_i(\Delta\Omega) \setminus A$ is an upper contour set. One can show that upper contour sets are closed in the product topology.[13]

3. Strategic topologies on hierarchies of beliefs

We are interested in a topology on types which is derived from strategic behavior. Dekel, Fudenberg, and Morris (2006a) introduced one such topology connected to correlated interim rationalizability. Here, for both solution concepts, we will investigate several natural alternative definitions and show that they are all equivalent.

3.1. Strategic topology.

Perhaps the most natural approach is to derive the topology from the rationalizable correspondences. It is known from more familiar contexts, such as complete information games, that the ε-rationalizable correspondence is upper hemi-continuous and the strict ε-rationalizable correspondence is lower hemi-continuous. Recall that if Y and Z are two topological spaces, then a correspondence $F : Y \to Z$ is upper hemi-continuous if for every open $U \subseteq Z$, the strong inverse image $\{y \in Y : F(y) \subseteq U\}$ is open, and lower hemi-continuous if the weak inverse image $\{y \in Y : F(y) \cap U \neq \emptyset\}$ is open. This leads to our first definition.

Definition 5 (Strategic Topology). *The* strategic topology *on $U_i(X)$ is the coarsest topology such that for every $\varepsilon \geq 0$, the ε-rationalizable correspondence is upper hemi-continuous and the strict ε-rationalizable correspondence is lower hemi-continuous.*

[13]Indeed, suppose that $u_i \preceq v_i^n$ for each n. Then, for each action a_i, if $a_i \in R(u_i|G,\varepsilon)$, then $a_i \in R(v_i^n|G,\varepsilon)$ for each n. If $v_i^n \to v_i$, then, because rationalizable correspondence is u.h.c., $a_i \in R(v_i|G,\varepsilon)$.

A sub-basis[14] for the strategic topology thus consists of sets of the form

$$\{u_i : R(u_i|G,\varepsilon) \subseteq B\}$$
$$\{u_i : R^\circ(u_i|G,\varepsilon) \cap B \neq \emptyset\}$$

for all $\varepsilon > 0$ games G and subsets B of actions in G. Note that upper hemi-continuity obtains if and only if the first class of subsets is included and lower hemi-continuity if and only if the second class of subsets is included. Since rationalizable correspondence is upper hemi-continuous in product topology, the sets of the first class are open in product topology. Sets from the second class are not necessarily open.

3.2. **h-topology.** Another familiar property of these correspondences leads to an alternative definition. For any game G and action a_i, define

$$h\left(a_i, G, u_i\right) = \inf\{\varepsilon : a_i \in R_i(u_i|G,\varepsilon)\}$$

In games with complete information, where there is a single state and hence u_i is fixed, the function h is a continuous function of the payoffs. It is therefore natural to seek a topology on types which reproduces this continuity.

Definition 6 (h-topology)**.** *The* h-topology *is the coarsest topology on $U_i(X)$ such that for each fixed G and a_i, the function $h(a_i, G, \cdot)$ is continuous.*

A sub-basis for the h-topology is thus the following collection of open sets for $\varepsilon > 0$

$$\{u_i : h(a_i, G, u_i) > \varepsilon\},$$
$$\{u_i : h(a_i, G, u_i) < \varepsilon\}.$$

Again, the two subcollections separately correspond to upper and lower hemi-continuity respectively.

3.3. **C-topology.** Next, we can motivate a definition by considering sequences that "should" converge.

Definition 7. *A sequence u_i^k lower-converges to a limit type u_i if for every action a_i that is 0-rationalizable for u_i, there is a sequence $\hat{\varepsilon}^k \downarrow 0$ such that a_i is $\hat{\varepsilon}^k$-rationalizable for u_i^k. The sequence upper-converges to u_i if for every action a_i that is not 0-rationalizable for u_i, there is a k' such that a_i is not 0-rationalizable for u_i^k for all $k > k'$.*

[14]Recall that a collection of sets $\mathcal{B}$ is a *basis* for a topology if every open set is a union of sets in $\mathcal{B}$. A collection $\mathcal{B}$ is a *sub-basis* if $\mathcal{B}$ together with all of its finite intersections forms a basis. The resulting topology is said to be *generated* by $\mathcal{B}$.

These convergence notions were proposed by Dekel, Fudenberg, and Morris (2006a). While the requirement that these sequences converge does not in general "define" a topology, it suggests a natural topology which we call the C-topology.

Definition 8 (C-topology)**.** *The C-topology is generated by the following class of sets for $\varepsilon > 0$*

$$\{u_i : a_i \notin R(u_i|G, \varepsilon)\},$$
$$\{u_i : a_i \in R_i^\circ(u_i|G, \varepsilon)\}.$$

The first class clearly corresponds to upper-convergence. The connection between the second class and lower-convergence deserves some explanation. Note that a_i is ε-rationalizable if and only if it is strictly $\tilde{\varepsilon}$-rationalizable for all $\tilde{\varepsilon} > \varepsilon$. Thus, the collection $\{u_i : a_i \in R_i^\circ(u_i|G, \tilde{\varepsilon})\}$ for $\tilde{\varepsilon} > \varepsilon$ is a family of neighborhoods for the limit type u_i. The sequence u_i^k enters such a neighborhood iff a_i is $\hat{\varepsilon}^k$-rationalizable for some $\hat{\varepsilon}^k < \tilde{\varepsilon}$.

3.4. **Metric topology.** Finally, we define a metric, which leads to a metric topology on the space of hierarchies. For any two games G, G' with the same action sets $A = A_1 \times A_2$, we denote a distance between games

$$\delta(G, G') = \max_i \sup_{a \in A, \omega \in \Omega} |g_i(a, \omega) - g_i'(a, \omega)|.$$

Let $\mathcal{G}_m$ be the subclass of games with at most m actions whose payoffs are bounded by 1, i.e. $\max_{a \in A, \omega \in \Omega} |g_i(a, \omega)| \leq 1$. Because Ω is compact, space $\mathcal{G}_m$ is separable in the topology on games induced by $\delta(G, G')$. In particular, there is a countable subset $\mathcal{G}_m^* \subseteq \mathcal{G}_m$ such that for each $G \in \mathcal{G}_m$, for any $\varepsilon > 0$, there exists $G' \in \mathcal{G}_m^*$ so that $d(G, G') \leq \varepsilon$. Let $\mathcal{G}^* = \bigcup \mathcal{G}_m^*$. Because $\mathcal{G}^*$ is countable, there exists enumeration of all elements of $\mathcal{G}^*$. Let $G_1, G_2, ... \in \mathcal{G}^*$ be such an enumeration. Then, for any $0 < \beta < 1$, let

$$d(u_i, u_i') = \sum_{k=1}^{\infty} \beta^k \sup_{a_i \in A_i} |h(a_i, G_k, u_i) - h(a_i, G_k, u_i')| \tag{3.1}$$

As in Dekel, Fudenberg, and Morris (2006a), one shows that the above is a proper metric.[15]

Definition 9 (Metric Topology)**.** *The* metric topology *is the topology generated by the metric (3.1).*

[15] Dekel, Fudenberg, and Morris (2006a) work with a slightly different definition of metric topology

$$d_{DFM}(u_i, u_i') = \sum_{m=1}^{\infty} \beta^m \sup_{G \in \mathcal{G}_m} \sup_{a_i \in A_i} |h(a_i, G, u_i) - h(a_i, G, u_i')|$$

These two topologies are equivalent when Ω is finite (as it is assumed in Dekel, Fudenberg, and Morris (2006a)).

3.5. **Equivalence.** We show that all definitions of topology from above are equivalent.

Theorem 2. *The strategic topology is equivalent to topologies h and C and the metric topology.*

The Theorem is proved in the Appendix B.

4. Critical Types

4.1. **Regular and critical hierarchies.** We have defined a critical type to be a type with a hierarchy of belief such that changes in beliefs at arbitrarily high order can have a discontinuous effect on rationalizable behavior. Precisely, at a critical type, there is either a failure of upper hemi-continuity of the ε-rationalizable correspondence or there is a failure of lower hemi-continuity of the strict ε-rationalizable correspondence, relative to the product topology on hierarchies of belief. Given our definition/characterization of the strategic topology as the weakest topology yielding continuity of these correspondences, we have the following formal definition of a critical type.

Definition 10. *We say that hierarchy $u_i \in U_i(\Delta\Omega)$ is* regular *if for any set $u_i \in V_i \subseteq U_i(\Delta\Omega)$, V_i open in the strategic topology, there is a set U_i, open in the product topology, such that $u_i \in U_i \subseteq V_i$. We say that hierarchy is* critical *if it is not regular.*

We are going to say that type is critical (or regular) if it has critical (or regular) hierarchy of beliefs. The critical types are those around which the product topology is strictly weaker than the strategic topology. In the remainder of this section, we will characterize critical types in terms of a version of common belief.

4.2. **Common Belief.** Fix subsets of hierarchies, $W_j \subseteq U_j(\Delta\Omega)$, for $j = 1, 2$. The set of hierarchies for player j that p-believe in W_{-j} is given by

$$B_j^p(W_{-j}) = \{u_j \in U_j(\Delta\Omega) : u_j(\Delta\Omega \times W_{-j}) \geq p\}.$$

For the product event $W = W_1 \times W_2$, we define

$$B_j^p(W) = W_j \cap B_j^p(W_{-j}) \tag{4.1}$$

$$B^p(W) = B_j^p(W) \times B_{-j}^p(W) \subseteq U(\Delta\Omega). \tag{4.2}$$

Note that $B^p(W) \subseteq W$. Common p-belief in W occurs when both players p-believe in W, and both players p-believe in $B^p(W)$, and .... This concept was introduced by Monderer and Samet (1989). Formally,

$$C^p(W) = \bigcap_{k \geq 1} [B^p]^k (W).$$

We have the following version of the original characterization due to Monderer and Samet (1989).

Lemma 6. *Let $W \subseteq U(\Delta\Omega)$ be a product event. Then $C^p(W)$ is a product event, i.e. $C^p(W) = \prod_{j=1,2} C^p_j(W)$ and*

$$C^p_j(W) = W_j \cap B^p_j C^p_{-j}(W) = B^p_j \left(\bigcap_{k \geq 0} [B^p]^k (W) \right).$$

Abusing notation, when $W_j \in U_j(\Delta\Omega)$ we can view it implicitly as the product event $W_j \times U_{-j}(\Delta\Omega)$ and write

$$B^p(W_j) := B^p(W_j \times U_{-j}(\Delta\Omega)$$
$$C^p_j(W_j) := C^p_j \left(W_j \times U_{-j}\left(\Delta\Omega\right)\right).$$

4.3. **Characterization of Critical Types.** The main result of the paper characterizes the set of regular hierarchies.

Theorem 3. *A hierarchy $u^0_i \in U_i(\Delta\Omega)$ is critical if and only if there exists $p > 0$ and a closed upper contour proper subset $W_i \subset U_i(\Delta\Omega)$, such that*

$$u^0_i \in C^p_i(W_i).$$

Proof. The proof applies Lemma 1 and Lemma 2 stated in the introduction. The proofs of these lemmas are in the following subsections. Suppose that $u^0_i \in C^p_i(W_i)$ for some closed upper contour proper subset $W_i \subset U_i(\Delta\Omega)$. Let $\varepsilon = \frac{p}{4(1-p)}$. Then, by Lemma 2, there exists $V \subset U_i(\Delta\Omega)$ and a game G with an action a_i, such that

$$a_i \in R_i(u_i|G, 0)$$

but

$$a_i \notin R_i(v_i|G, \varepsilon)$$

for any $v_i \notin C^{p/2}_i(V)$. By Lemma 1, there is a sequence of hierarchies $u^n_i \to u^0_i$, and $u^n_i \notin C^{p/2}_i(V)$. Hence, u^0_i is critical.

