ON ESTIMATION OF SEMI/NONPARAMETRIC CONDITIONAL MOMENT MODELS

Xiaohong Chen (Yale University)

Talk Based on Two Papers

- Chen, X. and D. Pouzo (08): “Efficient estimation of semiparametric conditional moment models with possibly nonsmooth residuals”.
- Chen, X. and D. Pouzo (07): “Estimation of nonparametric conditional moment models with possibly nonsmooth moments”.

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- Closely related work
  - Blundell, Chen and Kristensen (07, Econometrica) on shape-invariant semi/nonparametric Engel curves with endogenous total expenditure.
  - Ai and Chen (03, Econometrica) on efficient estimation with smooth residuals.
  - Chen (07, Handbook of Econometrics, vol. 6B) survey on method of sieves.

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  - Convergence Rate of Nonparametric Parts.
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  - Semiparametric Efficiency, Confidence Region.

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- Conclusion and Future Work

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- $\alpha_0 \equiv (\theta_0, h_0(.))$ are unknown parameters of interest,
- θ are finite dimensional parameters,
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- $h(.) = (h_1(.), ..., h_q(.))$ are functions, $h_j(.)$ could depend on Y, X, θ, other h_{-j} or latent variables.
- $\rho(.)$ is a $d_\rho \times 1$ -vector of generalized residual functions, with known functional form up to unknown $\alpha \equiv (\theta, h(.))$.
- $\rho(.)$ may be **nonlinear**, pointwise **non-smooth** w.r.t. α.

Examples

- Ex 1 (Shape-invariant Engel curve IV regression, BCK):

$$E[Y_{1l} - h_{1l}(Y_2 - X_1'\theta_1) - X_1'\theta_{2,l}|X_1, X_2] = 0, l = 1, ..., d_\rho,$$

- $\rho_l(Z, \alpha) = Y_{1l} - h_{1l}(Y_2 - X_1'\theta_1) - X_1'\theta_{2,l}$.
- $E[\rho(Z, \alpha_0(\cdot))|X] = 0$, $\rho = (\rho_1, ..., \rho_{d_\rho})'$; Para. of interest are $\alpha = (\theta_1, \theta_{2,1}, ..., \theta_{2,d_\rho}, h_1, ..., h_{d_\rho})'$.

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- Ex 2 (Engel curve quantile IV regression): for $\gamma \in (0,1)$,

$$E[1\{Y_{1l} \leq h_{1l}(Y_2 - X_1'\theta_1) + X_1'\theta_{2,l}\}|X_1, X_2] = \gamma$$

- $\rho_l(Z,\alpha) = 1\{Y_{1l} \leq h_{1l}(Y_2 - X_1'\theta_1) + X_1'\theta_{2,l}\}$.

Examples (cont.)

- Ex 3 (Consumption-based asset pricing models):

$$E(M_{t+1}R_{l,t+1}|w_t) = 1, l = 1, ..., d_\rho,$$

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- Hansen - Singleton (82) assume power utility $E\sum_{t=1}^{\infty} \delta^t \frac{(C_t)^{1-\gamma}-1}{1-\gamma}$; hence $M_{t+1} = \delta(\frac{C_{t+1}}{C_t})^{-\gamma}$.

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- Chen - Ludvigson (04) consider a semiparametric utility $E\sum_{t=1}^{\infty}\delta^t\frac{(C_t-H_t)^{1-\gamma}-1}{1-\gamma}$, where $H_t = C_t h(\frac{C_{t-1}}{C_t}, ..., \frac{C_{t-L}}{C_t})$ is unknown habit level at time t.
- Para. of interest: $\alpha = (\delta, \gamma, h())$ and $E[\frac{\partial^2 h(x_1,...,x_L)}{\partial x_1^2}]$.

Example 3 (cont.)

- rewrite semiparametric asset pricing model as $E[\rho_i(\mathbf{z}_{t+1}, \delta_0, \gamma_0, h_0)|\mathbf{w}_t] = 0$, $i = 1, ..., d_\rho$, where

$$\rho_i(\mathbf{z}_{t+1}, \delta_0, \gamma_0, h_0) \equiv \delta \left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right)^{-\gamma} R_{i,t+1} \widetilde{F}_{i,t+1} - 1,$$

$$\begin{aligned} \widetilde{F}_{i,t+1} \quad &\equiv \quad 1 - \sum_{j=0}^{L} \delta^j \left(\frac{C_{t+1+j} - H_{t+1+j}}{C_{t+1} - H_{t+1}} \right)^{-\gamma} \frac{\partial H_{t+1+j}}{\partial C_{t+1}} \\ &\quad + \sum_{j=0}^{L} \delta^{j-1} \left(\frac{C_{t+j} - H_{t+j}}{C_{t+1} - H_{t+1}} \right)^{-\gamma} \frac{\partial H_{t+j}}{\partial C_t} \frac{1}{R_{i,t+1}}. \end{aligned}$$

More General Class of Models

- $m_j(X_{j,t}, \alpha_0) \equiv E[\rho_j(Z_t, \theta_0, h_0(\cdot))|X_{j,t}] = 0$, $j = 1, ..., d_\rho$,
- $\{(Y_t, X_t) : t = 1, ..., n\}$ either i.i.d. or stationary weakly dependent time series data.
- X_j is "IV" for j-th equation, but may be endogenous to j' equation for $j' \neq j$. Some of the X_j may be constant.

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- **Examples**: structural models of incomplete information, simultaneous equations, control function approach, panel data models, missing data, measurement errors via IV approach, treatment effects. Estimation of smooth functionals defined via expectations: $E[Y_1 - h_0(Y_2)|X] = 0$ and $E[\theta_0 - a(Y_2)\partial h_0(Y_2)] = 0$.

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- Ai - Chen (05) on efficiency under correct specification; Ai - Chen (07) on estimation under misspecification.

Back to Semi/nonparametric Conditional Moment Models

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- **Issues**: when $h(.)$ may depend on endogenous Y,
  - identification of $\alpha_0 = (\theta_0, h_0(.))$;
  - estimation of h_0 at nonparametric rate;
  - $\sqrt{n}$ normality of estimators of smooth functionals;
  - efficient estimation of θ_0 under correct specification;
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  - misspecified models, model comparison, testing.
- **Difficulty**: estimation of h may be ill-posed, and $\rho()$ may not be pointwise smooth wrt α.

Brief Literature Review

- The model **without** unknown h: $E[\rho(Z,\theta_0)|X]=0$.

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- Lots of papers about theoretical and practical issues on estimating θ_0 and huge amount of applications !!!
- Sargan (?), Hansen (82, 85, 05), Hansen - Singleton (82), Hansen et al. (95), Hansen et al. (96), Chamberlain (87), Robinson (88), Newey (93), Imbens (97), Imbens et al. (98), Kitamura et al. (04), Antoine et al. (06), Smith (00), Zellner (91), Newey - Smith (04), Newey - McFadden (94), Pakes - Pollard (89), Manski (94), Mantzkin (94), Powell (94), Carrasco - Florens (00), Gallant - Tauchen (00), Stock - Wright (00), Andrews - Stock (05),...
- Estimating equations in statistics: Hyde, Owen, van der Vaart, ...

Literature Review (cont.)

