Necessity of Rational Asset Price Bubbles in Two-Sector Growth Economies

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Abstract

We present plausible economic models in which an equilibrium with rational asset price bubbles exists but equilibria with asset prices equal to fundamental values do not. These economies feature multiple sectors with faster economic growth than dividend growth. In our two-sector endogenous growth model, entrepreneurs have access to a production technology subject to idiosyncratic investment risk (tech sector) and trade a dividend-paying asset (land). When leverage is relaxed beyond a critical value, the unique trend stationary equilibrium exhibits a phase transition from the fundamental regime to the bubbly regime with growth, implying the inevitability of bubbles with loose financial conditions.

Keywords: bubble, endogenous growth, leverage, phase transition, transversality condition.

JEL codes: D52, D53, G12.

1 Introduction

This paper considers whether asset price bubbles—situations where the asset price exceeds its fundamental value defined by the present value of the dividend stream—are possible or *inevitable* in rational equilibrium models. Although asset price bubbles are commonly discussed in the popular press and there is some

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empirical support,[1] the dominant view of modern financial theory is that there are fundamental difficulties in generating asset price bubbles in dividend-paying assets. This paper challenges this view by providing robust example economies in which asset price bubbles are necessary for equilibrium existence and connects the emergence of bubbles to economic growth and loose financial conditions.

The fundamental difficulties in generating asset price bubbles can be summarized by the seminal Santos and Woodford (1997) Bubble Impossibility Theorem: their Theorem 3.3 states that if the present value of the aggregate endowment is finite, then the price of an asset in positive net supply or with finite maturity equals its fundamental value. Since most assets in reality are in positive net supply (e.g., stocks and land) or have finite maturity (e.g., bonds and options), in order to generate bubbles in realistic settings, it is necessary to construct models in which the present value of the aggregate endowment is infinite. Apart from stylized overlapping generations (OLG) models in which individual optimality and infinite present value of the aggregate endowment may be consistent due to finite lives, it is necessary to consider models with financial frictions. With sufficient financial constraints, individual optimality and infinite present value of the aggregate endowment may be consistent because financial constraints can prevent agents from capitalizing the infinite present value of endowments.[2]

However, existing rational bubble models with financial frictions have three severe shortcomings. First, the literature has almost exclusively focused on "pure bubbles", i.e., assets that pay no dividend and hence are intrinsically useless like fiat money.[3] It is difficult to apply the theory for empirical or quantitative analysis because pure bubble assets other than fiat money or cryptocurrency are hard to find in reality; it is more realistic to consider bubbles attached to dividend-paying assets such as land or housing. Second, these models suffer from equilibrium indeterminacy: there exists an equilibrium in which the bubble asset has no value, and there also exist a continuum of bubbly equilibria, making model predictions non-robust. Third, by its very nature the existence of rational bubbles rests on financial frictions, and thus bubbles are more likely to arise when financial condi-

[1]Kindleberger (2000, Appendix B) documents 38 bubbly episodes in the 1618–1998 period.

[2]In some models with financial frictions such as Geanakoplos (2010) and Fostel and Geanakoplos (2012), the fundamental theorem of asset pricing fails because financial constraints bind and first-order conditions hold with inequalities for all agents. In such models, the asset price may exceed the valuation of any agent even in two period models. In this paper we only consider models in which the fundamental theorem of asset pricing holds.

[3]Examples are Bewley (1980), Scheinkman and Weiss (1986), Kocherlakota (1992, 2009), Farhi and Tirole (2012), Aoki et al. (2014), Hirano and Yanagawa (2017), and Guerron-Quintana et al. (2022), among others. See Miao (2014) for a review of the pure bubble literature.

tions get tighter, which contradicts stylized facts that bubbly episodes tend to be associated with loose financial conditions (Kindleberger, 2000).