Now, suppose that $u^0_i \notin C^p_i(W_i)$ for every closed upper contour subset $W_i \subset U_i(\Delta\Omega)$ and every $p > 0$. We will show that u^0_i is regular, i.e. for any set $V \ni u^0_i$, such that V is open in the strategic topology, there is a set O with $u^0_i \in O \subseteq V$ and O is open in the product topology.

We can assume w.l.o.g. that V is an element of the subbasis of the strategic topology. By Theorem 2, strategic topology is equivalent to C-topology (Definition 8). Suppose that first for some $\varepsilon > 0$, game G and action a_i

$$V = \{u_i : a_i \notin R(u_i|G, \varepsilon)\}.$$

Then our result follows from the fact that the rationalizable correspondence is u.h.c. on the universal type space $U_i(\Delta\Omega)$ (see remarks after Theorem 1).

Next, suppose that for some $\varepsilon > 0$, game G and action $a_i \in A_i$

$$V = \{u_i : a_i \in R_i^\circ(u_i|G, \varepsilon)\}.$$

By the definition of the correspondence $R^\circ(\cdot)$ in equation (2.2), there is $0 < \varepsilon' < \varepsilon$, such that

$$u_i^0 \in \{u_i : a_i \in R_i(u_i|G, \varepsilon')\} \subseteq V.$$

By Lemma 4, there is a game G' and action a', such that

$$\begin{aligned} u_i^0 &\in \{u_i : a' \in R_i(u_i|G', 0)\} \\ &\subseteq \{u_i : a' \in R_i(u_i|G', \varepsilon - \varepsilon')\} \\ &= \{u_i : a_i \in R_i(u_i|G, \varepsilon')\} \subseteq V. \end{aligned}$$

Let $W_i \subset U_i(\Delta\Omega)$ be the closed upper contour set and $p^* > 0$ be the probability given by Lemma 3, and let $p = \min\left(p^*, \frac{\varepsilon - \varepsilon'}{6}\right) > 0$. Since $u_i^0 \notin C_i^p(W_i)$, Lemma 3 implies that there is an open neighborhood O such that

$$u_i^0 \in O \subseteq \{u_i : a' \in R_i(u_i|G', \varepsilon - \varepsilon')\} \subseteq V.$$

. □

Recall that subset of topological space is residual if it is a countable intersection of dense and open sets.

Theorem 4. *The set of regular types forms a residual subset (in the product topology) of* $U_i(\Delta\Omega)$.

Proof. For any $p > 0$ and nonempty closed $W_i \subset U_i(\Delta\Omega)$, the set $C_i^p(W_i)$ is closed as the intersection of closed sets. Hence, $U_i(\Delta\Omega) \backslash C_i^p(W_i)$ is open. By Lemma 1, it is also dense. Notice also that if $W \subseteq W'$, then $C_i^p(W) \subseteq C_i^p(W')$.

Note that for any two open sets $W_i \subseteq W_i'$, any $p' \leq p$,

$$C_i^p(W_i) \subseteq C_i^{p'}(W_i').$$

Find a sequence of open sets $U_i^1, U_i^2, ... \subseteq U_i(\Delta\Omega)$ such that for any open lower contour set $U' \subseteq U_i(\Delta\Omega)$, there is n, such that $U_i^n \subseteq U'$. Such a sequence exists, since the space $U_i(\Delta\Omega)$ is separable and metrizable, hence, it has countable basis. The set of regular hierarchies of player i is equal to

$$\bigcap_{p>0, U_i \subseteq U_i(\Delta\Omega), U_i \text{ is open and lower-contour}} U_i(\Delta\Omega) \setminus C_i^p(U_i(\Delta\Omega) \setminus U_i)$$
$$= \bigcap_n \bigcap_m U_i(\Delta\Omega) \setminus C_i^{1/n}(U_i(\Delta\Omega) \setminus U_i^m)$$

and is therefore residual as an intersection of a countable family of open and dense sets. □

4.4. **Proof of Lemma 1.** To show that set $U_i(\Delta\Omega)/C_i^p(V)$ is dense, let u_i be an arbitrary hierarchy for i. For each odd integer k, we will construct a hierarchy $z_i \notin C_i^p(V)$ which agrees with u_i up to order $k-1$. The sequence of such hierarchies $(z)^k$ converges to u_i in the product topology.

Let $T = (T_j, \mu_j)$ be any type space such that there is a type t_i^* for player i which has hierarchy u_i and there is also a type for i which has a hierarchy that is not in V. We begin by constructing an alternate type space T' which has represents the same hierarchies as T but has a convenient structure. The idea is to "factorize" T into infinitely many replicas.

For each odd positive integer k, let T^k be a set and $\eta^k : T_i \to T^k$ a bijective mapping between T^k and i's space of types in T. Similarly, for k even, let η^k be a bijection between T_{-i} and a set T^k. We construct a type space T' in which the set of types for player i is

$$T_i' = \bigcup_{k\,\text{odd}} T^k$$

and the set of types for player $-i$ is

$$T_{-i}' = \bigcup_{k\,\text{even}} T^k.$$

The belief mapping μ' is derived from μ as follows. For each k, and for $j = 1, 2$

$$\mu_j'(\eta^k(t_j)) = \Delta(\mathrm{id}_{\Delta\Omega} \times \eta^{k+1})[\mu_j(t_j)]$$

It is immediate that the hierarchy of beliefs for any type $\eta^k(t_j)$ is identical to that of t_j.

We now take any type y_i whose hierarchy is not in V. Fix any odd $\bar{k} \geq 3$. We construct a new type space from T' by redefining the belief mapping. Define $\hat{\mu}_{-i} = \mu_{-i}'$ and for player i,

$$\hat{\mu}_i(\varphi^k(t_i)) = \begin{cases} \mu_i'(y_i) & \text{if } k = \bar{k} \\ \mu_{-i}'(\eta^k(t_i)) & \text{otherwise.} \end{cases}$$

Consider the type space with type sets T' and belief mappings $\hat{\mu}$. In this type space, every type for i in $T^{\bar{k}}$ has the hierarchy of y_i. Every type $\eta^{\bar{k}-1}(t_{-i})$ for $-i$ in $T^{\bar{k}-1}$ has the same first-order belief as t_{-i} but is certain that the opponent's hierarchy is that of y_i. Every type $\eta^{\bar{k}-2}(t_i)$ for i in $T^{\bar{k}-2}$ has the same first- and second-order beliefs as t_i but is certain that the opponent is certain that his own hierarchy is that of y_i. Continuing inductively, the type

$\eta^1(t_i^*)$ has a hierarchy of beliefs z_i which coincides with that of t_i^*,(i.e. u_i) up to order $\bar{k}-1$, but is certain that the opponent is certain that ... that his own hierarchy is that of y_i. That is,

$$z_i \in \underbrace{B_i^1 B_{-i}^1 \dots B_i^1 B_{-i}^1}_{\bar{k}-1 \text{ times}} (\{y_i\}),$$

and so in particular, $z_i \notin C_i^p(V)$ for any $p > 0$ since $y_i \notin V$. □

4.5. **Proof of Lemma 2.** We begin with the following preliminary result, proved in Section D.4. It provides a game with three important features. First, the action set has a product structure and the first dimension of i's action is irrelevant for $-i$'s payoffs. Second, the rationalizable correspondence has a product structure. Finally, there is a distinguishing subset of actions for player i that are rationalizable only for a proper subset of types that includes W.

Lemma 7. *Fix player i. Let $W_i \subseteq U_i(\Delta\Omega)$ (strict inclusion) be a closed upper contour set. For any type $u_i^* \notin W_i$, there is $\varepsilon > 0$, a game $G = (A_j, g_j)$, such that $A_i = A_i^0 \times A_i^1$ and*

(1) *Payoffs of player $-i$ do not depend on the a_i^0-dimension of player's i action: for any $a_{-i} \in A_{-i}$, any $a_i^1 \in A_i^1$, any $a_i^0, \hat{a}_i^0 \in A_i^0$, any ω,*

$$g_{-i}\left(a_{-i}, \left(a_i^0, a_i^1\right), \omega\right) = g_{-i}\left(a_{-i}, \left(\hat{a}_i^0, a_i^1\right), \omega\right).$$

(2) *There are correspondences $A^0 : U_i(\Delta\Omega) \rightrightarrows A_i^0, A^1 : U_i(\Delta\Omega) \rightrightarrows A_i^1$, such that for any u_i,*

$$R_i\left(u_i|G, 0\right) = A^0\left(u_i\right) \times A^1\left(u_i\right).$$

(3) *There is a nonempty subset $A^{0*} \subseteq A_i^0$,*

$$\left[A^{0*} \times A^1\left(u_i\right)\right] \subseteq R_i\left(u_i|G, 0\right) \text{ for any } u_i \in W_i,$$
$$\left[A^{0*} \times A^1\left(u_i^*\right)\right] \cap R_i\left(u_i^*|G, \varepsilon\right) = \emptyset.$$

To prove Lemma 2, consider the following game G^*, derived from the game G given in Lemma 7. The set of actions for j is $A_j \times \{0, 1\}$, i.e. the product of the actions from G, with a binary coordinate which we denote z_j. The payoffs to an action profile

$$a^* = \left(\left(a_i, z_i\right), \left(a_{-i}, z_{-i}\right)\right)$$

are as follows.

$$g_i^*(a^*,\omega) = g_i(a,\omega) + \begin{cases} 1 & \text{if } z_i = z_{-i} = 1, \\ \frac{-p}{1-p} & \text{if } z_i = 1 \text{ and } z_{-i} = 0, \\ 0 & \text{if } z_i = 0. \end{cases}$$

$$g_{-i}^*(a^*,\omega) = g_{-i}(a,\omega) + \begin{cases} 1 & \text{if } z_{-i} = z_i = 1 \text{ and } a_i^0 \in A^{0*}, \\ \frac{-p}{1-p} & \text{if } z_{-i} = 1 \text{ and } (a_i^0 \notin A^{0*} \text{ or } z_i = 0), \\ 0 & \text{if } z_{-i} = 0. \end{cases}$$

The game G^* is simply G augmented with a binary coordination game where coordination is desirable only if an element of the distinguished set A^{0*} is chosen by player i.

Define the following sets of hierarchies.

$$V = \left\{u_i : A^{0*} \times A^1(u_i) \subseteq R_i(u_i|G,0)\right\},$$
$$V^\varepsilon = \left\{u_i : A^{0*} \times A^1(u_i) \cap R_i(u_i|G,\varepsilon) \neq \emptyset\right\}.$$

By the third part of Lemma 7, $W \subseteq V \subseteq V^\varepsilon$ and V^ε is a proper subset of $U_i(\Delta\Omega)$. Let

$$Z = A^{0*} \times A^1 \times \{1\}$$

be the set of actions for i in G^* whose first coordinate belongs to A^{0*} and whose last coordinate is $z_i = 1$. We show in two steps that:

- If $u_i \in C_i^p(V)$, then $Z \cap R_i(u_i|G^*,0) \neq \emptyset$
- If $u_i \notin C_i^p(V^\varepsilon)$, then $Z \cap R_i(u_i|G^*,\varepsilon) = \emptyset$.