- The model **with** unknown h: $E[\rho(Z, \theta_0, h_0)|X] = 0$.
- A large special class (**no endogeneity**): $E[\rho(Z, \theta, h) - \rho(Z, \theta_0, h_0)|X] = \rho(Z, \theta, h) - \rho(Z, \theta_0, h_0)$.

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- Semiparametric M-estimation problem, including MLE, Least Squares, nonlinear LS, quantile regression, etc
- Asymptotic theory on consistency, convergence rate, semiparametric efficiency, limiting distribution have been developed.
- Horowitz (98), Pagan - Ullan (99), Robinson (88, 93), Ichimura (93), Powell (94), Hardle - Linton (94), Andrews (94), Manski (94), Newey (94), Chen - Shen (98), Linton - Mammen (05), Ichimura - Lee (06), etc.
- BKRW (93), van der Vaart - Wellner (96), Fan-Gijbels, Fan-Yao, van de Geer, ...

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- **Aim**: root-n consistency, normality and efficiency of $\hat{\theta}$.

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- Estimating both h and θ by imposing the model.
  - Smooth but possibly nonlinear $\rho()$: Ai - Chen (03, 05), Chen - Ludvigson (04), Otsu (07).
  - Linear $\rho()$: FJvB (07), Severini - Tripathi (07).

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- **Difficulty**: recovering h is nonlinear, may be ill-posed.

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  - Unknown function h could depend on endog. Y.
- Consider penalized Sieve Minimum Distance (SMD).
- Establish consistency, convergence rates of h that may be (nonlinear) ill-posed.
- Obtain asymp. normality of $\hat{\theta}$, and weighted bootstrap.
- Show efficiency of optimally weighted $\tilde{\theta}$, and profile criterion is asymp. Chi-square.
- Ex 1: Partially linear quantile IV regression.
- Ex 2: Average derivative of nonparametric quantile IV.

Review: Sieve Minimum Distance

- $m(X, \alpha) \equiv E[\rho(Z, \theta, h)|X]$, $\Sigma(X)$ is a p.d. matrix.
- Then $m(X, \alpha_0) = 0$ iff $\alpha_0 \in \mathcal{A}$ is the unique solution to

$$\inf_{\alpha \in \mathcal{A}} E\left[m(X, \alpha)'[\Sigma(X)]^{-1} m(X, \alpha)\right].$$

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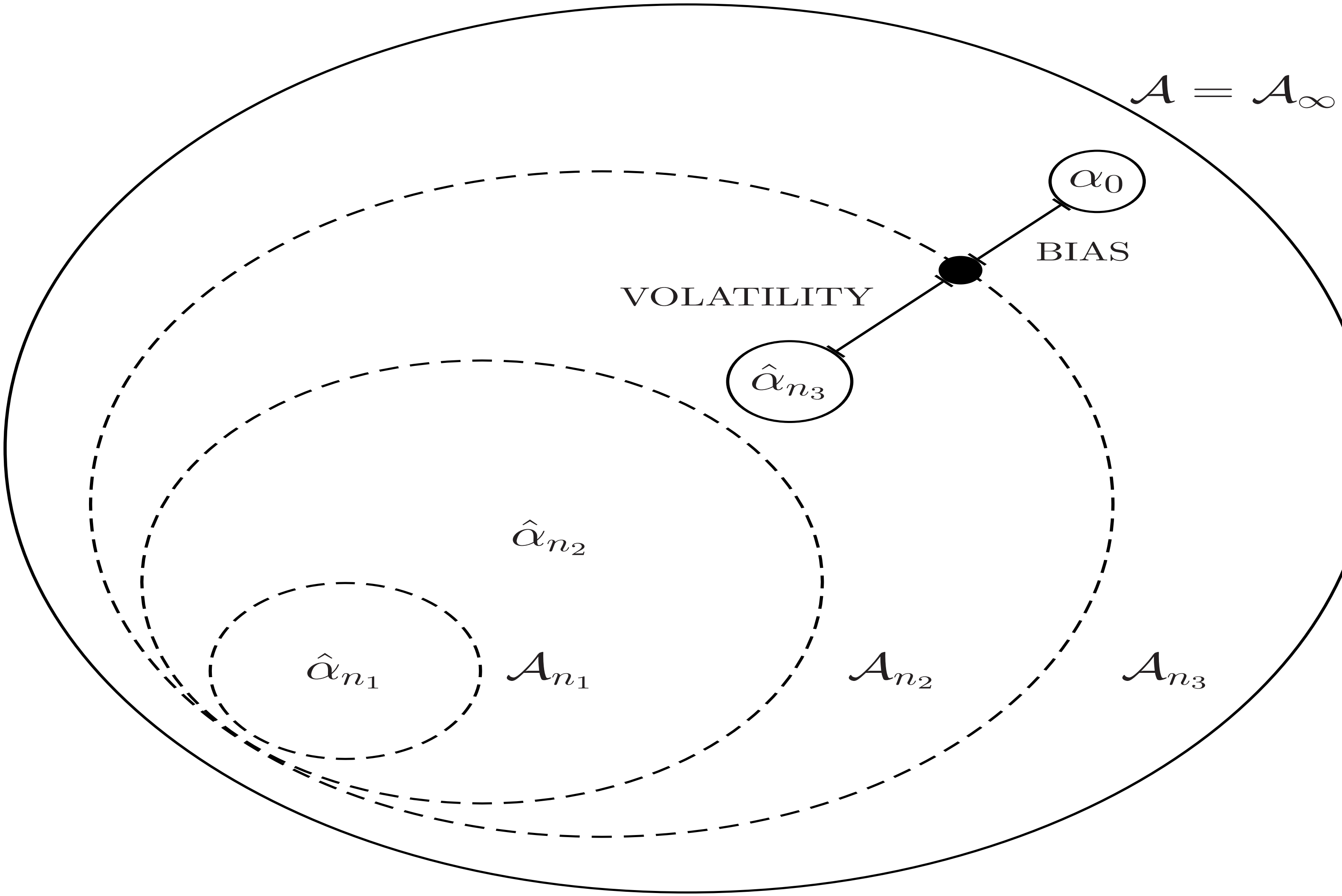
- Newey-Powell (89, 03), Ai-Chen (99, 03) propose SMD estimator $\hat{\alpha}_n$ that solves

$$\min_{\alpha \in \mathcal{A}_n} \frac{1}{n} \sum_{t=1}^{n} \left[\widehat{m}(X_t,\alpha)'[\widehat{\Sigma}(X_t)]^{-1} \widehat{m}(X_t,\alpha)\right]$$

- $\widehat{m}(X,\alpha)$ and $\widehat{\Sigma}(X)$ are any consistent estimators of $m(X,\alpha)$ and $\Sigma(X)$ respectively.
- $\mathcal{A}_n$ is a finite dimensional compact sieve space for $\mathcal{A}$.

SMD Estimation (cont.)



Examples of Sieves

- Finite-dimensional linear sieves $\mathcal{H}_n$ is of the form $\{h(.) = \sum_{k=1}^{k_n} \beta_k p_k(.)\}$, with $p_k(.)$ a known basis, e.g.
  1. Polynomials: $p_k(Y) = Y^k$
  2. Sine (Cosine): $p_k(Y) = Sin(k\pi Y)\ (Cos(k\pi Y))$
  3. B-Splines: $p_k(X) = 2^{k_{1n}/2} B_r(2^{k_{1n}} Y - k)$
  4. Polynomial splines, wavelets, finite elements, Hermite poly., Laguerre poly.

