Long-term components of risk prices[1]

Lars Peter Hansen

Tjalling C. Koopmans Lectures, September 2008

[1] Related papers:Hansen,Heaton and Li, JPE, 2008; Hansen and Scheinkman, forthcoming Econometrica; Hansen, Fisher-Schultz Lecture

Koopmans and recursive preferences

Koopmans initiated an important line of research on recursive preferences that pushed beyond the additive discounted utility framework.

References:

- Stationary Ordinal Utility and Impatience - Econometrica 1960
- Stationary Utility and the Time Perspective - Econometrica 1964 with Diamond and Williamson

Koopmans and recursive utility

Utility representation:

$$V_t = \Phi[U(C_t), V_{t+1}]$$

as a generalization of

$$V_t = U(C_t) + \beta V_{t+1}$$

where C_t is the current period consumption vector, V_t is the "continuation value" or what Koopmans called the "prospective" utility.

UNCERTAINTY

Kreps-Porteus representation

$$V_t = \Phi\left[U(C_t), E\left(V_{t+1}|\mathcal{F}_t\right)\right]$$

as a generalization of expected utility

$$V_t = U(C_t) + \beta E\left(V_{t+1}|\mathcal{F}_t\right).$$

Do not “reduce” intertemporal compound consumption lotteries. Intertemporal composition of risk matters.

I will feature a convenient special case

$$V_t^* = (1-\beta)\log C_t + \frac{\beta}{1-\gamma}\log E\left(\exp\left[(1-\gamma)V_{t+1}^*\right]|\mathcal{F}_t\right)$$

where I have taken a monotone transformation of the continuation value. Links risk sensitive control and recursive utility.

Empirical Macroeconomics

- Identify macroeconomic shocks: w_t
- Quantify responses to those shocks. How does the future macro or financial economic vector y_{t+j} depend on the current shock w_t?
- Compare models with alternative mechanism by which these shocks are transmitted to macro time series

How can asset pricing contribute?

- The macroeconomic shocks are the ones that cannot be diversified.
- Investors that are "exposed" to such shocks require compensation for bearing this risk.

Add (shadow) pricing counterparts to these impulse responses.

- Pricing dual to an impulse response: What is the current period "price" of an exposure to future macroeconomic growth rate shocks?
- Compare how alternative economic models assign prices to exposures even when these exposures are not in the center of the support of the historical time series. Structural model in the sense of Hurwicz.

Some history

The manner in which risk operates upon time preference will differ, among other things, according to the particular periods in the future to which the risk applies.

Irving Fisher (Theory of Interest, 1930)

RESPONSE AND RISK PRICE TRAJECTORIES

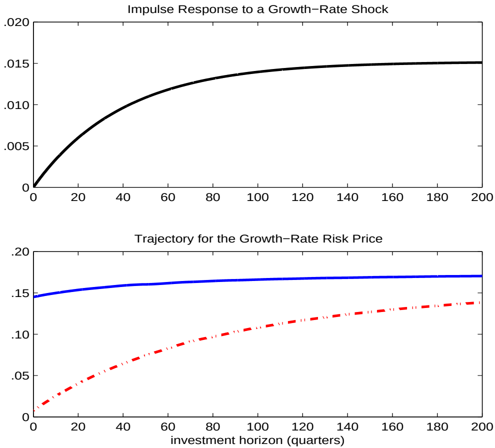


FIGURE: The horizontal axis is given in quarterly time units. The top panel gives the impulse-responses of the logarithm of consumption and the bottom panel gives corresponding risk prices for two alternative models. Blue assumes recursive utility and red assumes power utility.

Remainder of the talk

1. My approach to characterizing risk-price dynamics.
2. Mathematical support for value decompositions.
3. Model comparisons and long-run components to value.
4. Recursive utility versus power utility: a comparison.
5. Recursive utility: a "robust" interpretation.