In order to do this, we use the definition of rationalizable correspondence as the maximal assessment on some type space that has the best-response property. From now on, fix a type space $T = (T_j, \mu_j)$. By Theorem 1, we can assume that each hierarchy of beliefs is represented in this type space. Recall that any type space can be mapped into $U(\Delta\Omega)$ via the mappings $\phi_j^T : T_j \to U_j(\Delta\Omega)$. If, for example, $\phi_i^T(t_i) \in B_i^p(W_{-i})$, for some $W_{-i} \subseteq U_{-i}(\Delta\Omega)$, then we will conserve on notation and write, e.g. $t_i \in B_i^p(W_{-i})$, and we will use repeatedly the fact that the mappings $\phi_i^T(t_i)$ preserve beliefs (see Theorem 1), so that, for example, $t_i \in B_i^p(W_{-i})$ is equivalent to $\mu_i(t_i)\left[\left(\phi_{-i}^T\right)^{-1}(W_{-i})\right] \geq p$.

Step 1: We first show that the assessment α, where

$$\alpha_j(t_j) = \begin{cases} R_j(t_j|G,0) \times \{1\} & \text{for } t_j \in C_j^p(V), \\ R_j(t_j|G,0) \times \{0\} & \text{for } t_j \notin C_j^p(V). \end{cases}$$

has the best response property. Thus, by Definition 1, for $t_i \in C_i^p(V)$, $R_i(t_i|G,0) \times \{1\} \subseteq R_i(t_i|G^*,0)$. Since $C_i^p(V) \subseteq V$, the definition of V then implies $Z \cap R_i(t_i|G^*,0) \neq \emptyset$. Let us start with player $j = -i$ and type t_{-i}. Take any $a_{-i} \in R_{-i}(t_{-i}|G, 0)$. By definition, the rationalizable correspondence for G has the best-response property, so let σ_i be a behavioral strategy of player i in game G that makes a_{-i} a best response for t_{-i} and that

$$\sigma_i(t_i)(R_i(t_i|G, 0)) = \sigma_i(t_i)\left(A^0(t_i) \times A^1(t_i)\right) = 1 \text{ for any } t_i.$$

We used property 2 of Lemma 7 in the above equality. By property 1 we can choose σ_i so that for any type $t_i \in C_i^p(V) \subseteq V$

$$\sigma_i(t_i)\left(A^{0*} \times A^1(t_i)\right) = 1.$$

Now we use σ_i to define a behavioral strategy for player i in G^*. Let

$$\sigma_i^*(t_i) = \begin{cases} (\sigma_i(t_i), 1) & \text{for } t_i \in C_i^p(V), \\ (\sigma_i(t_i), 0) & \text{for } t_i \notin C_i^p(V). \end{cases}$$

Let q be the probability type t_{-i} assigns to the event $t_i \in C_i^p(V)$. We will first show that the action $(a_{-i}, 1)$ is a best-reply to σ_i^* for any type $t_{-i} \in C_{-i}^p(V)$. In this case, by Lemma 6, $q \geq p$. Since a_{-i} is a best-reply to σ_i, for any $a'_{-i} \in A_{-i}$, the difference in payoffs between actions $(a_{-i}, 1)$ and $(a_{-i}, 0)$.

$$\pi_{-i}((a_{-i}, 1), \sigma_i^*|t_{-i}) - \pi_{-i}((a_{-i}, 0), \sigma_i^*|t_{-i}) \geq q - \frac{p}{1-p}(1-q) \geq 0.$$

Next, consider $t_{-i} \notin C_{-i}^p(V)$. In this case, by Lemma 6, $q \leq p$. We now show that the action $(a_{-i}, 0)$ is a best-reply to σ_i^*. For any $a'_{-i} \in A_{-i}$,

$$\pi_{-i}((a_{-i}, 0), \sigma_i^*|t_{-i}) - \pi_{-i}((a_{-i}, 1), \sigma_i^*|t_{-i}) \geq \frac{p}{1-p}(1-q) - q \geq 0.$$

The same argument applies for player $j = i$.

Step 2: Let $q = p - 2\varepsilon(1-p)$. Observe that

$$U_i(\Delta\Omega) \backslash C_i^q(V^\varepsilon) = U_i(\Delta\Omega) \setminus V^\varepsilon \cup \bigcup_{k\geq 1} U_i(\Delta\Omega) \setminus \left[B_i^q B_{-i}^q\right]^k (V^\varepsilon)$$

By definition, $Z \cap R_i(t_i|G^*, \varepsilon) = \emptyset$ for any type $t_i \in U_i(\Delta\Omega) \backslash V^\varepsilon$. We are going to show by induction on k that no action $(a_j, 1)$ is ε-interim rationalizable for any type $t_i \notin \left[B_i^q B_{-i}^q\right]^k (V^\varepsilon)$. By the induction hypothesis, if $t_{-i} \notin B_{-i}^q \left[B_i^q B_{-i}^q\right]^{k-1} (V^\varepsilon)$, then t_{-i} assigns probability at most q to the set of types t_i for i such that the intersection $Z \cap R_i(t_i|G^*, \varepsilon)$ is non-empty. Thus, for any $a_{-i} \in A_{-i}$, for any strategy σ_i of player i, such that $\sigma_i(t_i) \in \Delta R_i(t_i|G^*, \varepsilon)$,

$$\pi_{-i}((a_{-i}, 1), \sigma_i^*|t_{-i}) - \pi_{-i}((a_{-i}, 0), \sigma_i^*|t_{-i}) \leq -\frac{p}{1-p} + q\left(1 + \frac{p}{1-p}\right) = \frac{q-p}{1-p} = -2\varepsilon.$$

Thus, $(a_{-i}, 1)$ is not ε-interim rationalizable for t_{-i}. Now let t_i be a type for i such that $t_{-i} \notin B_i^q B_{-i}^q \left[B_i^q B_{-i}^q\right]^{k-1} (V^\varepsilon)$. The previous argument implies that t_{-i} assigns probability at most q to types of $-i$ for whom $z_{-i}=1$ is part of a ε-rationalizable action. Hence, for any action $a_i \in A_i$, for any strategy σ_{-i} such that $\sigma_{-i}(t_{-i}) \in \Delta R_{-i}(t_{-i}|G^*,\varepsilon)$

$$\pi_i((a_i,1),\sigma^*_{-i}|t_i) - \pi_i((a_i,0),\sigma^*_{-i}|t_i) \leq -\frac{p}{1-p} + q\left(1+\frac{p}{1-p}\right) = \frac{q-p}{1-p} = -2\varepsilon.$$

By property 2 of Lemma 7, $R_i(t_i|G,\varepsilon) = A^0(t_i) \times A^1(t_i)$. By definition, for any $t_i \notin V^\varepsilon$, $A^0(t_i) \cap A^{0*} = \varnothing$. Hence, for any $t_i \notin V^\varepsilon$, $Z \cap R_i(t_i|G^*,\varepsilon) = \varnothing$ completing the proof of step 2.

4.6. **Proof of Lemma 3.** Fix player i and a game G. Define a collection of subsets of actions

$$\mathcal{A}_\varepsilon = \{A'_i \subseteq A_i : R_i(u_i|G,\varepsilon) = A'_i \text{ for some } u_i.\}$$

The collection $\mathcal{A}_\varepsilon$ is a non-empty collection of non-empty sets and it is ordered with respect to ε in the following way: for any $\varepsilon' < \varepsilon$, any $A'_i \in \mathcal{A}_\varepsilon$, there is $A''_i \in \mathcal{A}_{\varepsilon'}$ such that $A''_i \subseteq A'_i$. Since the number of all actions is finite, the number of all subsets of actions is finite and, therefore, there is $\varepsilon^A > 0$, such that for all $0 \leq \varepsilon \leq \varepsilon^A$, we have $\mathcal{A}_\varepsilon = \mathcal{A}_0$ Let A^*_i be a minimal element of $\mathcal{A}_0$, i.e. $A_i \in \mathcal{A}_0$ and there is no $Z_i \subsetneq A^*_i$ belonging to $\mathcal{A}_0$. Define

$$U^A = \left\{u_i : A^*_i = R_i\left(u_i|G,\varepsilon^A\right)\right\}.$$

Then, U^A is open because the correspondence $R_i(\cdot|G,\varepsilon) : U_i(\Delta\Omega) \rightrightarrows A_i$ is u.h.c. It is a non-empty lower contour set because of the choice of set A^*_i. Finally, for any $0 \leq \varepsilon \leq \varepsilon^A$, for any $u_i \in U^A$, $A^*_i = R(u_i|G,\varepsilon)$.

Take any $p \leq \frac{\varepsilon^A}{6}$. By Lemma 6, if $u^*_i \notin C^p_i(W_i)$, then

$$\begin{aligned} u^*_i &\in U_i(\Delta\Omega) \backslash C^p_i(W_i) \\ &= U_i(\Delta\Omega) \backslash B^p_i\left(\bigcap_{k\geq 0} [B^p]^k (W_i)\right) \\ &= U_i(\Delta\Omega) \backslash \bigcap_{k\geq 0} B^p_i\left([B^p]^k (W_i)\right) \\ &= \bigcup_{k\geq 0} U_i(\Delta\Omega) \backslash B^p_i\left([B^p]^k (W_i)\right). \end{aligned}$$

Lemma 3 follows from the next result, proved in the appendix.

Lemma 8. *Consider the closed, upper contour proper subset*

$$W_i = U_i(\Delta\Omega) \backslash U^A$$

For any $p < \frac{\varepsilon^A}{6}$, any player j, any $k \geq 0$, any

$$u^0_j \notin B^p_j\left([B^p]^k (W_i)\right),$$

any sequence $u_j^n \to u_j^0$ convergent in the product topology, and any action $a_j \in A_j$, such that $a_j \in R\left(u_j^0|G,0\right)$, there is n^* sufficiently high, so that for any $n \geq n^*$, $a_j \in R\left(u_j^n|G,6p\right)$.

Thus, for any $u_i^* \notin C_i^p(W_i)$, there is an open neighborhood $V \ni u_i^*$, such that if action a_i is interim rationalizable for any type with a hierarchy u_i^*, then, it is $6p$-interim rationalizable for any type with a hierarchy $u_i \in V$.

5. Common prior and critical hierarchies

We will show that almost all types from types spaces with a common prior are critical. Let $T = (T_i, \mu_i)$ be a type space over X. Say that $\psi \in \Delta\left(T_i \times T_{-i}\right)$ is a *common prior* on T if for any bounded measurable function $f : T_i \times T_{-i}$, any player i

$$\psi\left[f\left(t_i, t_{-i}\right)\right] = \int \left(\mu_i\left(t_i\right)\left[f\left(t_i, t_{-i}\right)\right]\right) d\psi_i\left(t_i\right),$$

where $\psi_i = \operatorname{marg}_{T_i} \psi$. This is a non-standard definition. We do not require in particular that the common prior is a measure also over uncertainty Ω. In a sense, ours is a weaker definition and, as a consequence, the subsequent result is stronger.

Theorem 5. *Suppose that ψ is a common prior on a type space $T = (T_i, \mu_i)$. Then, for each player i, t_i has critical hierarchy ψ_i-almost surely.*

The Theorem is a corollary to three lemmas. The first lemma says that any common prior on type space over Ω corresponds to a common prior on the universal type space $U\left(\Delta\Omega\right)$.