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- Finite-dimensional compact sieves $\mathcal{H}_n$ could take the form $\{h(.) = \sum_{k=1}^{k_n} \beta_k p_k(.), ||D^r h||_{L^p} \leq log(k_n)\}$.

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Penalized SMD Estimators

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2. $\hat{P}_n() \geq 0$: Penalty, either lower semicompact (e.g., Sobolev norm) or convex (e.g., L^2), may be random.
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- If $\lambda_n = 0$ and $\mathcal{H}_n$ compact, penalized SMD becomes the SMD of Newey-Powell and Ai-Chen.

Penalized SMD (cont.)

- If $n^{-1}\sum_{i=1}^{n}\widehat{m}(X_i,\alpha)'\widehat{\Sigma}(X_i)^{-1}\widehat{m}(X_i,\alpha)$ is convex in $h \in \mathcal{H}$, and $\mathcal{H}$ is closed convex (but not compact under $||\cdot||_s$),
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- is equivalent to the penalized SMD with a linear sieve $clsp(\mathcal{H}_n^c)$:

$$\inf_{\theta\in\Theta}\left\{\inf_{h\in clsp(\mathcal{H}_n^c)} n^{-1}\sum_{i=1}^{n}\widehat{m}(X_i,\alpha)'\widehat{\Sigma}(X_i)^{-1}\widehat{m}(X_i,\alpha)+\lambda_n\widehat{P}_n(h)\right\}$$

Penalized SMD (cont.)

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- $\widehat{\Sigma}(x,\alpha)$: series LS estimator $[\widehat{\sigma}(x,\alpha)]_{jk}=\sum_{s=1}^{n}\rho_j(Z_s,\alpha)\rho_k(Z_s,\alpha)p^{J_n}(X_s)'(P'P)^{-1}p^{J_n}(x)$, or any consistent nonparametric estimator of $Var[\rho(Z,\alpha)|X]$.

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  2. Start at $\hat{\alpha}^{(1)}=\hat{\alpha}$, and solve $\hat{\alpha}^{(2)}$ by

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- Method II: locally continuous updated Penalized SMD on $\mathcal{N}_{0n}$.

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- $$Y_1 = X_1 + \overbrace{\Phi\left(\frac{Y_2 - \mu_{y_2}}{\sigma_{y_2}}\right)}^{h_0} + U.$$
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- $\hat{m}$: P-SPL(3,3) $\times$ P-COL(9). $\lambda_n \in \{0.001, 0.01, 0.1\}$.
- # of MC iter: 500, # of Obs: 1000.

Monte Carlo: Robustness Analysis f = G-DEN and $\gamma = 0.5$

Endogeneity:	No	Yes	Yes	Yes
Basis h:	PSpl(2,6)	PSpl(2,6)	PSpl(2,6)	BSpl(8)
Penalization:	$\|\|D^1 h\|\|_2$	$\|\|D^1 h\|\|_2$	$\|\|D^2 h\|\|_2$	$\|\|D^1 h\|\|_2$
$E[\theta]$	0.9999	1.0009	1.0015	1.0081
$V[\theta]$	0.0002	0.0011	0.0013	0.0067
2.5% CI	0.96	0.93	0.93	0.90
97.5% CI	1.02	1.07	1.07	1.19
$BIAS^2[\theta] \times 10^3$	0.0000	0.0008	0.0023	0.0060
$MISE[h]$	0.0017	0.0087	0.0144	0.0960
$IBIAS^2[h]$	0.0000	0.0030	0.0067	0.0139
$IVAR[h]$	0.0016	0.0056	0.0137	0.0821