1. CHARACTERIZING RISK-PRICE DYNAMICS

- Use Markov formulations and martingale methods to produce decompositions of model implications.
- Allow for nonlinear time series models - stochastic volatility, stochastic regime shifts.
- Use the long-term as a frame of reference.

Stochastic growth and discount factors will be state dependent. Explore the implications of this dependence when we alter the forecast or payoff horizon.

WHY USE THE LONG TERM AS A FRAME OF REFERENCE?

- growth uncertainty has important consequences for welfare
- stochastic component growth can have a potent impact on asset values
- economics more revealing for modeling long-run phenomenon

Components of asset values

1. One period returns: bundles of state-contingent payoffs in a single period or Arrow securities.
   - An economic model predicts prices for the components of single-period payoffs - assigns values to one period risk exposures.
2. Intertemporal counterpart: price bundled consumption claims across states and time periods; durable assets.
   - An economic model predicts prices of intertemporal cash flows or hypothetical consumption processes - assigns values to risk exposures at alternative future points in time.

Asset valuation and stochastic discounting

$$\pi = \sum_{t=0}^{\infty} E\left[S_t G_t | x_0\right]$$

where π is the date zero price of a "cash flow" or "dividend" process $\{G_t\}$ that grows stochastically over time.

$\{S_t\}$ is a stochastic discount factor process. Encodes both discounting and adjustments for risk. Satisfies consistency constraints - Law of Iterated Values.

- Economic models imply stochastic discount factor

  $$S_t \quad \text{Intertemporal investors}' \text{ MRS}$$

  Work of Koopmans, Kreps and Porteus and others expanded the array of models of investor preferences.
- Dynamics of pricing are captured by the time series behavior of the stochastic discount factor.

CHALLENGES

Extract dynamic pricing implications in a revealing way.

Compare models and model ingredients.

2. Mathematical setup

- $\{X_t : t \geq 0\}$ be a continuous time Markov process on a state space $\mathcal{D}$. This process can be stationary and ergodic.
- $X = X^c + X^d$
- X^c is the solution to $dX_t^c = \mu(X_{t-})dt + \sigma(X_{t-})dW_t$ where W is an $\{\mathcal{F}_t\}$ Brownian motion and $X_{t-} = \lim_{\tau \downarrow 0} X_{t-\tau}$.
- X^d with a finite number of jumps in any finite interval.

Simple distinction between small shocks and big shocks.
Discrete time works too, but Markov structure is central.

ADDITIVE FUNCTIONAL - DEFINITION

- Construct a scalar process $\{Y_t : t \geq 0\}$ as a function of X_u for $0 \leq u \leq t$.
- An additive functional is parameterized by (β, γ, κ) where:
    - I) $\beta : \mathcal{D} \to \mathbb{R}$ and $\int_0^t \beta(X_u)du < \infty$ for every positive t;
    - II) $\gamma : \mathcal{D} \to \mathbb{R}^m$ and $\int_0^t |\gamma(X_u)|^2 du < \infty$ for every positive t;
    - III) $\kappa : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, $\kappa(x, x) = 0$.

$$Y_t = \underbrace{\int_0^t \beta(X_u)du}_{\text{smooth}} + \underbrace{\int_0^t \gamma(X_u) \cdot dW_u}_{\text{small shocks}} + \underbrace{\sum_{0 \leq u \leq t} \kappa(X_u, X_{u-})}_{\text{big shocks}}$$

- Process Y is nonstationary and can grow linearly.
- Sums of additive functionals are additive. Add the parameters.

MULTIPLICATIVE FUNCTIONAL - DEFINITION

- Let $\{Y_t : t \geq 0\}$ be an additive functional.
- Construct a multiplicative functional $\{M_t : t \geq 0\}$ as

$$M_t = \exp(Y_t)$$

- Process M is nonstationary and can grow exponentially.
- products of multiplicative functionals are multiplicative. Multiply the parameters.

Use multiplicative functionals to model state dependent growth and discounting.