This paper makes two theoretical contributions. First, we show the *necessity* or *inevitability* of rational asset price bubbles under certain conditions. To make this point as clear as possible, we present several stylized economies with exogenous growth and a dividend-paying asset such that (i) there exists an equilibrium in which the asset price exceeds its fundamental value (*bubbly equilibria*), but (ii) there exist no equilibria in which the asset price equals its fundamental value (*fundamental equilibria*). These examples suggest that rational bubbles are no longer exotic but need to be embraced because it is the only way to restore equilibrium existence, at least in some economies. Second, we present a two-sector endogenous growth model with financial frictions (leverage constraint) and rational asset price bubbles that circumvents all aforementioned shortcomings of pure bubble models. Namely, in our model (i) the bubble is attached to a dividend-paying asset, (ii) the equilibrium is determinate, and (iii) asset price bubbles necessarily emerge as the leverage constraint is relaxed. To our knowledge, our paper is the first that shows the necessity of rational asset price bubbles for equilibrium existence and a positive connection between loose financial conditions (financial accelerator) and asset price bubbles.

Our endogenous growth model is a simple incomplete-market dynamic general equilibrium model with infinitely-lived heterogeneous agents. The economy consists of two sectors, the endowment and production sectors. The endowment sector is a long-lived asset that pays dividends, which we metaphorically refer to as "land". Typical examples are residential real estate, farmland, natural resources, publicly traded stock, etc. The production sector consists of a continuum of agents (entrepreneurs) having access to a production technology, which we refer to as the "tech" sector. Each period, agents are hit by productivity shocks and decide how much capital to invest in their own production technology using leverage and how much to save or borrow using risk-free bonds or land.

In this model, there are two possibilities for the long run behavior of aggregate wealth and asset prices. One possibility is that the economy converges to the steady state, where the output from the tech sector and the dividend from the land sector are of the same order of magnitude. Another possibility is that the financial leverage of investing entrepreneurs in the tech sector is sufficiently high so that aggregate capital grows indefinitely and faster than dividends from the land sector. We find that which regime the economy falls into has significant asset pricing implications. We prove (Theorem 3.6) that in any rational expectations equilibrium, the land price always equals (exceeds) its fundamental value when the time series of aggregate wealth is bounded (unbounded). Therefore we have the following simple dichotomy: the land price necessarily contains a bubble if and only if aggregate wealth is unbounded.

The intuition for this Bubble Characterization Theorem is relatively simple. When aggregate wealth is bounded, so is the land price because it cannot exceed aggregate wealth. Then the present value of land in the far distant future converges to zero (the transversality condition holds) and the land price equals the present value of the dividend stream, i.e., the fundamental value. On the other hand, when aggregate wealth is unbounded, the aggregate wealth of low productive agents must also be unbounded because they must be able to finance the capital investment by high productive agents. But because land is the only store of value (other than capital) in the aggregate and held by low productive agents, the land price must also be unbounded. This implies that the land price eventually exceeds its fundamental value, and a backward induction argument shows that a bubble arises in every period.

The logic discussed above is of course vacuous unless we provide robust examples such that aggregate wealth could be bounded or unbounded. To complete the analysis, we define a long run equilibrium concept in which the economy converges to a (properly defined) steady state and establish theoretical results such as existence, uniqueness, and determinacy of equilibria and study their properties. In particular, we (i) obtain necessary and sufficient conditions for the existence of fundamental and bubbly trend stationary equilibria (Theorems 4.2 and 4.3), (ii) prove the uniqueness and determinacy of bubbly long run equilibrium (Theorem 4.4), (iii) prove the existence of a critical value for the leverage limit above which only bubbly equilibria exist (Theorem 4.5), and (iv) prove that the wealth distribution has a Pareto upper tail and wealth inequality is higher (Pareto exponent is smaller) in the bubbly regime (Theorem 4.7). Thus our model shows that once the financial accelerator gets strong enough, the door to the bubble economy is inevitably opened. The intuition why loose financial conditions lead to bubbles (unlike in existing papers) is that production in the tech sector and the land price reinforce each other, generating the financial accelerator. So long as the financial accelerator is small enough, the land price just reflects the fundamental value and the economy converges to the stationary equilibrium with bounded wealth. Once leverage gets high enough, the financial accelerator gets sufficiently strong that the land price grows faster than dividends, exceeding the fundamental value.