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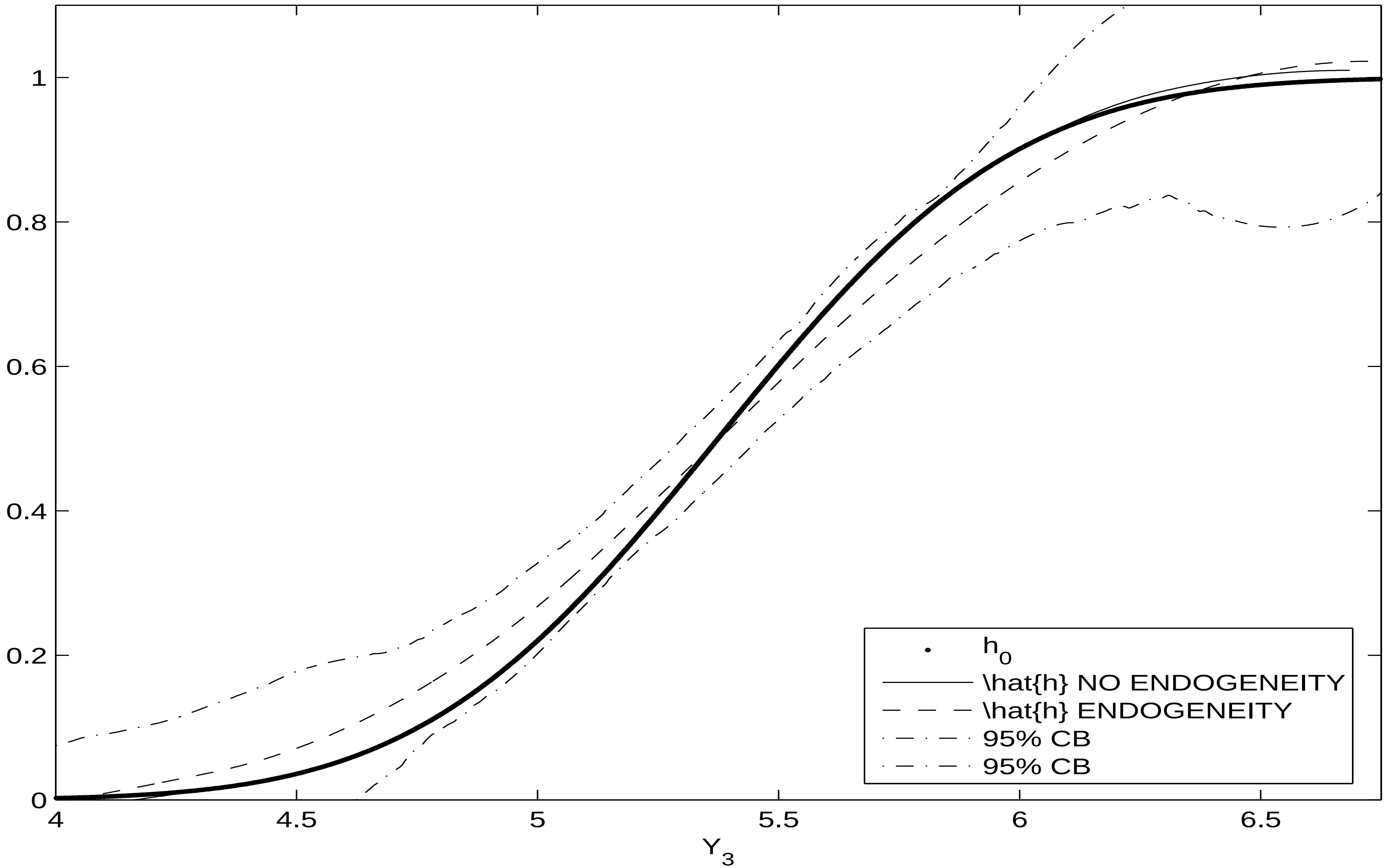
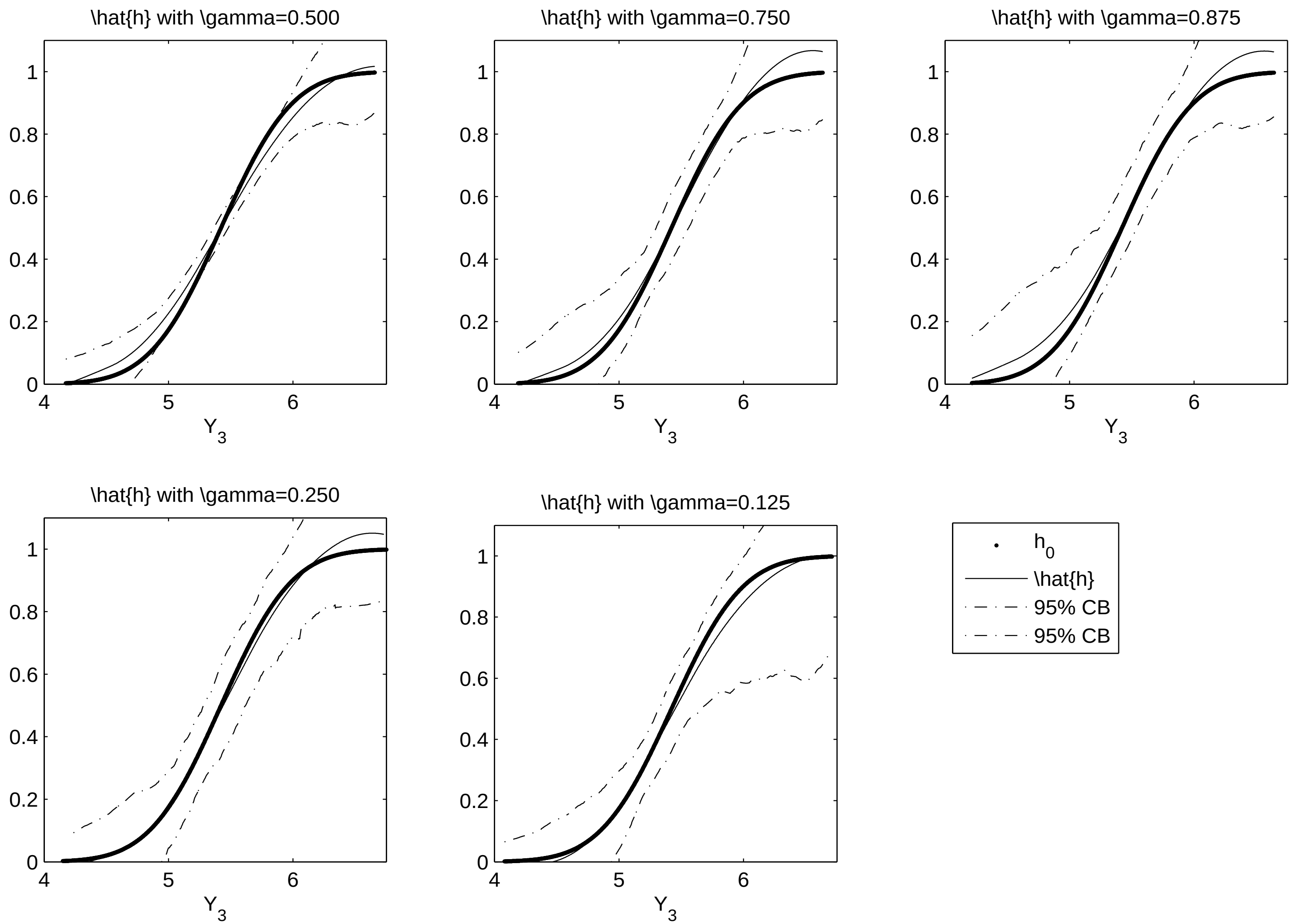


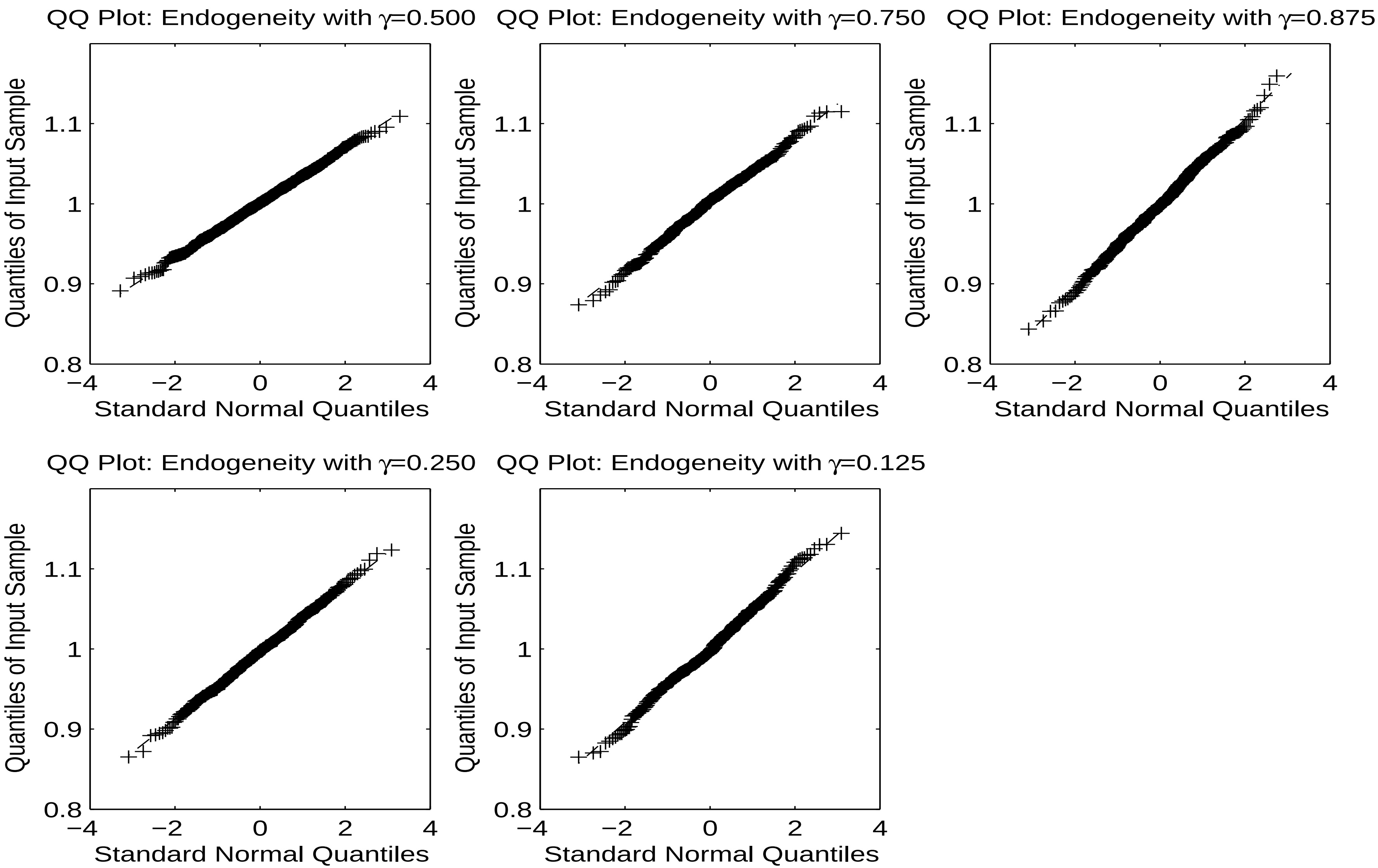
Monte Carlo: f = G-DEN and $\gamma \in \{0.125, 0.25, 0.50, 0.75, 0.875\}$