Discrete-time counterpart

Additive functional

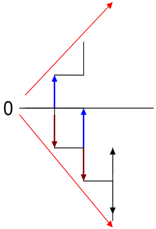
$$Y_t = \sum_{j=1}^{t} \kappa(X_j, X_{j-1})$$

Multiplicative functional

$$M_t = \prod_{j=1}^{t} \exp[\kappa(X_j, X_{j-1})]$$

Use multiplicative functionals to model state dependent growth and discounting.

ILLUSTRATION



$Y_0=0$

$Y_1 = Y_0$ +/- 1

$Y_2 = Y_1$ +/- 1

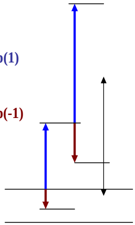
…

Range of Y grows + or -

A Simple Discrete-Time *Additive* Functional Y

$k(X_t, X_{t-1})$ =+/- 1

Note: X’s can be temporally dependent



A Simple *Multiplicative* Functional M made from exponentiating Y

$Y_t = \exp[k(X_0,X_1)]\exp[k(X_1, X_2)]\ldots\exp[k(X_{t-1},X_t)]$

$Y_0 = 0$ $M_0 = \exp(0) = 1$

$Y_1 = Y_0$ +/- 1 $M_1 = \exp(0)\exp(+/- 1)$

$Y_2 = Y_1$ +/- 1 $M_2 = \exp(0)\exp(+/-1)\exp(+/- 1)$

Range of M is non-negative.

Multiplicative decomposition

$$M_t = \underset{\substack{\uparrow \\ \textbf{exponential trend}}}{\exp(\rho t)} \quad \underset{\substack{\uparrow \\ \textbf{martingale}}}{\hat{M}_t} \quad \underset{\substack{\uparrow \\ \textbf{ratio}}}{\left[\frac{e(X_0)}{e(X_t)}\right]}. \tag{1}$$

- ρ is a deterministic growth rate;
- $\hat{M}_t$ is a multiplicative martingale;
- e is a strictly positive function of the Markov state;

Observations

- Reminiscent of a permanent-transitory decomposition from time series. Important differences!
- Not unique and co-dependence between components matters.

Why?

- In valuation problems there are two forces at work - stochastic growth G and stochastic discounting S. Study product SG.
- Decompose pricing implications of a model as represented by a stochastic discount factor S.
- Term structure of risk prices - look at value implications of marginal changes in growth exposure as represented by changes in G.

Frobenius-Perron Theory/ Martingales

- Solve,

$$E\left[M_t e(X_t) | X_0 = x\right] = \exp(\rho t) e(x)$$

where e is strictly positive. Eigenvalue problem.

- Construct martingale

$$\hat{M}_t = \exp(-\rho t) M_t \left[\frac{e(X_t)}{e(X_0)}\right].$$

- Invert

$$M_t = \exp(\rho t) \hat{M}_t \left[\frac{e(X_0)}{e(X_t)}\right].$$

MULTIPLICATIVE MARTINGALES

Decomposition:

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{e(X_0)}{e(X_t)}\right].$$

Observations about $\hat{M}$.

1. Converge - often to zero - raise to powers for refined analysis - Chernoff
2. Change of measure
   - preserves Markov structure
   - at most one is stochastically stable - Hansen-Scheinkman

Stochastic stability

$$\exp(-\rho t) E\left[M_t f(X_t) | X_0 = x\right] = e(x) \hat{E}\left[\frac{f(X_t)}{e(X_t)} | X_0 = x\right]$$

Under stochastic stability and the moment restriction:

$$\hat{E}\left[\frac{f(X_t)}{e(X_t)}\right] < \infty,$$

the right-hand side converges to:

$$e(x) \hat{E}\left[\frac{f(X_t)}{e(X_t)}\right]$$

Common state dependence independent of f.