Our analysis is largely positive and the exposition follows the theorem-proof style. To focus on the theoretical aspects as cleanly and clearly as possible, we consider a relatively simple model without aggregate uncertainty and abstract from applications except for simple illustrative examples. We plan to provide an application of how changes in the collateral value or productivity generate recurrent asset price bubbles and their collapse in future work. The rest of the paper is organized as follows. Section 2 presents simple example economies in which bubbles are necessary for equilibrium existence. Section 3 introduces our endogenous growth model and derives equilibrium dynamics and asset pricing implications. Section 4 proves the existence, determinacy, and inevitability of bubbly equilibria and characterizes the stationary wealth distribution. Section 5 provides a literature review. Appendices A and B contain proofs and Appendix C presents various extensions.

2 Necessity of rational asset price bubbles

In this section, to illustrate the necessity of rational asset price bubbles under certain conditions, we present several simple example economies with a bubbly equilibrium but with no fundamental equilibria. To circumvent equilibrium indeterminacy associated with pure bubbles, throughout this section we consider a bubble attached to a dividend-paying asset.

2.1 Preliminaries

We consider an infinite-horizon, deterministic economy with a homogeneous good and time indexed by $t = 0, 1, \dots$. Let $q_t > 0$ be the Arrow-Debreu price, i.e., the date-0 price of the consumption good delivered at time t, with the normalization $q_0 = 1$. Consider a long-lived asset with dividend $D_t \geq 0$, with strict inequality infinitely often to rule out zero prices. Letting $P_t > 0$ be the ex-dividend price of the asset (in units of time-t good), the absence of arbitrage implies

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}).$$

Iterating this equation forward and using $q_0 = 1$, we obtain

$$P_0 = \sum_{t=1}^{T} q_t D_t + q_T P_T.$$

Letting $T \to \infty$, we obtain

$$P_0 = \sum_{t=1}^{\infty} q_t D_t + \lim_{T\to\infty} q_T P_T = V_0 + \lim_{T\to\infty} q_T P_T,$$

where the present value of the dividend stream $V_0 := \sum_{t=1}^{\infty} q_t D_t$ is the *fundamental value* of the asset. Thus whether the asset price P_0 equals its fundamental value V_0 or exceeds it (the asset price contains a bubble) depends on whether the *transversality condition*

$$\lim_{T\to\infty} q_T P_T = 0 \tag{2.1}$$

holds or not.[4]

The Santos and Woodford (1997) Bubble Impossibility Theorem (henceforth SW) roughly states that an asset price bubble is impossible (the transversality condition (2.1) holds) if the asset is in positive net supply and the present value of the aggregate endowment is finite. See Kocherlakota (1992) for an early contribution and Miao (2020, §13.6) for a textbook treatment.

2.2 Example economies with only bubbly equilibria

We now present stylized example economies such that asset price bubbles are necessary for equilibrium existence. In the discussion below, we say that an equilibrium is *fundamental* (*bubbly*) if the asset price equals (exceeds) its fundamental value defined by the present value of the dividend stream.

2.2.1 Overlapping generations model

We consider a simple overlapping generations endowment economy as in Samuelson (1958). An agent born at time t lives for two periods and has the constant relative risk aversion (CRRA) utility function

$$\frac{(c_t^y)^{1-\gamma}}{1-\gamma} + \beta \frac{(c_{t+1}^o)^{1-\gamma}}{1-\gamma},$$

where $\beta > 0$ is the discount factor, $\gamma > 0$ is the relative risk aversion coefficient, and c_t^y, c_{t+1}^o are consumption when young and old. The initial old care only about their consumption c_0^o. There is a unit supply of a long-lived asset that pays a

[4]The term "transversality condition" has two meanings: one is the transversality condition (2.1) for asset pricing (e.g. Miao, 2020, Eq. (13.21)) and the other is that for optimality in infinite-horizon dynamic programming (e.g. Miao, 2020, §7.5). The meaning should be clear from the context. In this paper the transversality condition for optimality is always satisfied.

constant dividend $D > 0$ in every period. The aggregate endowment at time t (including dividend) is $(A+B)G^t$, where $G > 1$ and $A, B > 0$. The asset is initially owned by old agents. The asset market is competitive and frictionless.