γ	0.125	0.250	0.500	0.750	0.875
$E[\theta]$	1.0009	0.9981	1.0009	1.0008	0.9992
$V[\theta]$	0.0023	0.0018	0.0011	0.0017	0.0028
$BIAS^2[\theta] \times 10^3$	0.0008	0.0034	0.0008	0.0006	0.0007
CI 2.5%	0.90	0.91	0.93	0.91	0.89
CI 97.5%	1.10	1.07	1.07	1.08	1.09
$IBIAS^2_{MC}[h]$	0.0022	0.0015	0.0030	0.0030	0.0044
$IVar_{MC}[h]$	0.0221	0.0287	0.0056	0.0147	0.0173
$IMSE^2_{MC}[h]$	0.0244	0.0302	0.0087	0.0177	0.0217

$\widehat{h}$: f = G-DEN and $\gamma \in \{0.125, 0.25, 0.5, 0.75, 0.825\}$



QQ-Plot: f = G-DEN and $\gamma \in \{0.125, 0.25, 0.5, 0.75, 0.825\}$



Monte Carlo: f = G-DEN, f = G-KER and $\gamma \in \{0.25, 0.50, 0.75\}$

γ f:	0.500 G-DEN	0.500 G-KER	0.750 G-KER	0.250 G-KER
$E[\theta]$	1.0009	1.0016	1.0013	0.9974
$V[\theta]$	0.0011	0.0010	0.0019	0.0016
2.5% CI	0.93	0.94	0.91	0.91
97.5% CI	1.07	1.06	1.08	1.09
$BIAS^2[\theta] \times 10^3$	0.0008	0.0021	0.0015	0.0064
$MISE[h]$	0.0087	0.0152	0.0499	0.0400
$IBIAS^2[h]$	0.0030	0.0048	0.0050	0.0058
$IVAR[h]$	0.0056	0.0094	0.0449	0.0341

Monte Carlo: f = G-DEN, $n = 125, 250, 500, 1000$ for $\gamma = 0.75$

n	125	250	500	1000
$E[\theta]$	1.0364	0.9926	1.0028	1.0008
$V[\theta]$	0.0278	0.0099	0.0039	0.0017

Monte Carlo: Estimators* for C.I. of $\hat{\theta}_n$ for f = G-DEN

Quantile:	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 0.25$
$E_{MC}[\theta]$	1.0009	1.0008	0.9981
$V_{MC}[\theta]$	0.0011	0.0017	0.0018
2.5% CI	0.93	0.91	0.91
97.5% CI	1.07	1.08	1.07
2.5% CI - BOOT	0.92	0.90	0.91
97.5% CI - BOOT	1.08	1.09	1.08
2.5% CI - χ^2	0.93	0.91	0.91
97.5% CI - χ^2	1.05	1.07	1.06

Application: Quantile IV Engel Curves

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- X_1: No Kids ($X_1 = 0$), Kids ($X_1 = 1$); Y_2: Total Expenditures. X_2 : Gross Earnings. $L = 7$ (food-in, food-out, alcohol, fares, fuel, leisure, travel).

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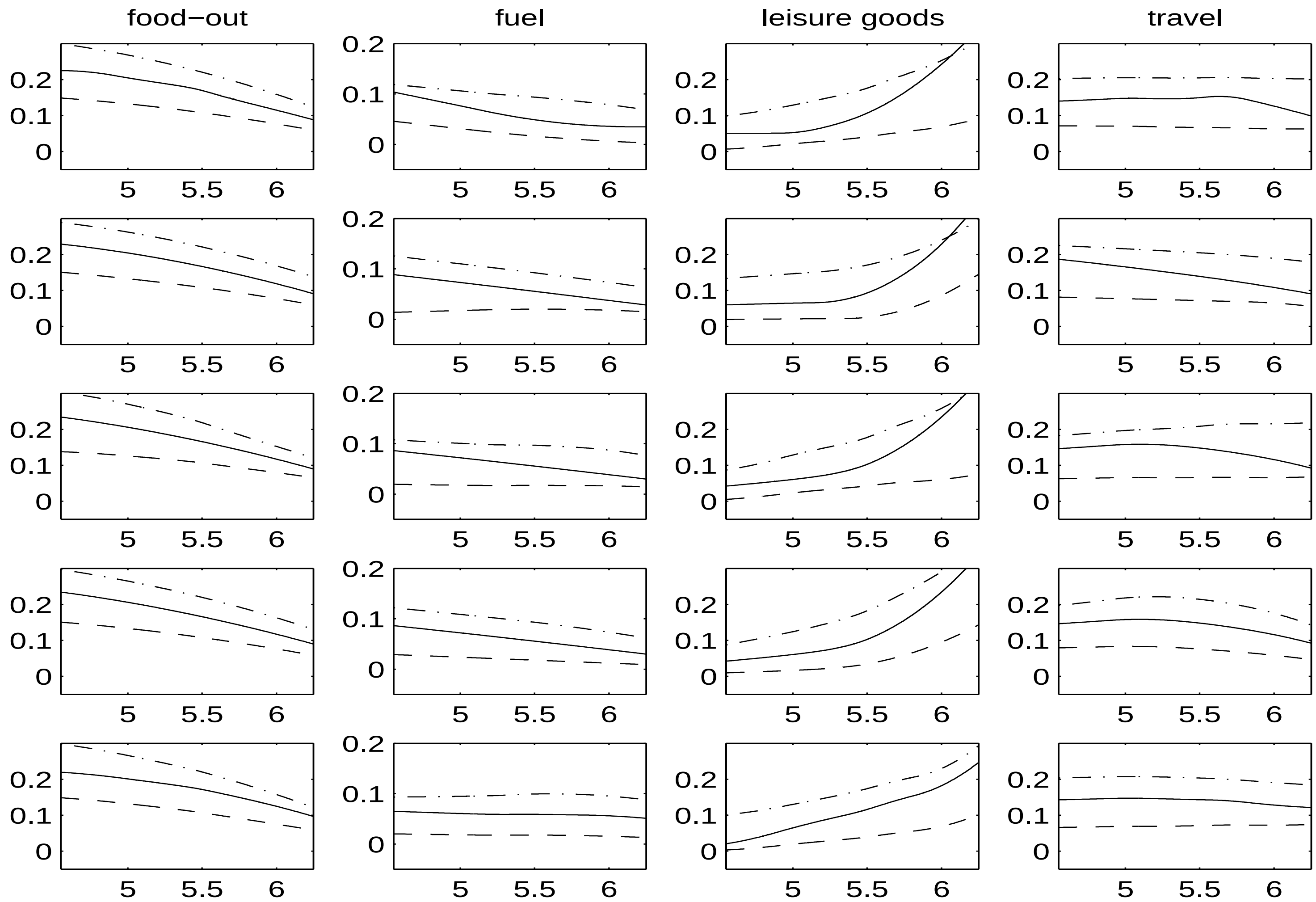
- $\widehat{m}(X, \alpha)$: P-Spline(5,10). $\mathcal{H}_n$: P-Spline(2,5).
- $P(h)$: $||\nabla^k h||^j_{L^j(d\hat{\mu})} \equiv n^{-1} \sum_{i=1}^n |\nabla^k h(Y_{2i})|^j$ for $k = 1, 2$ and $j = 1, 2$.