Lemma 9. *For any common prior ψ on type space T, there is a common prior ψ^* on the universal type space $U\left(\Delta\Omega\right)$, such that for any measurable $E_i \subseteq U_i\left(\Delta\Omega\right)$ for each player $i = 1, 2$,*

$$\psi^*\left(E_1 \times E_2\right) = \psi\left(\left(u_1^T\right)^{-1}\left(E_1\right) \times \left(u_2^T\right)^{-1}\left(E_2\right)\right).$$

The second result says that the support of a common prior on $U\left(\Delta\Omega\right)$ can be approximated by closed upper contour sets.

Lemma 10. *For any common prior ψ^* on the universal type space $U\left(\Delta\Omega\right)$, any $\epsilon > 0$, there are upper contour closed proper subsets $V_j \subset U_j\left(\Delta\Omega\right)$, such that*

$$\psi^*\left(V_1 \times V_2\right) \geq 1 - \varepsilon.$$

The following is a version of one-side of the critical path lemma by Morris-Shin.

Lemma 11. *Let ψ^* be a common prior on type space $U\left(\Delta\Omega\right)$. For any measurable sets $V_i \subseteq U_i\left(\Delta\Omega\right)$, there are measurable subsets $S_i \subseteq V_i$, such that*

$$\psi^*\left(S_1 \times S_2\right) \geq \frac{3}{2}\psi^*\left(V_1 \times V_2\right) - \frac{1}{2}, \tag{5.1}$$

and for any player i, any type $u_i \in S_i$,

$$u_i\left(\Delta\Omega \times S_{-i}\right) \geq \frac{1}{4}. \tag{5.2}$$

Hence, $S_1 \times S_2 \subseteq C^{1/4}\left(S_1 \times S_2\right) \subseteq C^{1/4}\left(V_1 \times V_2\right).$

We can prove the Theorem:

Proof of Theorem 5. Suppose that ψ^* is a common prior on the universal type space $U\left(\Delta\Omega\right).$ By Lemma 10, for any $\varepsilon > 0$, there are closed, upper contour, proper subsets $V_j \subset U_j\left(\Delta\Omega\right)$, such that $\psi^*\left(V_1 \times V_2\right) \geq 1-\varepsilon$. Next, Lemma 11 implies that

$$\psi^*\left(C^{1/4}\left(V_1 \times V_2\right)\right) \geq 1 - \frac{3}{2}\varepsilon.$$

Hence, for any player i, with ψ^*-probability at least $1-\frac{3}{2}\varepsilon$, player i's hierarchy is critical. Since the latter is true for any $\epsilon > 0$, it means that ψ^*-almost all hierarchies are critical. Therefore, if ψ is a common prior on type space T over Ω, then, Lemma 9 implies that ψ-almost all types are critical. □

5.1. **Proof of Lemma 9.** Recall that the types in T are mapped via ϕ^T to the universal type space $U(\Delta\Omega)$. This lemma is an immediate consequence of the fact that ϕ^T preserves beliefs. See Theorem 1.

5.2. **Proof of Lemma 10.** We begin with an observation. Let $\zeta_i^T : T_i \to U_i\left(\Omega\right)$ be the Mertens-Zamir mapping assigning types to their hierarchies of beliefs over Ω. Dekel, Fudenberg, and Morris (2006a) show the following Lemma. (To be precise, they state this result for interim correlated rationalizability, but their proof applies unchanged to interim independent rationalizability.)

Lemma 12. *[Dekel, Fudenberg, and Morris (2006a)]For any two types t_i, t_i', if $\zeta_i^T\left(t_i\right) \neq \zeta_i^T\left(t_i'\right)$, then there is a game G and an action a_i such that $a_i \in R_i\left(t_i|G,0\right)$ and $a_i \notin R\left(t_i'|G,0\right)$.*

The Lemma has two implications. First, together with Theorem 1, it implies that there is a continuous onto mapping $\upsilon_i : U_i\left(\Delta\Omega\right) \to U_i\left(\Omega\right)$, such that for any type space and a type $t_i \in T$,

$$\upsilon_i\left(\phi_i^T\left(t_i\right)\right) = \zeta_i^T\left(t_i\right).$$

Second, for any hierarchies $u_i' \succeq u_i$, any type space and types $t_i, t_i' \in T_i$, such that $\phi_i^T\left(t_i\right) = u_i$ and $\phi_i^T\left(t_i'\right) = u_i'$ and for any game G, it must be that $\zeta_i^T\left(t_i\right) = \zeta_i^T\left(t_i'\right)$.

For any $\rho > 0$, $w_i \in U_i\left(\Omega\right)$, let

$$V_i\left(w_i, \rho\right) := U_i\left(\Delta\Omega\right) \backslash \upsilon^{-1}\left(B\left(w_i, \rho\right)\right),$$

where $B(w_i, \rho)$ is an open ball in the universal type space $U_i(\Omega)$ with center at w_i and radius at ρ. By the above, each $V(w_i, \rho)$ is a nonempty, closed, upper contour subset of $U_i(\Delta\Omega)$. For ρ small enough, $V(w_i, \rho)$ is a proper subset of $U_i(\Delta\Omega)$. Finally, since the cardinality of $U_i(\Delta\Omega)$ is infinite, for any $\varepsilon > 0$, there exist w_j and $\rho > 0$ for any $j = 1, 2$, such that $\psi^*(V_i \times V_{-i}) \geq 1 - \varepsilon$.

5.3. **Proof of Lemma 11.** Define inductively sets: $V_i^{(0)} = V_i$

$$V_i^{(n+1)} = \left\{ u_i \in V_i^{(n)} : u_i\left(\Delta\Omega \times V_{-i}^{(n)}\right) \geq \frac{1}{4} \right\}.$$

To save on redundant repetitions, in what follows we drop the term "$\Delta\Omega\times$" from the product over which beliefs are defined. Let

$$S_i = \bigcap_{n \geq 0} V_i^{(n)}.$$

Since the sequence of sets $\left(V_i^{(n)}\right)$ is decreasing, for any player i, any $u_i \in S_i$, (5.2) holds.

Recall that ψ_i denotes the marginal prior over types of player i. Notice that

$$V_1^{(n+1)} \times V_2^{(n+1)} = V_1^{(n)} \times V_2^{(n)} \backslash \bigcup_{i=1,2} \left(V_i^{(n)} \backslash V_i^{(n+1)}\right) \times V_{-i}^{n}.$$

By definition,

$$\begin{aligned} \psi^*\left(\left(V_i^{(n)} \backslash V_i^{(n+1)}\right) \times V_{-i}^{n}\right) &= \int_{U_i(\Delta\Omega)} u_i\left(\Delta\Omega \times V_{-i}^{(n)}\right) d\psi_i^*(u_i) \\ &\leq \frac{1}{4}\psi_i^*\left(V_i^{(n)} \backslash V_i^{(n+1)}\right). \end{aligned}$$

Hence,

$$\begin{aligned} &\psi^*\left(V_1^{(n+1)} \times V_2^{(n+1)}\right) \\ &\geq \psi^*\left(V_1^{(n)} \times V_2^{(n)}\right) - \frac{1}{4}\psi_1^*\left(V_1^{(n)} \backslash V_1^{(n+1)}\right) - \frac{1}{4}\psi_2^*\left(V_2^{(n)} \backslash V_2^{(n+1)}\right). \end{aligned}$$

By passing to the limit, we obtain

$$\psi^*(S_1 \times S_2) \geq \psi^*(V_1 \times V_2) - \frac{1}{4}\left(\psi_1^*(V_1 \backslash S_1) + \psi_2^*(V_2 \backslash S_2)\right).$$

On the other hand, for each player i,

$$\psi_i^*(V_i \backslash S_i) \leq 1 - \psi_i^*(S_i) \leq 1 - \psi^*(S_1 \times S_2).$$

Together with the above inequality, this implies that

$$\frac{1}{4}\left(\psi_1^*(V_1 \backslash S_1) + \psi_2^*(V_2 \backslash S_2)\right) \leq \frac{1}{2}\left(1 - \psi^*(V_1 \times V_2)\right),$$

and Equation 5.1 follows.

6. Interim Correlated Rationalizability

All of the results in this paper have counterparts for the solution concept of interim correlated rationalizability introduced by Dekel, Fudenberg, and Morris (2006a).

Theorem 3 has a simpler statement in this case. Hierarchy $w_i \in U_i(\Omega)$ is critical iff there exists a $p > 0$ and a closed, proper subset $U \subset U_i(\Omega)$ such that $w_i \in C_i^p(U)$. When we consider correlated rationalizability, then the partial order $\succeq$ is trivial, and the relevant universal type space is $U_i(\Omega)$. Also, the statement for correlated rationalizability is true for any number of players.

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Appendix A. Example of a Regular Hierarchy

The basic idea was described in the introduction. Let Q be a countable dense subset of $U_i(\Delta\Omega)$ (recall that $U_i(\Delta\Omega)$ is compact.) Let $\mathbb{N}$ denote the set of natural numbers and let $\mathbb{N}_{\text{even}}$ and $\mathbb{N}_{\text{odd}}$ denote the sets of even and odd numbers. Fix mappings $y : \mathbb{N} \to Q$ and $z : \mathbb{N} \to \mathbb{N}_{\text{even}}$ such that the mapping

$$m \to (y(m), z(m))$$

is a bijection between $\mathbb{N}$ and $Q \times \mathbb{N}_{\text{even}}$. Such mappings exist because $Q \times \mathbb{N}_{\text{even}}$ is countable. A regular hierarchy will be constructed by "stacking" all finite hierarchies in Q. The arrangement is coordinated by mappings $m \to (y(m), z(m))$. Begin with $m = 1$. The first $z(1)$ orders of beliefs are taken equal to the first $z\,(1)$ orders of beliefs of hierarchy $y\,(1)$. Move to $m = 2$. Construct $z\,(1) + z\,(2)$ orders of beliefs by assuming that player i 1-beliefs that player $-i$ 1-beliefs that ... ($z\,(1)$ times) that $\acute{\imath}$'s first $z\,(2)$ orders of beliefs are equal to the first $z\,(2)$ orders of beliefs of hierarchy $y\,(2)$. This corresponds to stacking the first $z\,(2)$ order belief of hierarchy $y\,(2)$ behind $z\,(1)$ order belief of hierarchy $y\,(1)$. Thus, at each step m, we will have constructed orders of beliefs ranging from 1 to $\sum_{m'<m} z\,(m)$.