Hyperbolic approximation in valuation horizon:

$$\frac{1}{t} \log E\left[M_t f(X_t) | X_0 = x\right] \approx \rho + \frac{1}{t}\left(\log e(x) + \log \hat{E}\left[\frac{f(X_t)}{e(X_t)}\right]\right)$$

Long-term cash flow risk

$$\rho(M) = \lim_{t\to\infty} \frac{1}{t} \log E\left[M_t | X_0 = x\right].$$

- Cash flow return over horizon t:

$$\frac{E\left(G_t | X_0 = x\right)}{E\left(S_t G_t | X_0 = x\right)}.$$

- long-term expected rate of return (risk adjusted):

$$\rho(G) - \rho(SG).$$

- long-term expected excess rate of return (risk adjusted):

$$\rho(G) + \rho(S) - \rho(SG)$$

using $G = 1$ as a long run risk free reference.

3. Model comparison

Factorization

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{e(X_0)}{e(X_t)}\right].$$

$$\text{If} \quad M_t^* = M_t \left[\frac{f(X_t)}{f(X_0)}\right],$$

then M and M^* share the same martingale component.

$$\begin{array}{ccccc} M & = & S & & G \\ & & \uparrow & & \uparrow \\ & & \text{discount} & & \text{growth} \end{array}$$

Observations:

- Applied to G - long-term components of consumption processes or cash flows - in the limit these dominate pricing.
- Applied to S - long-term model components of valuation - models with common martingale components share the same long-term value implications.

Transient Model Components

Bansal-Lehmann style decomposition:

$$S_t^* = S_t \left[\frac{f(X_t)}{f(X_0)} \right].$$

Same martingale component

Examples

- Habit persistence models - big differences between empirical macro and empirical asset pricing models
- Solvency constraint models
- Recursive utility models, long-term risk prices coincide with those from a power utility model
- Preference shock models and social externalities,
  - i) Santos-Veronesi - asset pricing - transient value implications relative to power utility model.
  - ii) Campbell-Cochrane - asset pricing - more subtle limiting analysis.

RISK PRICE FIGURE REVISITED

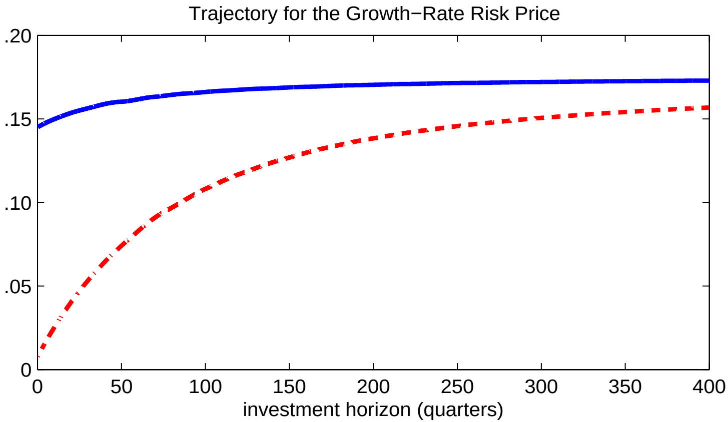


FIGURE: The horizontal axis is given in quarterly time units. Blue assumes recursive utility and red assumes power utility.

Term structure of risk prices

Risk price (for horizon j investment) is the marginal change in the risk premia of a martingale cash flow with payoff in j time periods. Build martingales from alternative macroeconomic shock processes.

Term structure emphasizes the time dependence on the horizon of the payoffs being priced.

Risk price in a log-linear model is the cumulative response of the the stochastic discount factor process to a shock. Constructed much more generally in a nonlinear Markov environment.

The dynamics for risk prices are encoded in the dynamics of the stochastic discount factor process.