θ_l **for** $l = 1, \ldots, 7$ **for different penalty and** $\gamma = 0.50$

$\hat{P}_n(h)$	$\lvert\lvert\nabla^2 h\rvert\rvert^2_{L^2(d\hat{\mu})}$	$\lvert\lvert\nabla^2 h\rvert\rvert_{L^1(d\hat{\mu})}$	$\lvert\lvert\nabla h\rvert\rvert^2_{L^2(d\hat{\mu})}$	$\lvert\lvert\nabla^2 h\rvert\rvert^2_{L^2(d\hat{\mu})}$	$\lvert\lvert\nabla$
λ_n	0.001	0.001	0.001	0.0003	0
$\hat{\theta}_1$	0.4133	0.3895	0.5479	0.43136	0.
food-i	0.0200	0.0267	-0.0056	0.00989	0.
food-o	0.0010	0.0006	0.0019	0.00033	0.
alc'ol	-0.0195	-0.0123	-0.0171	-0.02002	-0.
fares	0.0106	-0.0031	-0.0001	-0.00009	-0.
fuel	-0.0027	0.0027	0.0004	-0.00198	-0.
lei're	0.0208	0.0214	0.0380	0.02582	0.
travel	-0.0207	-0.0218	-0.0084	-0.00622	-0.

Quantile IV Engel curves $\gamma = 0.25$ (dash), 0.50 (solid), 0.75 (dot-dash)

(1) $||\nabla^2 h||^2_{L^2(d\hat{\mu})}$, $\lambda_n = 0.001$; (2) $||\nabla^2 h||_{L^1(d\hat{\mu})}$, $\lambda_n = 0.001$; (3) $||\nabla h||^2_{L^2(d\hat{\mu})}$, $\lambda_n = 0.001$, (4) $\lambda_n = 0.003$; (5) $||\nabla h||^2_{L^2(leb)}$, $\lambda_n = 0.005$.



Quantile IV Engel Curves ("Pooled" sample) (cont.)

γ	0.250	0.500 (BCK)	0.750
θ_1	0.669714	0.415483 (0.4088)	0.381019
θ_{21} - Food In	0.002548	0.013011 (0.0191)	0.037018
θ_{22} - Food Out	0.000504	0.000508 (-0.0002)	-0.000270
θ_{23} - Alcohol	-0.001969	-0.005315 (-0.0285)	0.046248
θ_{24} - Fares	-0.026957	-0.001056 (-0.0011)	0.001449
θ_{25} - Fuel	-0.010338	-0.006796 (-0.0038)	0.013448
θ_{26} - Leisure	0.003206	0.035873 (0.0496)	0.052509
θ_{27} - Travel	-0.034212	-0.036183 (-0.0399)	-0.045201

Table 1: θ_1 *and* $\theta_{2l},\ l = 1, ..., 7$

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- Strong Norm, $||\cdot||_s$ on $\mathcal{H}$ is the "standard" norm associated to the Banach space $\mathcal{H}$, e.g., L^p norms.

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- Chen-Pouzo (07) establish general consistency results of the penalized SMD estimator $\widehat{\alpha}_n$ without imposing identification of α_0, permitting flexible penalization function $\widehat{P}_n(h)$, and allowing for any consistent estimator $\widehat{m}(X,\alpha)$ of $m(X,\alpha)$

Weak Pseudo-Norm

- Under ill-posedness, the convergence rate under $||\cdot||_s$ is typically slower than $n^{-1/4}$.
- Ai - Chen (03) introduce a "weaker" pseudo-metric $||\cdot||$ (i.e., $||\alpha|| \leq ||\alpha||_s$): $||\alpha - \alpha'||^2 \equiv$

$$E\left[\frac{dm(X,\alpha_0)}{d\alpha}[\alpha - \alpha']'[\Sigma(X)]^{-1}\frac{dm(X,\alpha_0)}{d\alpha}[\alpha - \alpha']\right]$$

where $\frac{dm(X,\alpha_0)}{d\alpha}[u] \equiv \lim_{t\to 0} \frac{E[\rho(Z,(1-t)\alpha_0+t(\alpha_0+u))|X]}{t}$.

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  where $\frac{dm(X,\alpha_0)}{d\alpha}[u] \equiv \lim_{t\to 0}\frac{E[\rho(Z,(1-t)\alpha_0+t(\alpha_0+u))|X]}{t}$.
- Ex: NPIV $E[Y_1 - h_0(Y_2)|X] = 0$. $||\alpha||_s^2 = E[h(Y_2)^2]$, $||\alpha||^2 = E[(E[h(Y_2)|X])^2]$.

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- Ai - Chen (03) introduce a "weaker" pseudo-metric $||\cdot||$ (i.e., $||\alpha|| \leq ||\alpha||_s$): $||\alpha - \alpha'||^2 \equiv$

$$E\left[\frac{dm(X,\alpha_0)}{d\alpha}[\alpha - \alpha']'[\Sigma(X)]^{-1}\frac{dm(X,\alpha_0)}{d\alpha}[\alpha - \alpha']\right]$$

  where $\frac{dm(X,\alpha_0)}{d\alpha}[u] \equiv \lim_{t\to 0}\frac{E[\rho(Z,(1-t)\alpha_0 + t(\alpha_0+u))|X]}{t}$.
- Ex: NPIV $E[Y_1 - h_0(Y_2)|X] = 0$. $||\alpha||_s^2 = E[h(Y_2)^2]$, $||\alpha||^2 = E[(E[h(Y_2)|X])^2]$.
- Ai - Chen (03) obtain $||\hat{\alpha}_n - \alpha_0|| = o_p(n^{-1/4})$.