Rather than to offer a direct construction of a hierarchy, it will be easier to construct a type space and a type with a given hierarchy. For each $q \in Q$, there is a type space $\hat{T}^q$ containing a type $\hat{t}_i^q$ such that $\phi_i(\hat{t}_i^q) = q$. Let $\hat{\mu}_j^q$ denote the belief mappings in $\hat{T}^q$. Similar to the proof of Lemma 1 we shall construct for each $q \in Q$ type sets T_i^q and T_{-i}^q as follows. For each $c \in \mathbb{N}$, let $T_j^q(c)$ be a copy of $\hat{T}_j^q$ and

$$\eta_j^{q,c} : \hat{T}_j^q \to T_j^q(c)$$

be an isomorphism. Define

$$T_i^q = \bigcup_{c \in \mathbb{N}_{\text{odd}}} T_i^q(c),$$

$$T_{-i}^q = \bigcup_{c \in \mathbb{N}_{\text{even}}} T_{-i}^q(c).$$

We now consider a type space T^* where for $j = 1, 2$

$$T_j^* = \bigcup_{q \in Q} T_j^q.$$

We define for each natural number m

$$c(m) = \sum_{m' \leq m} \mathbf{1}_{y(m')=y(m)} z(m')$$

and we let E be the image of the mapping $m \to (y(m), c(m))$, i.e.

$$E = \{(q, c) : (q, c) = (y(m), c(m)) \text{ for some } m\}.$$

We can now define the beliefs of an arbitrary $t_j \in T_j^*$. We proceed in stages c. By construction $t_j = \eta_j^{q,c}(\hat{t}_j) \in T_i^q(c)$ for some $q \in Q$, $c \in \mathbb{N}$ and $\hat{t}_j \in \hat{T}_j^q$. First consider the case that $(q, c) \notin E$. In this case we take beliefs of type t_j to be equal to the beliefs of its representative $\hat{t}_j$ in the type space $\hat{T}^q$, except that they are concentrated on the corresponding representative in $T_{-j}^q(c+1)$,

$$\mu_j(t_j) = \Delta\left(\mathrm{id}_{\Delta\Omega} \times \eta_{-j}^{q,c+1}\right)\left[\hat{\mu}_j^q(\hat{t}_j)\right]. \tag{A.1}$$

Next, suppose $(q, c) \in E$ so that $(q, c) = (y(m), c(m))$. Observe that c must be even and take any type $t_i \in T_i^q(c)$. At such a "transition stage," all of the types have the same belief and they assign probability 1 to the representative of the distinguished type $\hat{t}_i^{y(m+1)}$. Define $q' = y(m+1)$ and $c' = c(m+1) - z(m+1) + 1$. We take $\mu_i(t_i)$ to be any belief which assigns probability 1 to the type $\eta_{-i}^{q',c'}(\hat{t}_{-i}^{q'})$.

We will prove that every type in T^* is regular. We will use the following lemma whose proof is straightforward and omitted. If a type u_i exhibits finite-order mutual certainty of some event V, then V must have non-empty intersection with any event W in which u_i has common-belief.

Lemma 13. *Let V and W be subsets of the universal type space. Suppose that for some u_i, both*

(1) $u_i \in (B_i^1 B_{-i}^1)^k V$ *for some k, and*
(2) $u_i \in C_i^p(W)$ *for some $p > 0$.*

Then $W \cap V \neq \emptyset$.

Let u_i be an arbitrary type in T^* and let c^* denote the copy $T_i^q(c)$ in which u_i resides. For any $q \in Q$, let $V^n(q_i)$ denote the set of hierarchies which coincide with q up to order n. We first show that for every q in Q and for every sufficiently large n, there exists a k such that $u_i \in (B_i^1 B_{-i}^1)^k V^n(q)$.

To prove this, let m be such that $y(m) = q$, $c(m) - z(m) \geq c^*$ and denote $z(m) = n$. Consider the type $\eta_j^{q,c}(\hat{t}_j^q)$ where $c = c(m) - z(m) + 1$. Because c is odd, this is a type for player i, i.e. $j = i$. Furthermore, $c \neq c(m)$ and hence $(q, c) \notin E$. Thus, by Equation A.1, This type has the beliefs of $\hat{t}_i^q$, concentrated on $T_{-i}^q(c+1)$, a copy of T_{-i}^q. Furthermore, the next $z(m) - 1$ stages of the type space are also copies of T_{-i}^q, and so the type $\eta_j^{q,c}(\hat{t}_j^q)$ has a hierarchy of belief which coincides with that of $\hat{t}_i^q$ up to order $z(m) = n$. Recall that $\hat{t}_i^q$ has hierarchy q. Thus the hierarchy of $\eta_j^{q,c}(\hat{t}_j^q)$ belongs to $V^n(q)$.

Now consider any type in stage $c - 1$, i.e. the transition stage $c(m) - z(m)$. Any such type is certain that the opponent is of type $\eta_j^{q,c}(\hat{t}_j^q)$ and therefore that the opponent has a hierarchy of belief in $V^n(q)$. By induction backward through the stages, we conclude that all types for player i in stages below c, including u_i, have hierarchies in $(B_i^1 B_{-i}^1)^k V^n(q)$ for some k.

We are now in a position to prove that u_i is a regular type. Let W be any closed subset of $U_i(\Delta\Omega)$ and suppose $u_i \in C_i^p(W)$ for some $p > 0$. Fix $q_i \in Q$. Since $u_i \in (B_i^1 B_{-i}^1)^k V^n(q_i)$, by Lemma 13, $W \cap V^n(q_i) \neq \emptyset$ and moreover this is true for all sufficiently large n. Let $w^n \in W \cap V^n(q_i)$. Since $w^n \in V^n(q_i)$, the sequence w^n converges to q_i in the product topology. Since $w^n \in W$ and W is closed, we have $q_i \in W$. Since q_i was arbitrary, we have shown that $Q \subset W$. Since Q is dense and W is closed we conclude $W = U_i(\Delta\Omega)$.

This establishes that u_i does not have common-belief in any closed *proper* subset of $U_i(\Delta\Omega)$ and therefore u_i is regular. □

A.1. **Counterexample to the Sufficient Condition.** In the introduction we argued that any type which has common-belief in some closed, proper subset of Mertens-Zamir hierarchies is a critical type. Here we sketch how to construct a critical type which has no common belief in any closed proper subset of Mertens-Zamir hierarchies. The construction builds on the regular types described above.

We construct a type space T^{**} as follows. The types for j are $T_j^* \times \Omega$. Thus each type has a "label" in Ω. The beliefs of a type $(t_j^*, \omega) \in T_j^{**}$ are derived from those of t_j^* as follows. Let $\iota : T_{-j}^* \times \Omega \to T_{-j}^{**} \times \Omega$ be given by $\iota_j = \left(\mathrm{id}_{T_{-j}^*} \times \mathrm{id}_\Omega\right) \times \mathrm{id}_\Omega$.

$$\mu_j^{**}(t_j^*, \omega) = \Delta\iota_j \left[\mu_j^*(t_j^*)\right].$$

Observe that this specification implies that the conditional beliefs over Ω conditional on the opponent's type having label ω assign probability 1 to the state ω. This means in particular that all of the types in T^{**} have the extreme Delta-hierarchies within their given Mertens-Zamir equivalence class: it is common-knowledge that both players believe that their opponent's type is perfectly correlated with the payoff-relevant state.

It can be shown that Delta-Hierarchies with this property are maximal under the partial ordering $\succeq$. Thus, the hierarchies represented in T^{**} form an upper-contour set. Furthermore, because the hierarchies in T^*, were dense, and because the transformation to T^{**} does not alter the Mertens-Zamir hierarchy of any type, the Mertens-Zamir hierarchies in T^{**} are dense in the set of all Mertens-Zamir hierarchies. It follows that each hierarchy in T^{**} has common-belief in a closed upper contour set and is therefore critical, but there is no closed proper subset of Mertens-Zamir hierarchies which can be common-belief for any hierarchy in T^{**}.

Appendix B. Equivalence of topologies

Proposition 1. *The strategic topology is equivalent to both topologies h and C.*

Proof. First note that

$$\{u_i : h(a_i, G, u_i) > \varepsilon\} = \{u_i : a_i \notin R(u_i|G, \varepsilon)\}$$

and

$$\{u_i : h(a_i, G, u_i) < \varepsilon\} = \{u_i : a_i \in R_i^\circ(u_i|G, \varepsilon)\}$$

so that the h and C topologies are equivalent. Next,

$$\{u_i : R^\circ(u_i, \varepsilon) \cap B \neq \emptyset\} = \cup_{a \in B}\{u_i : a \in R^\circ(u_i|G, \varepsilon)\}$$
$$\{u_i : R(u_i, \varepsilon) \subset B\} = \cap_{a \notin B}\{u_i : a \notin R(u_i|G, \varepsilon)\}$$

so that the strategic topology is at least as fine as the C topology. Also,

$$\{u_i : a \in R^\circ(u_i|G, \varepsilon)\} = \{u_i : R^\circ(u_i, \varepsilon) \cap \{a\} \neq \emptyset\}$$
$$\{u_i : a \notin R(u_i|G, \varepsilon)\} = \{u_i : R(u_i, \varepsilon) \subset \neg\{a\}\}$$

so that the C topology is at least as fine as the strategic topology. □

The next lemma establishes an important continuity property of the mappings h.

Lemma 14. *If G and G' are two games with the same set of actions A_i for each player, then*

$$|h_i(a_i, G, t_i) - h_i(a_i, G', t_i)| \leq 2\delta(G, G')$$

for all t_i and for all $a_i \in A_i$.

Proof. Recall the definition of function π_i in (2.1) and observe that for all t_i and a_i

$$\left|\pi_i^G(a_i, \sigma_{-i}|t_i) - \pi_i^{G'}(a_i, \sigma_{-i}|t_i)\right| \leq \delta(G, G'). \tag{B.1}$$

Pick any $\varepsilon > 0$ and consider the ε-rationalizable correspondence $R(\cdot|G, \varepsilon)$ for game G. For each $(t_i, a_i) \in R(\cdot|G, \varepsilon)$, there is a behavioral strategy $\sigma_{-i} \in \Sigma_{-i}(R(\cdot|G, \varepsilon))$ of the opponents, such that a_i is an ε-best-response of type t_i in game G. By (B.1), a_i is a $[\varepsilon + 2\delta(G, G')]$-best-response in game G'. Since this bound is independent of t_i and a_i, it follows that

$$R(\cdot|G, \varepsilon) \subset R(\cdot|G, \varepsilon + 2\delta(G, G'))$$

which implies

$$h(a_i, G', t_i) \leq h_i(a_i, G, t_i) + 2\delta(G, G').$$

The lemma follows from switching the roles of G and G' in the preceding argument. □

Finally we show that the strategic topology is equivalent to the metric topology. Thus, metric (3.1) generates the weakest topology consistent with continuity of the rationalizable correspondences. This provides connection between the metric topology used in Dekel, Fudenberg, and Morris (2006a) and the convergence properties they propose. In fact, we can dispense with their assumptions of finite Ω and uniformly bounded payoffs.

Proposition 2. *The metric topology is equivalent to the strategic topology.*

Proof. We show that for any hierarchy u_i and set $U \ni u_i$ open in one topology, there is set V open in the other topology and such that $u_i \in V \subseteq U$.

It is enough to show this relation for all sets U and V in the sub-bases of the respective topologies. For any hierarchy $u_i \in U_i$, any $\varepsilon > 0$, define

$$V^{\varepsilon}(u_i) = \{u_i' : d(u_i, u_i') < \varepsilon\}.$$

Then sets $V^{\varepsilon}(u_i)$ form a sub-basis for the metric topology: for any set V open in metric topology, for any $u_i \in V$, there is $\varepsilon > 0$, such that $u_i \in V^{\varepsilon}(u_i) \subseteq V$. By Proposition 1, the sets

$$\{u_i : h(a_i, G, u_i) > \varepsilon\} \text{ and } \{u_i : h(a_i, G, u_i) < \varepsilon\}$$

form a sub-basis of the strategic topology.