Risk prices continued

- parameterize (multiplicative) martingale cash flows : $G(\alpha)$
  - why? eliminate consumption or cash flow dynamics
  - how? construct them or extract them from macro or financial cash flows
- Compute risk prices

$$-\frac{1}{t}\frac{\partial \log E\left[S_t G_t(\alpha)|X_0 = x\right]}{\partial \alpha}$$

Observations

- Nonlinear pricing;
- Hyperbolic approximations in the payoff horizon;

RISK-PRICE FIGURE REVISITED

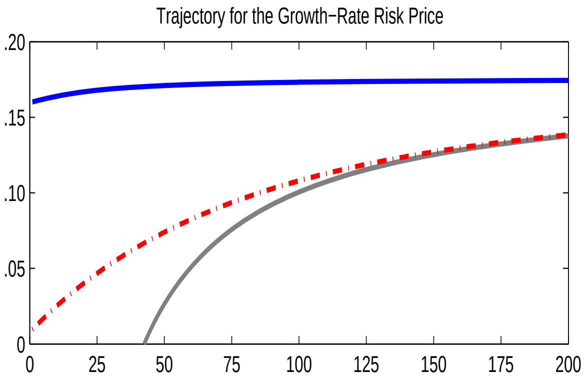


FIGURE: The horizontal axis is given in quarterly time units. Blue assumes recursive utility and red assumes power utility. Grey line is the hyperbolic approximation.

4. Recursive utility investors

$$V_t^* = (1-\beta)\log C_t + \frac{\beta}{1-\gamma}\log E\left(\exp\left[(1-\gamma)V_{t+1}^*\right]|\mathcal{F}_t\right)$$

- Risk sensitive control theory (Jacobson, Whittle) - linked to recursive utility theory (Kreps-Porteus and Epstein-Zin). Achieved by applying an exponential risk adjustment to continuation values. Hansen-Sargent.
- Intertemporal compound lotteries are no longer "reduced." The intertemporal composition of risk matters.
- Macroeconomic/asset pricing implications originally studied by Tallarini - increases risk prices while having modest implications for stochastic growth models - (stochastic counterparts to Koopmans' growth model).
- Asset pricing "success" achieved by imposing high risk aversion (Tallarini) or a predictable growth component (Bansal and Yaron).

Reminder

The study of asset pricing implications typically focus on one-period risk prices, but not on the entire term-structure of risk prices.

Asset pricing

In the discounted version of recursive risk-sensitive preferences, the stochastic discount factor is;

$$S_t^* = \exp(-\delta t) \left(\frac{C_0}{C_t} \right) \hat{V}_t$$

where $\hat{V}$ is a martingale component of the $\left\{ \left(\frac{V_t}{V_0} \right)^{1-\gamma} : t \geq 0 \right\}$ where V is the stochastic process of continuation values.

The process V and hence $\hat{V}$ are constructed from the underlying consumption dynamics. δ continues to be the subjective rate of discount and the inverse ratio of consumption growth reflects a unitary intertemporal elasticity of substitution in the preferences of the investor.

LIMITING STOCHASTIC DISCOUNT FACTOR

- Martingale component for consumption and continuation values:

$$\hat{V}_t = \exp(-\rho t) \left(\frac{C_t}{C_0} \right)^{1-\gamma} \frac{e(X_t)}{e(X_0)}.$$

- Limiting stochastic discount factor

$$S_t^* = \left(\frac{C_0}{C_t} \right) \hat{V}_t = \exp(-\rho t) \left(\frac{C_t}{C_0} \right)^{-\gamma} \left(\frac{e(X_t)}{e(X_0)} \right).$$

Different limiting risk-free interest rate but the same long-term risk prices.

Consumption dynamics

Suppose that X and Y evolve according to:

$$
\begin{aligned}
dY_t &= \nu + H_1 X_t^{[1]} dt + \sqrt{X_t^{[2]}} F dW_t \\
dX_t^{[1]} &= A_1 X_t^{[1]} dt + \sqrt{X_t^{[2]}} B_1 dW_t, \\
dX_t^{[2]} &= A_2 (X_t^{[2]} - 1) dt + \sqrt{X_t^{[2]}} B_2 dW_t
\end{aligned}
$$

Variables

- Y is the logarithm of consumption.
- $X^{[1]}$ governs the predictable growth rate in consumption.
- $X^{[2]}$ governs the macro volatility.