Convergence Rate in Weaker Metric

- **Thm 3.1** $\hat{\alpha}_n$ is penalized SMD with $||\hat{\alpha}_n - \alpha_0||_s = o_P(1)$. Then: For lower semicompact penalty,

$$||\hat{\alpha}_n - \Pi_n\alpha_0|| = O_P\left(\max\left\{\sqrt{\frac{J_n}{n}} + b_{J_n}, ||\Pi_n\alpha_0 - \alpha_0||, \sqrt{\lambda_n}\right\}\right)$$

For convex but non-lower semicompact penalty,

$$||\hat{\alpha}_n - \Pi_n\alpha_0|| = O_P\left(\max\left\{\sqrt{\frac{J_n}{n}} + b_{J_n}, ||\Pi_n\alpha_0 - \alpha_0||, o(\sqrt{\lambda_n})\right\}\right)$$

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- Without nonparametric endogeneity, the weaker and strong metrics are equivalent. Thm 3.1 leads to optimal convergence rates for penalized sieve M-estimators when $\rho()$ could be non-smooth.

Sieve Measure of Ill-posedness

- Define a *sieve measure of ill-posedness* as $\tau_n \equiv$

$$\sup_{\alpha \in \mathcal{A}_{osn}:\alpha \neq \Pi_n \alpha_0} \frac{||\alpha - \Pi_n \alpha_0||_s}{||\alpha - \Pi_n \alpha_0||} \asymp \frac{||\alpha - \Pi_n \alpha_0||_s}{\sqrt{E\left[\left(\frac{dm(X,\alpha_0)}{d\alpha}[\alpha - \Pi_n \alpha_0]\right)^2\right]}},$$

where $\mathcal{A}_{osn} = \{\alpha \in \mathcal{A}_n : ||\alpha - \Pi_n \alpha_0||_s = o(1)\}$.

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  where $\mathcal{A}_{osn} = \{\alpha\in\mathcal{A}_n : ||\alpha-\Pi_n\alpha_0||_s = o(1)\}$.

- This definition is a generalization of that in BCK (03) for nonparametric IV regression $E[Y_1 - h(Y_2)|X] = 0$:

$$\tau_n = \sup_{h_n\in\mathcal{H}_n:h_n\neq 0} \frac{\sqrt{E\{h_n(Y_2)\}^2}}{\sqrt{E\{E[h_n(Y_2)|X]\}^2}},$$

- In BCK, $\tau_n = 1$ iff Y_2 is measurable w.r.t. X.

Modulus of Continuity

- *Modulus of Continuity*:

$$\omega(\delta, \mathcal{A}_{os}) = \sup_{\{\alpha \in \mathcal{A}_{os} : ||\alpha - \alpha_0|| \leq \delta\}} ||\alpha - \alpha_0||_s$$

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- τ_n and $\omega_n(\delta, \mathcal{A}_{osn})$ measures do depend on choice of sieve space; only useful for finite-dimensional sieves.

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- **Thm 3.2** Under conditions of Thm 3.1 and A3.2, if $\max\left\{\frac{J_n}{n} + b^2_{m,J_n}, \lambda_n\right\} = \frac{J_n}{n}$, then:

  $$||\hat{\alpha}_n - \alpha_0||_s = O_P\left(||\alpha_0 - \Pi_n\alpha_0||_s + \tau_n \times \sqrt{\frac{J_n}{n}}\right)$$

  $$= O_P\left(||\alpha_0 - \Pi_n\alpha_0||_s + \omega_n(\sqrt{\frac{J_n}{n}}, \mathcal{A}_{osn})\right).$$

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  $$= O_P\left(||\alpha_0 - \Pi_n\alpha_0||_s + \omega_n(\sqrt{\frac{J_n}{n}}, \mathcal{A}_{osn})\right).$$
- Thm 3.2 directly extends BCK (07) on nonparametric IV regression to nonlinear or nonsmooth ill-posed problems.

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$$\max\left\{\sqrt{\frac{J_n}{n}} + b_{m,J_n}, \sqrt{\lambda_n}\right\} = \sqrt{\frac{J_n}{n}} + b_{m,J_n} = O(\sqrt{\lambda_n}),$$
- or for convex but non-lower semicompact penalty with
$$\max\{\sqrt{\frac{J_n}{n}} + b_{m,J_n}, \sqrt{\lambda_n ||\hat{\alpha} - \Pi_n \alpha_0||_s}\} = \sqrt{\frac{J_n}{n}} + b_{m,J_n},$$

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- Under conditions of Thm 3.1 and A3.2, we have:

$$||\hat{\alpha}_n - \Pi_n \alpha_0||_s = O_P\left(||h_0 - \Pi_n h_0||_s + \omega_n(\{\sqrt{\frac{J_n}{n}} + b_{m,J_n}\}, \mathcal{A}_{osi}$$

Sufficient Conditions for Convergence Rates

- A3.5: $\{q_j\}_{j=1}^{\infty}$ is a Riesz basis for a separable Hilbert space $(\mathcal{H}, ||\cdot||_s)$, and $\mathcal{H}_{os}$ is a subset of $\mathcal{H}$.

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- A3.6: Let $\mathcal{H}_n = clsp\{q_1, ..., q_{k(n)}\}$. There is a non-increasing positive sequence $\{b_j\}_{j=1}^{\infty}$ such that: (i) $||h||^2 \geq c\sum_{j=1}^{\infty} b_j |\langle h, q_j\rangle_s|^2$ for all $h \in \mathcal{H}_{osn}$; (ii) $C\sum_j b_j |\langle h_0 - \Pi_n h_0, q_j\rangle_s|^2 \geq ||h_0 - \Pi_n h_0||^2$.

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- Lemma: Let $\mathcal{H}_n = clsp\{q_1, ..., q_{k(n)}\}$, A3.5 and A3.6 hold. Then: A3.2 is satisfied, and

$$\tau_n \leq const./\sqrt{b_{k(n)}} \quad \text{and} \quad \omega_n(\delta, \mathcal{H}_{osn}) \leq const.\times\delta/\sqrt{b_{k(n)}}.$$

Sufficient Conditions for Convergence Rates

- let $a > 0$ be a finite constant. (i) If $b_j \asymp j^{-2a}$ then $\tau_n \leq const.(k(n))^a$. (ii) If $b_j \asymp \exp\{-j^a\}$ then $\tau_n \leq const. \exp\{\frac{1}{2}(k(n))^a\}$.

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- assume $||\alpha_0 - \Pi_n \alpha_0||_s = O(k(n)^{-\mu_h})$, $J_n = ck(n)$ for $c \geq 1$,
- if $\tau_n \leq const.(k(n))^a$, then $||\widehat{\alpha}_n - \alpha_0||_s = O_p(n^{-\frac{\mu_h}{2(a+\mu_h)+1}})$;
- if $\tau_n \leq const. \exp\{\frac{1}{2}(k(n))^a\}$, and $\mu_m = \infty$, then $||\widehat{\alpha}_n - \alpha_0||_s = O_p([\log(n)]^{-\mu_h/a})$.