Step I: Metric topology is finer than strategic topology. Take any game G, and let m be the number of actions in G. Let $\bar{G} \in \mathcal{G}_m$ be the game derived from G by normalizing the payoff function. Specifically, let $\bar{g}^* = \max_{a,\omega} |g(a,\omega)|$. Then $\bar{G}$ is the game with the same action set as G and payoff function $\bar{g}$ defined by $\bar{g} = \frac{g}{\bar{g}^*}$. Note that a_i is ε-rationalizable in G iff a_i is $\frac{\varepsilon}{\bar{g}^*}$-rationalizable in G^*. Write $\bar{\varepsilon} = \frac{\varepsilon}{\bar{g}^*}$.

Pick any hierarchy u_i and $\varepsilon > 0$. Suppose that

$$u_i \in \{u_i' : h(a_i, G, u_i') < \varepsilon\}.$$

Define

$$\gamma = \bar{\varepsilon} - h(a_i, \bar{G}, u_i) > 0.$$

Find a game $G_k \in \mathcal{G}_m^*$ such that

$$\delta\left(G_k, \bar{G}\right) \leq \frac{\gamma}{10}.$$

Here, k is the number of G_k in the ordering used to define metric (3.1). By Lemma 14, for any u_i' such that $d(u_i, u_i') \leq \beta^k \frac{\gamma}{10}$,

$$\begin{aligned} h(a_i, \bar{G}, u_i') &\leq h(a_i, G_k, u_i') + 2\delta\left(G_k, \bar{G}\right) \\ &\leq h(a_i, G_k, u_i) + \frac{\gamma}{10} + 2\delta\left(G_k, \bar{G}\right) \\ &\leq h(a_i, \bar{G}, u_i) + \frac{\gamma}{10} + 4\delta\left(G_k, \bar{G}\right) < \bar{\varepsilon}. \end{aligned}$$

Hence,

$$u_i \in V_i^{\beta^k \frac{\gamma}{10}}(u_i) \subseteq \left\{u_i' : h(a_i, \bar{G}, u_i') < \bar{\varepsilon}\right\} = \left\{u_i' : h(a_i, G, u_i') < \varepsilon\right\}.$$

Next, let

$$u_i \in \left\{u_i' : h(a_i, G, u_i') > \varepsilon\right\}.$$

Define

$$\gamma = h(a_i, \bar{G}, u_i') - \bar{\varepsilon} > 0.$$

Find a game $G_k \in \mathcal{G}_m^*$ such that

$$\delta\left(G_k, \bar{G}\right) \leq \frac{\gamma}{10}.$$

Then, for any u_i' such that $d(u_i, u_i') \leq \beta^k \frac{\gamma}{10}$,

$$\begin{aligned} h(a_i, \bar{G}, u_i') &\geq h(a_i, G_k, u_i') - 2\delta\left(G_k, \bar{G}\right) \\ &\geq h(a_i, G_k, u_i) - \frac{\gamma}{10} - 2\delta\left(G_k, \bar{G}\right) \\ &\geq h(a_i, \bar{G}, u_i) - \frac{\gamma}{10} - 4\delta\left(G_k, \bar{G}\right) > \bar{\varepsilon}, \end{aligned}$$

and

$$u_i \in V_i^{\beta^k \frac{\gamma}{10}}(u_i) \subseteq \left\{u_i' : h(a_i, G', u_i') > \bar{\varepsilon}\right\} = \left\{u_i' : h(a_i, G, u_i') > \varepsilon\right\}.$$

Step II: Strategic topology is finer than metric topology. For any ε, let $k(\varepsilon)$ satisfy $\frac{\beta^{k(\varepsilon)}}{1-\beta} \leq \frac{\varepsilon}{4}$. We can bound the distance between two hierarchies as follows

$$d(u_i, u_i') \leq \frac{\varepsilon}{4} + \sum_{k=1}^{k(\varepsilon)} \beta^k \sup_{a_i \in A_i} \left|h(a_i, G_k, u_i) - h(a_i, G_k, u_i')\right|.$$

Now pick any u_i and $\varepsilon > 0$, and consider the open set in the metric topology $V_i^{\varepsilon}(u_i)$. The previous inequality implies

$$\bigcap_{k \leq k(\varepsilon)} \bigcap_{a_i \in A_i} \left\{u_i' : h(a_i, G_k, u_i) - \frac{1}{k(\varepsilon)} \frac{\varepsilon}{2} < h(a_i, G_k, u_i') < h(a_i, G_k, u_i) + \frac{1}{k(\varepsilon)} \frac{\varepsilon}{2}\right\} \subseteq V_i^{\varepsilon}(u_i).$$

and the set on the left-hand side is a finite intersection of u_i-neighborhoods in the strategic topology.

$\square$

Appendix C. Proof of Lemma 5

Define assessment

$$R_i^u(\cdot|G,\varepsilon) := \bigcap_{\varepsilon'>\varepsilon} R_i(\cdot|G,\varepsilon').$$

We check that this assessment is closed with respect to best response property, i.e.

$$R_i^u = R_i. \tag{C.1}$$

By the second part of Theorem 1, there exists a type space $\left(L_i, \mu_i^L\right)$ over Ω and such that

- ε-rationalizable correspondence $R_i(\cdot|G,\varepsilon)$ is upper hemi continuous on L_i. In particular, define set $R_i(G,\varepsilon) \subseteq L_i \times A_i$ as

$$R_i(G,\varepsilon') = \{(l_i, a_i) : a_i \in R_i(l_i|G,\varepsilon')\}.$$

  Then, such a set is closed in L_i.
- for any type space and any type $t_i \in T_i$, there is a corresponding type $l_i \in L_i$, such that for any game G any $\varepsilon \geq 0$

$$R_i(t|G,\varepsilon) = R_i(l_i|G,\varepsilon). \tag{C.2}$$

We show that (C.1) holds on type space (L_i, μ_i). Because of (C.2), this is sufficient to establish the Lemma.

Suppose that action a_i is ε'-rationalizable for l_i and any $\varepsilon' > \varepsilon$. Then, there is a sequence of strategies $\sigma_{-i}^{\varepsilon'} : L_{-i} \to \Delta A_{-i}$, such that $a_i \in B\left(\varepsilon'|l_i, \sigma_{-i}^{\varepsilon'}\right)$ and $\sigma_{-i}^{\varepsilon'}$ is ε'-rationalizable, i.e.

$$\sigma_{-i}^{\varepsilon'}(l_{-i})\left(R_{-i}(l_{-i}|G,\varepsilon')\right) = 1.$$

The sequence of strategies generates corresponding conjectures $\psi_{-i}^{\varepsilon'} \in \Delta(\Omega \times L_{-i} \times A_{-i})$, such that

$$\operatorname*{marg}_{\Omega\times L_{-i}} \psi_{-i}^{\varepsilon'} = \mu_i^L(l_i)$$

and for any continuous function $f : L_{-i} \times A_{-i} \to R$,

$$\int_{L_{-i}\times A_{-i}} f(l_i, a_i)\, d\psi_{-i}^{\varepsilon'}(l_i, a_i) = \int_{L_{-i}} \left[\int_{A_{-i}} f(l_i, a_i)\, d\sigma_{-i}^{\varepsilon'}(l_i)(a_i)\right] d\psi_{-i}^{\varepsilon'}(l_i).$$

Moreover, each of the conjectures assigns probability 1 to the fact that the opponent strategy is ε'-rationalizable:

$$\psi_{-i}^{\varepsilon'}\left(\Omega \times R_{-i}^u(G,\varepsilon')\right) = 1.$$

By compactness, we can find a convergent subsequence with a limit ψ_{-i}^{ε}. Then, by continuity, a_i is ε-best response against conjecture $\psi_{-i}^{\varepsilon'}$ and

$$\psi_{-i}^{\varepsilon}\left(\Omega \times R_{-i}^u(G,\varepsilon)\right) \geq \lim_{\varepsilon'\to\varepsilon} \psi_{-i}^{\varepsilon'}\left(\Omega \times R_{-i}\left(G,\varepsilon'\right)\right) = 1,$$

which follows from the definition of the correspondence R^u and the fact that it is u.h.c. Define a strategy $\sigma^{\varepsilon}_{-i} : L_{-i} \to \Delta A_{-i}$ as

$$\sigma^{\varepsilon}_{-i}(l_{-i}) := \operatorname*{marg}_{A_{-i}} \psi^{\varepsilon}_{-i}(\cdot|l_{-i}).$$

Then, $\sigma^{\varepsilon}_{-i}(l_{-i}) \in R^{u}_{-i}(l_{-i}|G, \varepsilon)$ (one may need to modify $\sigma^{\varepsilon}_{-i}$ on the set of types l_{-i} with a total $\mu^L_i(l_i)$-mass equal to 0), and $a_i \in B\left(\varepsilon|l_i, \sigma^{\epsilon}_{-i}\right)$. This ends the proof.

Appendix D. Proofs of Section 4

D.1. **Proof of Lemma 6.** Note that for each $k \geq 1$,

$$[B^p]^k(W) = B^p_j [B^p]^{k-1}(W) \times B^p_{-j}[B^p]^{k-1}(W)$$

is a product event and

$$\begin{aligned} B^p_j [B^p]^k(W) &= B^p_j [B^p]^{k-1}(W) \cap B^p_j B^p_{-j}[B^p]^{k-1}(W) \\ &= B^p_j \left(B^p_{-j}[B^p]^{k-1}(W)\right). \end{aligned}$$

By definition

$$\begin{aligned} C^p(W) &= \bigcap_{k\geq 1} [B^p]^k(W) \\ &= \bigcap_{k\geq 1} \left(B^p_j [B^p]^{k-1}(W) \times B^p_{-j}[B^p]^{k-1}(W)\right) \\ &= \bigcap_{k\geq 0} B^p_j [B^p]^k(W) \times \bigcap_{k\geq 0} B^p_{-j}[B^p]^k(W) \end{aligned}$$

and so $C^p(W)$ is a product set and

$$\begin{aligned} C^p_j(W) &= \bigcap_{k\geq 0} B^p_j [B^p]^k(W) \\ &= B^p_j(W) \cap \bigcap_{k\geq 1} B^p_j [B^p]^k(W) \\ &= W_j \cap B^p_j(W_{-j}) \cap \bigcap_{k\geq 1} B^p_j \left(B^p_{-j}[B^p]^{k-1}(W)\right) \\ &= W_j \cap B^p_j \left(W_{-j} \cap \bigcap_{k\geq 1} B^p_{-j}[B^p]^{k-1}(W)\right) \\ &= W_j \cap B^p_j \left(\bigcap_{k\geq 0} B^p_{-j}[B^p]^k(W)\right) \\ &= W_j \cap B^p_j C^p_{-j}(W). \end{aligned}$$

The last equality follows from the first (reversing the roles of players.)

D.2. **Technical result.** Here, we prove a useful technical result.

Lemma 15. *Suppose that E is separable and metrizable, A is a finite set, and let $\{V_a\}_{a\in A}$ be an open covering of E. Let $\mu \in \Delta E$ be a measure over E. Consider a (measurable) mapping $\sigma : E \to \Delta A$ such that σ is adapted to the covering $\{V_a\}_{a\in A}$, i.e.*

$$\sigma(e)(a) > 0 \implies e \in V_a.$$

There is a sequence of continuous continuous mappings $\sigma^m : E \to \Delta A$, each adapted to $\{V_a\}_{a\in A}$ such that

$$\sigma^m \to \sigma, \quad \mu\text{-almost surely.}$$

Proof. By standard topological arguments, for each V_a, there exists a sequence of continuous functions $\alpha_a^m : E \to [0, 1]$ such that $\alpha_a^m(e) > 0$ if and only if $e \in V_a$ and α_a^m converges pointwise to the indicator function for V_a. Also, because E is separable and metrizable and ΔA is compact Polish, there is a sequence of continuous mappings $\tau^m : E \to \Delta A$, such that $\tau^m \to \sigma$, μ-almost surely.