Shocks dW

- $F_1 dW_t$ is the consumption shock.
- $B_1 dW_t$ is the consumption growth shock.
- $B_2 dW_t$ is the consumption volatility shock.

Growth-rate state variable

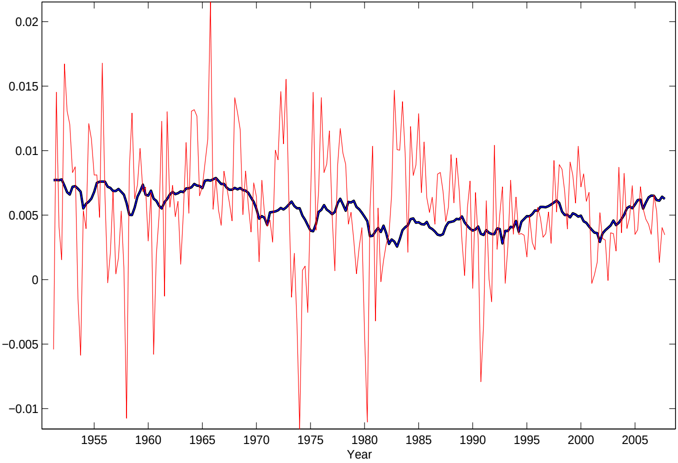


Figure: Consumption growth rate and growth-rate state variable.

VOLATILITY STATE VARIABLE

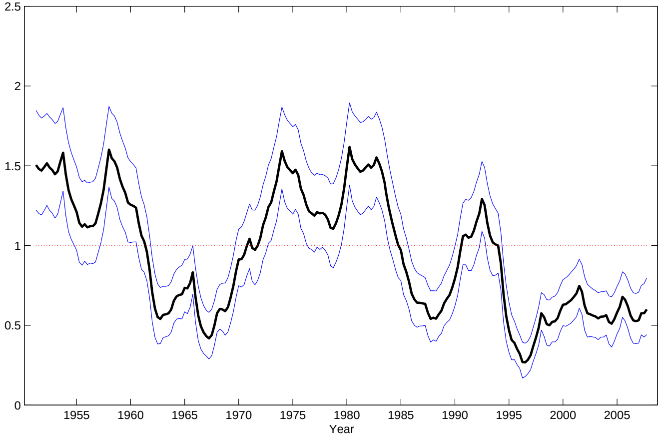


FIGURE: Smoothed volatility estimates and quartiles.

Risk price vectors

Recall that a risk price for horizon t is:

$$-\frac{1}{t}\frac{\partial \log E\left[S_t G_t(\alpha)|X_0 = x\right]}{\partial \alpha}$$

Power utility

$$S_t = \exp(-\delta t)\left(\frac{C_t}{C_0}\right)^{-\gamma}$$

Parameterized multiplicative martingale

$$d \log G_t(\alpha) = \sqrt{X_t^{[2]}}\alpha' dW_t - X_t^{[2]}\frac{|\alpha|^2}{2}$$

RISK PRICE LIMITS

- Local risk price vector (Breeden):

$$\sqrt{X_0^{[2]}}\gamma F.$$

- Long-term risk price vector:

$$\hat{E}\left(X_t^{[2]}\right) \quad [\underbrace{\gamma F}_{\textbf{local}} - \underbrace{(B_1)'\mathsf{r}_1}_{\textbf{growth}} - \underbrace{(B_2)'\mathsf{r}_2}_{\textbf{volatility}}]$$