Root-n Normality and Efficiency

- **Asymptotic Normality of $\hat{\theta}_n$:**

$$\begin{aligned} \sqrt{n} \quad & (\hat{\theta}_n - \theta_0) \Rightarrow \mathcal{N}(0, V^{-1}) \\ V^{-1} \quad = \quad & E[D_{w^*}(X)'\Sigma(X)^{-1}D_{w^*}(X)]^{-1} \times \\ & E[D_{w^*}(X)'\Sigma(X)^{-1}\Sigma_0(X)\Sigma(X)^{-1}D_{w^*}(X)] \times \\ & E[D_{w^*}(X)'\Sigma(X)^{-1}D_{w^*}(X)]^{-1}. \end{aligned}$$

with w^* as the minimizer of: $E[D_w(X)'\Sigma(X)^{-1}D_w(X)] =$

$$E\left[\left(\frac{dm(X,\alpha_0)}{d\alpha}[(\mathbf{1}, -w)]\right)' \Sigma^{-1}(X) \left(\frac{dm(X,\alpha_0)}{d\alpha}[(\mathbf{1}, -w)]\right)\right]$$

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$$

- **Efficiency:** $V_0 = \inf_w E[D_w(X)'\Sigma_0(X)^{-1}D_w(X)]$.

Weighted Bootstrap

- **Thm**: Let $\{W_i > 0\}_{i=1}^n$ be i.i.d. with $E[W_i] = 1$, $Var(W_i) = w_0$, and is indep. of the data $\{(Y_i', X_i')\}_{i=1}^n$.

$$\widehat{\alpha}_n^* \equiv \arg \inf_{\alpha \in \mathcal{N}_{0n}} \left\{ \frac{1}{n} \sum_{i=1}^n W_i \left\{ \widehat{m}(X_i, \alpha)' [\widehat{\Sigma}(X_i)]^{-1} \widehat{m}(X_i, \alpha) \right\} + \lambda_n P(h) \right.$$

Then: Conditional on the data, $\sqrt{\frac{n}{w_0}} \left(\widehat{\theta}_n^* - \widehat{\theta}_n \right)$ has the same limiting dist. as that of $\sqrt{n} \left(\widehat{\theta}_n - \theta_0 \right)$.

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- **W.B. Algorithm**: (1) Draw an i.i.d. sample $\{W_i > 0\}_{i=1}^n$ with $E(W_i) = 1$, $Var(W_i) = 1$, and compute $\widehat{\alpha}_n^*$; (2) Repeat step (1) many times (say N numbers of times) and compute the empirical quantiles of $(\widehat{\theta}_{n,q}^*)_{q=1}^N$.

Partially Linear Quantile IV

- $Y_{1i} = X_{1i}\theta_0 + h_0(Y_{2i}) + U_i$ with $F_{U|X}(0|X) = \gamma$.
  - $\mathcal{A} = [\underline{\theta}, \overline{\theta}] \times \mathcal{H}$.
  - $\mathcal{A}_n = [\underline{\theta}, \overline{\theta}] \times \{h \ : \ h(y_2) = q^{k_n}(y_2)'\beta\} \cap \mathcal{H}$.
  - $Y = (Y_1, Y_2)$, $X = (X_1, X_2)$, $dim(X) = 2$.

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  - $Y = (Y_1, Y_2)$, $X = (X_1, X_2)$, $dim(X) = 2$.
- $\rho(Z, \alpha) = \gamma - \mathcal{I}\{Y_1 - (X_1\theta + h(Y_2)) \leq 0\}$.
- $m(X, \alpha) = \gamma - \int F_{Y_1|Y_2,X}(x_1\theta + h(y_2)) f_{Y_2|X}(y_2, X) dy_2$.

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- $m(X, \alpha) = \gamma - \int F_{Y_1|Y_2,X}(x_1\theta + h(y_2)) f_{Y_2|X}(y_2, X) dy_2$.
- Case I: $(\Lambda_c^{r_h}(\mathcal{R}), ||h||_c = ||h \times w||_{L^\infty})$, $w(y) = (1 + y^2)^{-c}$, and $\lambda_n = 0$ ($\approx$ AC (03)).
- Case II: $(L^2(\mathcal{R}) \cap ||h||_c \leq M, ||h|_c = ||h||_{L^2})$ and $\lambda_n > 0$, $P(h) = ||D^s h||^2_{L^2}$ ($\approx$ HL (07)).

Partially Linear Quantile IV (cont.)

A: Low level standard assumptions:

- Smoothness and boundedness of $F_{Y_1|Y_2,X}$.
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- Identification conditions (CIN (07)).

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- Smoothness and boundedness of $F_{Y_1|Y_2,X}$.
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- Case I: A + $E[w^{-2}|X] \leq M < \infty$ then $|\widehat{\theta}_n - \theta_0| + \sup_y |(\widehat{h}_n - h_0)w(y)| = o_p(1)$.
- Case II: A + $\frac{n^{-2r_m/(2r_m+1)}}{\lambda_n} = o_p(1)$ then $|\widehat{\theta}_n - \theta_0| + ||\widehat{h}_n - h_0||_{L^2} = o_p(1)$.

Partially Linear Quantile IV (cont.)

- Case I: A + A3.5 - A3.6 + $\int f_{Y|X} w^{-2} \leq M < \infty$ then:
  - (i) If $b_k \leq \mu_k \asymp k^{-2a}$:
    $||\widehat{h}_n - h_0||_{L^2} = O_p(n^{-r_h/(2(a+r_h)+1)})$,
    with $k_n = O(n^{1/(2(a+r_h)+1)})$.
  - (ii) If $b_k \leq \mu_k \asymp \exp\{k^{-a}\}$ and $r_m = \infty$:
    $||\widehat{h}_n - h_0||_{L^2} = O_p([\ln(n)]^{-r_h/a})$,
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- Case II: polynomial rate coincides with HL (07).

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- Key Assumptions:

E.1: Specific rate for $\widehat{\alpha}_n$ under "strong" norm, i.e., $\sqrt{J_n/n}||\widehat{\alpha}_n - \alpha_0||_{L^2} = o(n^{-1/2})$.

E.2: $\widehat{h}_\theta$ satisfies stochastic equi-continuity type of restrictions $\forall\theta \; : \; |\theta - \theta_0| = O_p(n^{-1/2})$.

- $\widehat{h}_\theta$ is the profiled estimator of h, i.e., fixing θ, we solve the SMD problem for h.
- E.2 is easy to check in linear problems (e.g. linear IV semiparametric regression) but hard to verify for non-linear problems.

Partially Linear Quantile IV (cont.)

- Both Cases: if A - E hold and the problem is **mildly** ill-posed, then $\widehat{\theta}_n$ is Asymp. Normal with variance

$$\frac{1}{\gamma(1-\gamma)} E\left[\left(\int f_{Y|X}(X_1\theta_0 + h_0; y_2, X)(X_1 - w^*)dy_2\right)^2\right].$$

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- If $Y_2 = X_2$ and $f_{U|X} = f_U$ then $V = f_U^2(0)\frac{E[Var(X_1|X_2)]}{\gamma(1-\gamma)}$ which is optimal, Lee (03).
- Under **severely** ill-posedness some regularity restrictions on 2nd order approx. term are difficult to check.

Conclusion

- We propose penalized SMD estimators for semi/nonparametric conditional moment models, allowing for:
  - Possibly non-smooth generalized residual functions.
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  - Possibly non-compact infinite-dimensional parameter space and sieve spaces, with flexible penalties.
- Obtain efficiency and normality of parametric part.
- Establish convergence rate for nonparametric functions that may depend on endogenous variables.
- Future work:
  - Data-driven choice of smoothing parameters.
  - Time series extension.
  - Partially identified semi/nonparametric conditional moment models.