Note that for any $e \in E$, $\sum_{a\in A_i} \alpha_a^m(e) > 0$. Construct the sequence of mappings $\sigma^m : E \to \Delta A$ as follows. For any $e \in E$, for any $a \in A$, let

$$\sigma^m(e)(a) := \frac{\alpha_a^m(e)\left[\tau^m(e)(a) + \frac{1}{m}\right]}{\sum\limits_{a'\in A} \alpha_{a'}^m(e)\left[\tau^m(e)(a') + \frac{1}{m}\right]}.$$

By construction, σ^m is continuous. Moreover, for each m, $\sigma^m(e)(a) > 0$ if and only if $e \in V_a$. For μ-almost all $e \in V_a$,

$$\lim_{m\to\infty} \alpha_a^m(e)\,\tau^m(e)(a) \to \sigma(e)(a)$$

Thus $\sigma^m \to \sigma$, μ-almost surely. □

D.3. m**-rationalizable correspondence.** In order to prove Lemma 7, we need some results about m-rationalizable actions. We start with a definition. In Ely and Peski (2006), we showed that the set of rationalizable actions can be obtained through the iterative elimination of not best responses (similar fact for correlated rationalizability is used in Dekel, Fudenberg, and Morris (2006b)). This can be easily extended to ε-rationalizability. Let us denote by $R_i^m(\cdot|G, \varepsilon)$ the correspondence of actions obtained in the mth round of elimination for player i in game G. Then, the quoted results state that

$$R_i(\cdot|G, \varepsilon) = \bigcap_m R_i^m(\cdot|G, \varepsilon). \tag{D.1}$$

We refer to $R_i^m(\cdot|G, \varepsilon)$ as the correspondence of m-rationalizable actions.

It is easy to show that the set of type t_j's m-rationalizable correspondence depends only on type t_j's hierarchy of beliefs: for any player j, any types $t_j, t_j' \in T_i$, any game G and any $\varepsilon > 0$, if $\phi_j^T(t_j) = \phi_j^T(t_j')$, then $R_j^m(t_j|G,\varepsilon) = R_j^m(t_j'|G,\varepsilon)$.[16] Thus we can view $R_j^m(\cdot|G,\varepsilon)$ to be correspondence whose domain is $U_i(\Delta\Omega)$. By standard arguments, this correspondence is upper hemi-continuous. For any m, player j, game G, action a_j and hierarchy u_j, define

$$h_j^m(a_j, G, u_j) = \inf\{\varepsilon : a_j \in R^m(u_j|G,\varepsilon)\}.$$

Lemma 16. *$h_j^m(a_j, G, u_j)$ is continuous in u_j.*

Proof. Suppose that $u_j^n \to u_j^*$ and let $h = \liminf_{n\to\infty} h_j^m(a_j, G, u_j^n)$. Hence, for each $\varepsilon > 0$ there exists a subsequence $u_j^n \to u_j^*$ such that for each n, $h_j^m(a_j, G, u_j^n) \leq h + \varepsilon$, or, in other words, $a_j \in R(u_j^n|G, h+\varepsilon)$. By the upper hemi-continuity of the correspondence $R(\cdot|G, h+\varepsilon)$, $h_j^m(a_j, G, u_j) \leq h + \varepsilon$ for each $\varepsilon > 0$, and $h_j^m(a_j, G, .)$ is lower semi-continuous.

We prove upper semi-continuity by induction on m. When $m = 0$, by definition $h_j^0(a_j, G, u_j) \equiv 0$. Now assume that the Lemma holds for $m-1$. Fix a player j, a game G, an action a_j^* and a hierarchy u_j^*. Let $h^* = h_j^m(a_j^*, G, u_j^*)$. We are going to show that for any $\varepsilon > 0$, there is a neighborhood $V \ni u_j^*$, such that $h_j^m(a_j^*, G, u_j) < h^* + \varepsilon$ for any $u_j \in V$.

Let $L = (L_j, \mu_j)$ be the type space from the second part of Theorem 1. Find a type $t_j^* \in L_j$, such that $\phi_j^L(t_j) = u_j^*$. Let σ_{-j} be a strategy of player $-j$ that makes a_j^* an h^*-interim best response for type t_j^*. For any action $a_{-j} \in A_{-j}$, define

$$V_{a_{-j}} = \left\{u_{-j} : h_{-j}^m(a_{-j}, G, u_{-j}) < h^* + \varepsilon/2\right\}.$$

By the induction hypothesis, the collection $\left\{V_{a_{-j}}\right\}_{a_{-j}\in A_{-j}}$ is an open covering of L_{-j}. Also, if $\sigma_{-j}(t_{-j})(a_{-j}) > 0$ for some type t_{-j}, then $t_{-j} \in V_{a_{-j}}$. By Lemma 15, there is a sequence of continuous strategies σ_{-j}^n converging $\mu_j(t_j^*)$-almost surely to σ_{-j} such that for any $t_{-j} \in L_{-j}$ and action $a_{-j} \in A_{-j}$, if $\sigma_{-j}^n(t_{-j})(a_{-j}) > 0$, then $t_{-j} \in V_{a_{-j}}$.

Take any sequence of hierarchies $u_j^k \to u_j^*$ and find a sequence of types $t_j^k \in L_j$, such that $\phi_j^L(t_j^k) = u_j^k$ and $\lim_{k\to\infty} \mu_j(t_j^k) = \mu_j(t_j^*)$ in the weak* topology (such a sequence exists by the last part of Theorem 1). Then, for any action $a_j \in A_j$

$$\begin{aligned}&\lim_{n\to\infty}\lim_{k\to\infty} \mu_j(t_j^k)\left[g_j(a_j^*, \sigma_{-j}^n(t_{-j}), \omega) - g_j(a_j, \sigma_{-j}^n(t_{-j}), \omega)\right] \\ &= \lim_{n\to\infty} \mu_j(t_j^*)\left[g_j(a_j^*, \sigma_{-j}^n(t_{-j}), \omega) - g_j(a_j, \sigma_{-j}^n(t_{-j}), \omega)\right] \leq h^*.\end{aligned}$$

This implies that there is a neighborhood $V \ni u_j^*$ such that for all types $u_j \in V$, a_j^* is $(h^* + \varepsilon/2)$-interim rationalizable, i.e.

$$h_j^m(a_j^*, G) < h^* + \varepsilon.$$

□

[16]This can be shown using the product game construction in the proof of Lemma 19 below.

Lemma 17. *Suppose that $v_i \nleq u_i^*$. There are game $\bar{G} = \left(\bar{A}_j, \bar{g}_j\right)$, action $a_i \in \bar{A}_i$, $\varepsilon > 0$ and m such that*

$$a_i \in R_i^m \left(v_i|\bar{G}, 0\right) \text{ and } a_i \notin R_i^m \left(u_i^*|\bar{G}, \varepsilon\right).$$

Proof. By the definition of order, there is game $\bar{G}$ and action a_i, such that $a_i \in R_i \left(v_i|\bar{G}, 0\right)$ and $a_i \notin R_i \left(u_i^*|\bar{G}, 0\right)$. By Lemma 5, there is $\varepsilon > 0$, such that $a_i \notin R_i \left(u_i^*|\bar{G}, \varepsilon\right)$. By (D.1), there is m, such that the thesis of the Lemma holds. □

Lemma 18. *Suppose that $v_i \nleq u_i^*$. There are neighborhood $U_i \ni v_i$, game $\bar{G} = \left(\bar{A}_j, \bar{g}_j\right)$, action $a_i \in \bar{A}_i$, $\varepsilon > 0$ and m such that*

$$a_i \in R_i^m \left(u_i|\bar{G}, 0\right) \text{ for all } u_i \in U_i \text{ and } a_i \notin R_i^m \left(u_i^*|\bar{G}, \varepsilon\right).$$

Proof. Let $\bar{G}', a_i'$, be the game from Lemma 17. Because of Lemma 16, there is a neighborhood $U_i \ni v_i$ open in the product topology and $\varepsilon' > \varepsilon'' > 0$, such that

$$a_i' \in R_i^m \left(u_i|\bar{G}', \varepsilon''\right) \text{ for all } u_i \in U_i \text{ and } a_i' \notin R_i^m \left(u_i^*|\bar{G}', \varepsilon'\right).$$

Take $\varepsilon = \varepsilon' - \varepsilon'' > 0$ and apply Lemma 4. □

D.4. **Proof of Lemma 7.** We first prove a preliminary result.

Lemma 19. *Suppose that $v_i \nleq u_i^*$. There are $\varepsilon > 0$, a neighborhood $U_i \ni v_i$, a game $G = (A_j, g_j)$, such that $A_i = A_i^0 \times A_i^1$ and*

1. *For any $a_{-i} \in A_{-i}$, any $a_i^1 \in A_i^1$, any $a_i^0, a_i^{0\prime} \in A_i^0$, any ω,*
$$g_{-i}\left(a_{-i}, \left(a_i^0, a_i^1\right), \omega\right) = g_{-i}\left(a_{-i}, \left(a_i^{0\prime}, a_i^1\right), \omega\right).$$
2. *There are correspondences $A^0 : U_i(\Delta\Omega) \rightrightarrows A_i^0, A^1 : U_i(\Delta\Omega) \rightrightarrows A_i^1$, such that for all u_i,*
$$R_i\left(u_i|G, 0\right) = A^0\left(u_i\right) \times A^1\left(u_i\right).$$
3. *There is an action $a_i^{0*} \subseteq A_i^0$, such that*
$$\left\{a_i^{0*}\right\} \times A^1\left(u_i\right) \subseteq R_i\left(u_i|G, 0\right) \text{ for all } u_i \in U_i,$$
$$\left\{a_i^{0*}\right\} \times A^1\left(u_i^*\right) \cap R_i\left(u_i^*|G, \varepsilon\right) = \emptyset.$$

Proof. Let $u_i \in U_i$, game $\bar{G} = \left(\bar{A}_j, \bar{g}_j\right)$, action $a_i \in \bar{A}_i$, $\varepsilon > 0$ and m be as in Lemma 18. Define game $G = (A_j, g_j)$: for any player j, let $A_j = \left(\bar{A}_j\right)^m$ and

$$g_j\left(\left(a_j^1, ..., a_j^m\right), \left(a_{-j}^1, ..., a_{-j}^m\right), \omega\right) = \sum_{k=1}^{m-1} g_j\left(a_j^k, a_{-j}^{k+1}, \omega\right).$$

Notice that for any $\varepsilon' \geq 0$,

$$R_i\left(\cdot|G, \varepsilon'\right) = R_i^m\left(\cdot|\bar{G}, \varepsilon'\right) \times ... \times R_i^0\left(\cdot|\bar{G}, \varepsilon'\right).$$

Let $A_i^0 = \bar{A}_i$, $A_i^1 = (\bar{A}_i)^{m-1}$,

$$A^0(.) = R_i^m(\cdot|\bar{G},\varepsilon'),$$
$$A^1(.) = R_i^{m-1}(\cdot|\bar{G},\varepsilon') \times ... \times R_i^0(\cdot|\bar{G},\varepsilon'),$$

and $a_i^{0*} = a_i$. The thesis of the Lemma follows. □

We can prove Lemma 7. For any $v_i \in W_i$, $v_i \nleq u_i^*$ and we can apply Lemma 19 to find $\varepsilon^{(v_i)} > 0$, neighborhoods $U_i^{(v_i)} \ni v_i$, games $G^{(v_i)}$. Let $v_i^1, ..., v_i^K \in W_i$ be a finite sequence of hierarchies, such that $W_i \subseteq \bigcup_k U_i^{(v_i^k)}$ (it exists, because W_i is compact.) To shorten the notation, let $G^k = (A_j^k, g_j) := G^{(v_i^k)}$. Define game G as the product game $G = G^1 \times ...G^K$: let

$$A_i^0 := \prod_{k=1...,K} A_i^{0,k},\ A_i^1 := \prod_{k=1...,K} A_i^{1,k} \text{ and}$$
$$A_i^{0*} = \left\{a_i^0 \in A_i^0 : (a_i^0)_k = a_i^{0*,k} \text{ for some } k = 1,...,K\right\},$$

and for any $(a_j^k) \in A_j$, $(a_{-j}^k) \in A_{-j}$, let

$$g_j\left((a_j^k), (a_{-j}^k), \omega\right) = \sum_{k=1}^{K} g_j^k\left(a_j^k, a_{-j}^k, \omega\right).$$

Notice that for any $\varepsilon \geq 0$,

$$R(\cdot|G,\varepsilon) = R(\cdot|G^1,\varepsilon) \times ... \times R(\cdot|G^K,\varepsilon)$$

The thesis of the Lemma follows from the construction.