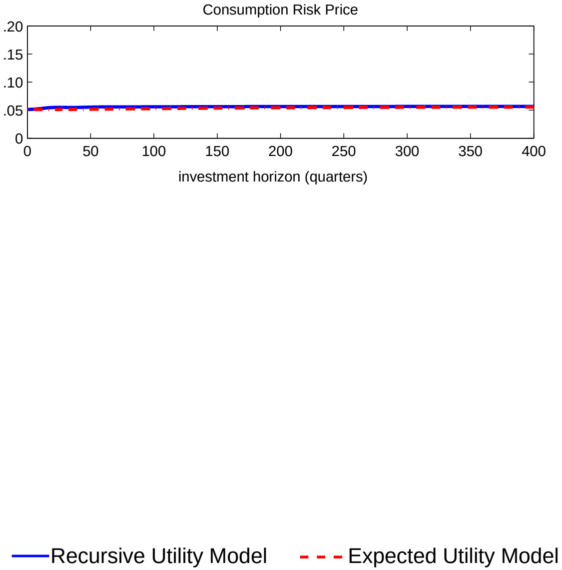
where

$$-\sqrt{X_t^{[2]}}(\mathsf{r}_1)'B_1 dW_t$$

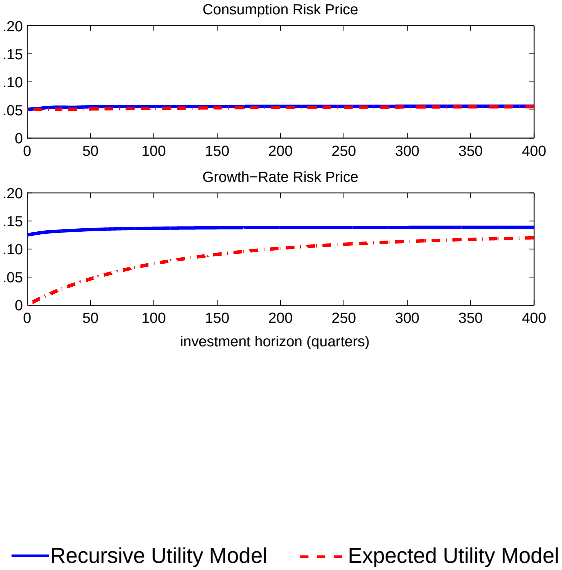
is the surprise movement in

$$\gamma H_1 \int_0^\infty E\left(X_{t+\tau}^{[1]} | X_t^{[1]}\right) d\tau$$

Risk prices


Consumption Risk Price
.20
.15
.10
.05
0
0
50
100
150
200
250
300
350
400
investment horizon (quarters)
Recursive Utility Model
Expected Utility Model

Risk prices



Consumption Risk Price
.20
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Growth−Rate Risk Price
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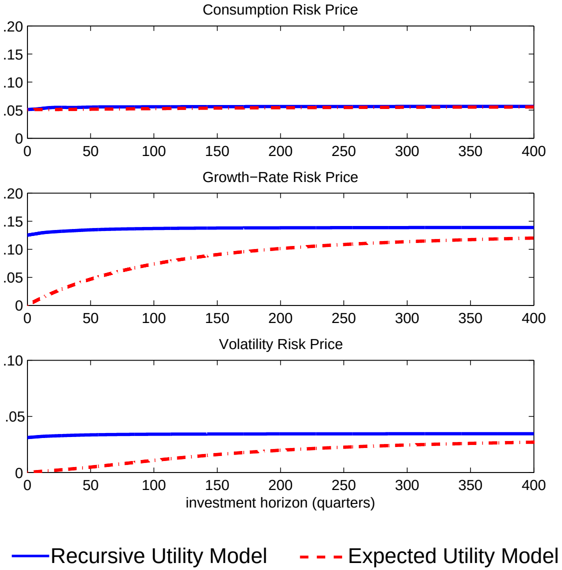
investment horizon (quarters)
Recursive Utility Model
Expected Utility Model