We specify individual endowments such that an asset price bubble arises in equilibrium. Conjecture that in equilibrium, individual consumption is

$$(c_t^y, c_t^o) = ((A-\eta)G^t, (B+\eta)G^t) \tag{2.2}$$

for some $0 \le \eta < A$. Conjecture that the asset price at time t is

$$P_t = \frac{D}{G-1} + \eta G^t, \tag{2.3}$$

where we conjecture that the gross interest rate is $R = G$, $\frac{D}{G-1} = \sum_{t=1}^{\infty} R^{-t}D$ is the fundamental value of the asset, and ηG^t is the bubble component. Conjecture that the young (old) buy (sell) the asset. Letting e_t^y (e_t^o) be the time t endowment of the young (old) agents, the budget constraints imply

$$\begin{aligned}
&\text{Old:} && (B+\eta)G^t + P_t \cdot 0 = (P_t + D)\cdot 1 + e_t^o \iff e_t^o = BG^t - \frac{G}{G-1}D, \\
&\text{Young:} && (A-\eta)G^t + P_t \cdot 1 = (P_t + D)\cdot 0 + e_t^y \iff e_t^y = AG^t + \frac{1}{G-1}D.
\end{aligned}$$

Let $B > 0$ be large enough such that $e_0^o = B - \frac{G}{G-1}D > 0$. To support the conjectured consumption and asset allocation as an equilibrium, it remains to verify the Euler equation (first-order condition) of the young, which is

$$\beta G \left(\frac{B+\eta}{A-\eta} G\right)^{-\gamma} = 1 \iff \eta = \frac{A(\beta G^{1-\gamma})^{1/\gamma} - B}{1 + (\beta G^{1-\gamma})^{1/\gamma}}. \tag{2.4}$$

For $\eta > 0$, it is necessary and sufficient that $\beta G^{1-\gamma} > (B/A)^\gamma$. Therefore we obtain the following proposition.

Proposition 2.1. *Let $\beta > 0$, $\gamma > 0$, $D > 0$, and $G > 1$ be given. Take any $A, B > 0$ such that*

$$\frac{G}{G-1}D < B < (\beta G^{1-\gamma})^{1/\gamma} A \tag{2.5}$$

and define $\eta > 0$ by (2.4). Then the consumption allocation (2.2) and asset price (2.3) constitute a bubbly equilibrium.

This example is consistent with SW because the present value of the aggregate endowment is infinite: $\sum_{t=0}^{\infty} R^{-t}(A+B)G^t = \infty$ because $G = R$.

We next ask if a fundamental equilibrium exists. Let P_t be the asset price and R_t be the gross interest rate from t to $t+1$ in any equilibrium. Assume that $\liminf_{t\to\infty} R_t > 1$ so that the fundamental value of the asset is finite and bounded. Using the budget constraint, consumption must satisfy

$$\begin{aligned}
&\text{Old:} && c_t^o + P_t \cdot 0 = (P_t + D)\cdot 1 + e_t^o \iff c_t^o = BG^t - \frac{1}{G-1}D + P_t,\\
&\text{Young:} && c_t^y + P_t \cdot 1 = (P_t + D)\cdot 0 + e_t^y \iff c_t^y = AG^t + \frac{1}{G-1}D - P_t.
\end{aligned}$$

Therefore the Euler equation of the young must satisfy

$$1 = \beta R_t (c_{t+1}^o / c_t^y)^{-\gamma} = \beta R_t \left(\frac{BG^{t+1} - \frac{1}{G-1}D + P_{t+1}}{AG^t + \frac{1}{G-1}D - P_t} \right)^{-\gamma} \tag{2.6a}$$

$$\iff R_t = \frac{1}{\beta} \left(\frac{BG^{t+1} - \frac{1}{G-1}D + P_{t+1}}{AG^t + \frac{1}{G-1}D - P_t} \right)^{\gamma} \to \frac{1}{\beta}(BG/A)^\gamma \tag{2.6b}$$

as $t \to \infty$. Note that in (2.5), $A > 0$ can be arbitrarily large. Then BG/A can be arbitrarily small, and hence $R_t \le 1$ is possible in (2.6b). In this case the fundamental value of the asset is infinite, so a fundamental equilibrium does not exist. Although this argument is slightly heuristic (because we assumed $\liminf_{t\to\infty} R_t > 1$), we may prove the following proposition.