D.5. **Proof of Lemma 8.** We assume here that

$$\max_{j,a,\omega} |g_j(a,\omega)| \leq 1. \tag{D.2}$$

This is w.l.o.g., as one can always scale the payoffs without affecting the set of rationalizable actions.

We prove the Lemma by induction on k and player j. In order to shorten the notation, define

$$E_j^k := U_j(\Delta\Omega) \backslash B_j^p\left([B^p]^k (W_i \times U_{-i}(\Delta\Omega))\right).$$

Note that, by construction, sets E_j^k are open. When $k = 0$ and $j = i$, then

$$E_j^k = E_i^0 = U_i^A.$$

Then, the Lemma is a consequence of the fact that U_i^A is open and for any hierarchy in U_i^A, any $\varepsilon \leq \varepsilon^A$, the set of ε-rationalizable actions is equal to A_i^*.

Assume that the thesis of Lemma 8 holds for $k \geq 0$ and $j = i$. Take any $u_j^0 \in E_j^k$ and suppose that a_{-i}^* is a rationalizable action at u_{-i}^0. Let $L = (L_j, \mu_j)$ be the type space from the second part of Theorem 1. Let $l_{-i}^0 \in L_{-i}$ be a type with a hierarchy $\phi_{-i}^L\left(l_{-i}^0\right) = u_{-i}^0$. Let $l_{-i}^n \in L_{-i}$ be a sequence of types of player $-i$ with hierarchies $\phi_{-i}^L\left(l_{-i}^n\right) = u_{-i}^n$. By construction of type space L_i (see Ely and Peski (2006)),

$$\mu_{-i}\left(l_{-i}^n\right) \rightarrow \mu_{-i}\left(l_{-i}^0\right) \tag{D.3}$$

in the sense of weak* topology.

Say that behavioral strategy $\sigma_j : L_j \rightarrow \Delta A_j$ is *δ-rationalizable*, if for any type $l_j \in L_j$, actions played by l_j are δ-rationalizable, $\sigma_j^0\left(l_j\right)\left(R\left(l_j|G,\delta\right)\right) = 1$. Let σ_i^0 be a 0-rationalizable strategy of player i, against which action a_{-i}^* is a best response of type l_{-i}^0. We need an intermediate result.

Lemma 20. *There is a sequence of $6p$-rationalizable strategies $\sigma_i^m : L_i \rightarrow \Delta A_i$, such that σ_i^m is continuous on $\left(\phi_i^L\right)^{-1}\left(E_i^k\right)$ and*

$$\sigma_i^m \rightarrow \sigma_i^0,\ \mu_{-i}\left(l_{-i}^0\right)\text{-almost surely.} \tag{D.4}$$

Proof. By the inductive hypothesis, there are sets of hierarchies

$$V_i\left(a_i\right) \subseteq \left(\phi_i^L\right)^{-1}\left(E_i^k\right),$$

such that for each a_i set $V_i\left(a_i\right)$ is open in the product topology and

$$\begin{aligned}&\left\{l_i \in \left(\phi_i^L\right)^{-1}\left(E_i^k\right) : a_i \in R_i\left(l_i|G,0\right)\right\}\\ &\subseteq V_i\left(a_i\right)\\ &\subseteq \left\{l_i \in \left(\phi_i^L\right)^{-1}\left(E_i^k\right) : a_i \in R_i\left(u_i|G,6p\right)\right\}.\end{aligned}$$

In particular, $\left\{V_i\left(a_i\right)\right\}_{a_i \in A_i}$ is an open cover of $\left(\phi_i^L\right)^{-1}\left(E_i^k\right)$. By Lemma 15, there is a sequence of continuous functions $\sigma_i^{m*} : \left(\phi_i^L\right)^{-1}\left(E_i^k\right) \rightarrow \Delta A_i$, such that for any $l_i \in \left(\phi_i^L\right)^{-1}\left(E_i^k\right)$,

$$\sigma_i^{m*}\left(l_i\right)\left(a_i\right) > 0 \text{ iff } l_i \in V_i\left(a_i\right) \text{ and}$$

and $\sigma_i^{m*} \rightarrow \sigma_i^0$ almost surely with respect to measure $\mu_{-i}^L\left(l_{-i}^0\right)\left(\cdot|\left(\phi_i^L\right)^{-1}\left(E_i^k\right)\right)$.

Define $\sigma_i^m : L_i \rightarrow \Delta A_i$ as

$$\sigma_i^m\left(l_i\right) = \begin{cases} \sigma_i^{m*}\left(l_i\right), & l_i \in \left(\phi_i^L\right)^{-1}\left(E_i^k\right), \\ \sigma_i^0\left(l_i\right) & \text{otherwise.}\end{cases}$$

Thus, σ_i^m is $6p$-rationalizable and continuous on $\left(\phi_i^L\right)^{-1}\left(E_i^k\right)$. $\square$

Let $\alpha^m : L_i \to [0,1]$ be a sequence of continuous functions, such that $\alpha_i^m(l_i) = 0$ for any type $l_i \notin \left(\phi_i^L\right)^{-1}\left(E_i^k\right)$ and $\lim_{m\to\infty}\alpha_i^m(l_i) = 1$ for any $l_i \in \left(\phi_i^L\right)^{-1}\left(E_i^k\right)$. Since E_i^k is open and ϕ_i^L is continuous, $\left(\phi_i^L\right)^{-1}\left(E_i^k\right)$ is open and such a sequence trivially exists. To abbreviate notation, for any function $f : L_i \times \Omega \to R$, any measure $\mu \in \Delta(L_i \times \Omega)$, write $\mu[f]$ as the expectation of f with respect to μ. For each $a_{-i} \in A_{-i}$,

$$\begin{aligned}
&\left|\mu_{-i}\left(l_{-i}^n\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\right] - \mu_{-i}\left(l_{-i}^0\right)\left[g_{-i}\left(a_{-i},\sigma_i^0,\omega\right)\right]\right| \\
&\le \left|\mu_{-i}\left(l_{-i}^n\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\right] - \mu_{-i}\left(l_{-i}^0\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\right]\right| \\
&+ \left|\mu_{-i}\left(l_{-i}^0\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\right] - \mu_{-i}\left(l_{-i}^0\right)\left[g_{-i}\left(a_{-i},\sigma_i^0,\omega\right)\right]\right| \\
&\le \left|\mu_{-i}\left(l_{-i}^n\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\alpha_i^m\right] - \mu_{-i}\left(l_{-i}^0\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\alpha_i^m\right]\right| \\
&+ \left|\mu_{-i}\left(l_{-i}^0\right)\left[(1-\alpha_i^m)\right]\right| + \left|\mu_{-i}\left(l_{-i}^n\right)\left[(1-\alpha_i^m)\right]\right| \\
&+ \left|\mu_{-i}\left(l_{-i}^0\right)\left[\left(g_{-i}\left(a_{-i},\sigma_i^m,\omega\right) - g_{-i}\left(a_{-i},\sigma_i^0,\omega\right)\right)\alpha_i^m\right]\right| \\
&+ \left|\mu_{-i}\left(l_{-i}^0\right)\left[\left(g_{-i}\left(a_{-i},\sigma_i^m,\omega\right) - g_{-i}\left(a_{-i},\sigma_i^0,\omega\right)\right)(1-\alpha_i^m)\right]\right|.
\end{aligned}$$

In the second inequality, we used (D.2). Because of the convergence (D.3), and the fact that $g_{-i}\left(a_{-i},\sigma_i^m(.),\omega\right)\alpha_i^m(.)$ is a continuous function of l_i,

$$\limsup_{n\to\infty}\left|\mu_{-i}\left(l_{-i}^n\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\alpha_i^m\right] - \mu_{-i}\left(l_{-i}^0\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\alpha_i^m\right]\right| = 0.$$

Because of (D.4),

$$\limsup_{m\to\infty}\left|\mu_{-i}\left(l_{-i}^0\right)\left[\left(g_{-i}\left(a_{-i},\sigma_i^m,\omega\right) - g_{-i}\left(a_{-i},\sigma_i^0,\omega\right)\right)\alpha_i^m\right]\right| = 0,$$

Since

$$\lim_{m\to\infty}\mu_{-i}\left(l_{-i}^0\right)\left[1-\alpha_i^m\right] = 1 - \mu_{-i}\left(l_{-i}^0\right)\left(\left(\phi_i^L\right)^{-1}\left(E_i^k\right)\right) \le p,$$

we have

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{n\to\infty}\left[\begin{array}{c}\left|\mu_{-i}\left(l_{-i}^0\right)\left[1-\alpha_i^m\right]\right| + \left|\mu_{-i}\left(l_{-i}^n\right)\left[1-\alpha_i^m\right]\right| \\ + \left|\mu_{-i}\left(l_{-i}^0\right)\left[\left(g_{-i}\left(a_{-i},\sigma_i^m,\omega\right) - g_{-i}\left(a_{-i},\sigma_i^0,\omega\right)\right)(1-\alpha_i^m)\right]\right|\end{array}\right] \\
&\le \limsup_{m\to\infty}\limsup_{n\to\infty} 4\mu_{-i}\left(l_{-i}^0\right)\left[1-\alpha_i^m\right] \le 4p.
\end{aligned}$$

All the above imply that

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{n\to\infty}\sup_{a_{-i}\in A_{-i}}\left|\mu_{-i}\left(l_{-i}^n\right)\left[g_{-i}\left(a_{-i},\sigma_i^m,\omega\right)\right] - \mu_{-i}\left(l_{-i}^0\right)\left[g_{-i}\left(a_{-i},\sigma_i^0,\omega\right)\right]\right| \\
&\le 4p.
\end{aligned}$$

This shows that for high n and k, a_{-i}^* is a $6p$-best response of l_{-i}^n against σ_i^m and it shows th inductive thesis for k and $j = -i$.

Assume that the thesis of (Lemma 8) holds for k and $j = -i$. A very similar argument to the one above demonstrates that the inductive thesis holds for $k+1$ and $j = i$. This ends the proof of the Lemma.

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