RISK PRICES



Consumption Risk Price
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Growth−Rate Risk Price
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.05
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250
300
350
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Volatility Risk Price
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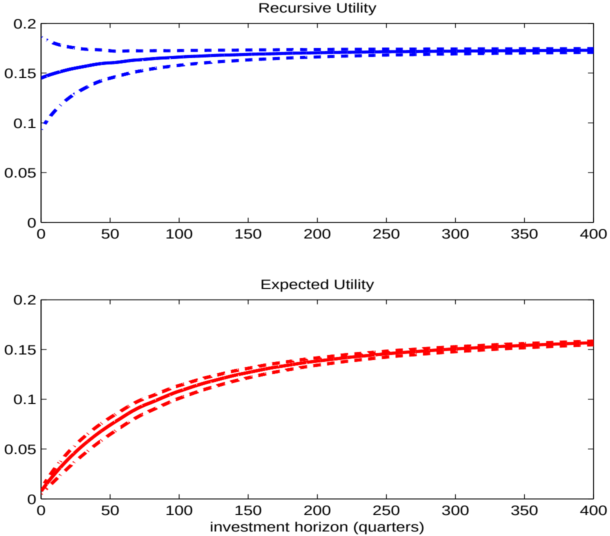
investment horizon (quarters)
Recursive Utility Model
Expected Utility Model

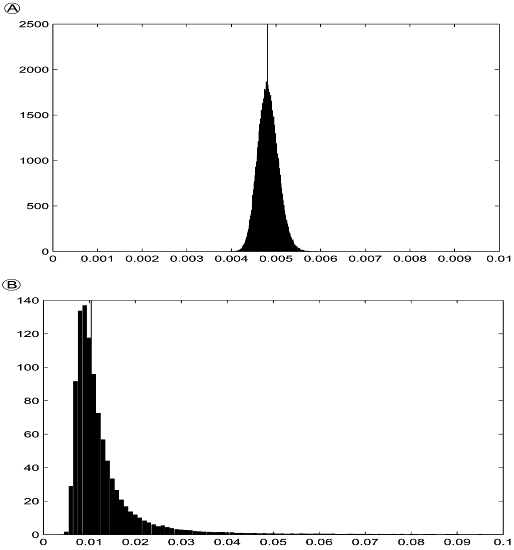
Quartiles for growth-rate risk prices



Other empirical specifications

- Explicit production with long-term uncertainty about technological growth.
- Regime shift models of volatility and growth - “great moderation”.

Estimation accuracy of risk prices



Source:Hansen, Heaton, Li (JPE)

5. High risk aversion or a concern for model misspecification?

The stochastic discount factor is;

$$S_t^* = \exp(-\delta t) \left(\frac{C_0}{C_t}\right) \hat{V}_t$$

where $\hat{V}$ is a martingale component of the $\left\{ \left(\frac{V_t}{V_0}\right)^{1-\gamma} : t \geq 0 \right\}$.

Martingale component in the stochastic discount factor implies a change of probability measure and manifests the alternative robust interpretation of risk-sensitive preferences.

- Lack of investor confidence in the models they use.
- Investors explore alternative specifications for probability laws subject to penalization.
- Martingale is the implied "worst case" model. Parameter γ determines the penalization.
- Related methods have a long history in "robust" control theory and statistics.

Where does this leave us?

- The flat term structure for recursive utility shows the potential importance of macro growth components on asset pricing.
- Typical rational expectations modeling assumes investor confidence and uses the “cross equation” restrictions to identify long-term growth components from asset prices. Instead do asset prices identify “subjective beliefs” of investors and risk aversion?
- Predictable components of macroeconomic growth and volatility are hard for an econometrician to measure from macroeconomic data.

Questions

- Where does investor confidence come from when confronted by weak sample evidence? Motivates my interest in modeling investors who have a concern for model specification.
- What about learning? Concerns about model specification of the type I described make reference to a single benchmark model and as a consequence abstract from learning.

Next talk

A formal exploration of learning and ambiguity

Thanks for coming!