Proposition 2.2. *Let everything be as in Proposition 2.1. If $A > \beta^{-1/\gamma} GB$, then there exist no fundamental equilibria.*

The intuition for Proposition 2.2 is as follows. When A is large, the young have a strong incentive to save, which suppresses the interest rate and raises the asset price. However, a positive interest rate is necessary for a finite asset value, yet in this case the fundamental value of the asset is bounded and the asset price may be insufficient to absorb the savings of the young. The only possibility to restore the equilibrium is thus to have an asset price bubble.

2.2.2 Infinite-horizon model

We next consider a model with infinitely-lived agents, which is an extension of Example 1 of Kocherlakota (1992). There are two agents with CRRA utility

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

where $\beta \in (0,1)$ is the discount factor and $\gamma > 0$ is the relative risk aversion coefficient. There is a unit supply of a long-lived asset that pays a constant dividend $D > 0$ in every period. The aggregate endowment at time t (including dividend) is $(A+B)G^t$, where $G > 1$ and $A > B > 0$. The asset is initially owned by agent 1. Suppose the asset cannot be shorted.

We specify individual endowments such that agent 1 is poor (rich) in even (odd) periods, and vice versa for agent 2. Conjecture that in equilibrium, individual consumption is

$$(c_{1t}, c_{2t}) = \begin{cases} ((B+\eta)G^t, (A-\eta)G^t) & \text{if } t\text{: even}, \\ ((A-\eta)G^t, (B+\eta)G^t) & \text{if } t\text{: odd} \end{cases}$$

for some $0 \le \eta < A$. Conjecture that the asset price and interest rate are the same as in the OLG model. Conjecture that every period, the poor (rich) agent sells (buys) the entire asset to smooth consumption. Letting e_t^p (e_t^r) be the time t endowment of the poor (rich) agent, the budget constraints imply

$$\begin{aligned} &\text{Poor:} \quad (B+\eta)G^t + P_t \cdot 0 = (P_t + D) \cdot 1 + e_t^p \iff e_t^p = BG^t - \frac{G}{G-1}D, \\ &\text{Rich:} \quad (A-\eta)G^t + P_t \cdot 1 = (P_t + D) \cdot 0 + e_t^r \iff e_t^r = AG^t + \frac{1}{G-1}D. \end{aligned}$$

Let $B > 0$ be large enough such that $e_0^p = B - \frac{G}{G-1}D > 0$. To support the conjectured consumption and asset allocation as an equilibrium, it remains to verify the Euler equations. Since the rich agent is unconstrained, the Euler equation must hold with equality. For the poor agent, the Euler equation may be an inequality. Since by assumption we have $R = G$, the Euler equations become

$$\begin{aligned} &\text{Poor:} \qquad \beta G \left(\frac{A-\eta}{B+\eta} G \right)^{-\gamma} \le 1, \\ &\text{Rich:} \qquad \beta G \left(\frac{B+\eta}{A-\eta} G \right)^{-\gamma} = 1. \end{aligned}$$

Note that the Euler equation for the rich is identical to that for the young in (2.4). On the other hand, for the Euler inequality for the poor agent to hold, it is necessary and sufficient that

$$1 \ge \left(\frac{B+\eta}{A-\eta} \right)^{\gamma} = \beta G^{1-\gamma}. \tag{2.7}$$

Finally, we verify the transversality condition for optimality. Since $c_t \sim G^t$

and $P_t \sim G^t$ as $t \to \infty$, we obtain the transversality condition for optimality $\beta^t u'(c_t) P_t \sim (\beta G^{1-\gamma})^t \to 0$ if and only if $\beta G^{1-\gamma} < 1$, in which case the Euler inequality for the poor (2.7) holds. Therefore we obtain the following proposition.

Proposition 2.3. *Let $\beta \in (0,1)$, $\gamma > 0$, and $D > 0$ be given. Take any $G > 1$ such that $\beta G^{1-\gamma} < 1$. Take any $A > B > 0$ such that (2.5) holds and define $\eta > 0$ by (2.4). Then the consumption allocation $(c_t^r, c_t^p) = ((A-\eta)G^t, (B+\eta)G^t)$ and asset price $P_t = \frac{D}{G-1} + \eta G^t$ constitute a bubbly equilibrium.*

By exactly the same argument as in the OLG model, this example is consistent with SW because the present value of the aggregate endowment is infinite, and fundamental equilibria do not exist under the condition of Proposition 2.2.

Note that the existence of a bubbly equilibrium in Proposition 2.3 rests on the assumption that the asset cannot be shorted (or more precisely, the shortsales constraint binds for at least one agent). If the shortsales constraint does not bind, then there exist no bubbly equilibria, which can be seen as follows. If the shortsales constraint does not bind, the model becomes a complete market model because the economy is deterministic and there is an asset that can be used to transfer wealth across time. Since agents have identical homothetic preferences, agents consume a constant fraction of aggregate endowment, which is $(A+B)G^t$ at time t. Therefore the Euler equation becomes

$$\beta R_t G^{-\gamma} = 1 \iff R_t = R = \frac{1}{\beta} G^{\gamma}.$$

Now $\beta G^{1-\gamma} < 1$ and $G > 1$ imply $1 < G < \frac{1}{\beta} G^{\gamma} = R$, so the present value of the aggregate endowment $\sum_{t=0}^{\infty} R^{-t}(A+B)G^t$ is finite. Therefore there cannot be a bubble according to SW.

Thus in this example, relaxing financial frictions (shortsales constraints) inevitably kills the bubble. This feature—that a bubble is more likely to arise with tight financial conditions—is common to all existing papers on rational bubbles. From the next section we study our own model, in which bubbles emerge under loose financial conditions.

3 Two-sector endogenous growth model

3.1 Setup

We consider a discrete-time infinite-horizon economy with a homogeneous good and heterogeneous agents.

Agents The economy is populated by a continuum of agents with mass 1 indexed by $i \in I = [0, 1]$.[5] A typical agent has the utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t, \tag{3.1}$$

where $\beta \in (0, 1)$ is the discount factor and $c_t \geq 0$ is consumption. The logarithmic utility is only for simplicity: Appendix C.1 extends to CRRA utility.

Production Each agent has access to an AK-type production technology. If agent i invests $k_{it} \geq 0$ units of capital into the technology at time t, it yields an output of $y_{i,t+1} = z_{it} k_{it}$ at time $t + 1$, where $z_{it} \geq 0$ is the productivity. Unless otherwise stated, we maintain the following assumption.

Assumption 1. *The productivity z_{it} is independent and identically distributed (IID) across agents with a continuous cumulative distribution function (cdf) $F_t : [0, \infty) \to [0, 1]$ satisfying $F_t(1) < 1$ and $\int_0^\infty z \, \mathrm{d}F_t(z) < \infty$.*

The IID and continuity assumptions are only for simplicity: Appendices C.2 and C.3 allow Markov dependence and discontinuities. The condition $F_t(1) < 1$ implies that positive net return on capital ($z > 1$) is possible, which is necessary to ensure that investment occurs in equilibrium. The condition $\int_0^\infty z \, \mathrm{d}F_t(z) < \infty$ implies that the mean productivity is finite, which is necessary to ensure that the aggregate output is finite. When $F_t(0) > 0$, there is a point mass $F_t(0)$ of agents with $z = 0$. These agents can be interpreted as savers.

Land There is a unit supply of a dividend-paying asset with infinite maturity. Throughout the rest of the paper, we metaphorically refer to this asset as "land" because we have in mind residential real estate or farmland as typical examples—assets that are useful but not directly used in production. Land pays dividend $D_t \geq 0$ at time t, which is deterministic. The (endogenous) land price at time t is denoted by P_t. The following assumption prevents land from becoming worthless.

Assumption 2. *The dividend satisfies $D_t > 0$ infinitely often.*

[5] It is well known that using the Lebesgue unit interval as the agent space leads to a measurability issue. We refer the reader to Sun and Zhang (2009) for a resolution based on Fubini extension. Another simple way to get around the measurability issue is to suppose that there are countably many agents and define market clearing as $\lim_{I \to \infty} \frac{1}{I} \sum_{i=1}^{I} x_{it} = X_t$, where x_{it} is agent i's demand at time t and X_t is the per capita supply.

Bond There are risk-free bonds with exogenous net supply B_t. The (endogenous) gross interest rate between time t and $t+1$ is denoted by R_t. The benchmark case $B_t = 0$ can be interpreted as a closed economy. However, we occasionally specify B_t to simplify the analysis. We can interpret the case $B_t \neq 0$ as the presence of foreign investors participating in the international capital market.

Budget constraint Suppressing the individual subscript, the budget constraint of a typical agent is

$$c_t + k_t + P_t x_t + b_t = z_{t-1} k_{t-1} + (P_t + D_t) x_{t-1} + R_{t-1} b_{t-1}, \tag{3.2}$$

where $c_t \geq 0$ is consumption at time t, $k_t \geq 0$ is investment in the production technology at time t, and $x_t, b_t \in \mathbb{R}$ are the land and bond holdings at time t. The condition $x_t, b_t \in \mathbb{R}$ implies that land and bonds can be held in arbitrary positive or negative positions.

Leverage constraint Agents are subject to the leverage constraint

$$k_t \leq \lambda_t (k_t + P_t x_t + b_t), \tag{3.3}$$

where $\lambda_t \geq 1$ is the exogenous leverage limit. Here $k_t + P_t x_t + b_t$ is total financial asset ("equity") of the agent. The leverage constraint (3.3) implies that total investment in the production technology cannot exceed some multiple of total equity. Note that since $k_t \geq 0$ and $\lambda_t \geq 1 > 0$, (3.3) implies that equity must be nonnegative: $k_t + P_t x_t + b_t \geq k_t/\lambda_t \geq 0$. Furthermore, since

$$P_t x_t + b_t \geq (1/\lambda_t - 1) k_t,$$

$k_t \geq 0$, and $\lambda_t \geq 1$, the leverage constraint imposes a joint shortsales constraint on land and bonds, although they can be shorted individually.

Equilibrium The economy starts at $t = 0$ with some initial distribution of endowment and land $\{(y_{i0}, x_{i,-1})\}_{i \in I}$, where $(y_{i0}, x_{i,-1}) > 0$ for all i. The definition of a rational expectations equilibrium is standard.

Definition 1 (Rational expectations equilibrium)**.** Given the initial condition $\{(y_{i0}, x_{i,-1})\}_{i \in I}$ and bond supply $\{B_t\}_{t=0}^{\infty}$, a *rational expectations equilibrium* consists of land prices $\{P_t\}_{t=0}^{\infty}$, interest rates $\{R_t\}_{t=0}^{\infty}$, and allocations $\{(c_{it}, k_{it}, x_{it}, b_{it})_{i \in I}\}_{t=0}^{\infty}$ such that the following conditions hold.

(i) (Individual optimization) Agents maximize the utility (3.1) subject to the budget constraint (3.2) and the leverage constraint (3.3), where for $t = 0$ we interpret $z_{-1}k_{-1} = y_0$ and $b_{-1} = 0$.

(ii) (Land market clearing) For all t, we have

$$\int_I x_{it} \, \mathrm{d}i = 1. \tag{3.4}$$

(iii) (Bond market clearing) For all t, we have

$$\int_I b_{it} \, \mathrm{d}i = B_t. \tag{3.5}$$

3.2 Equilibrium conditions

Asset price restrictions Since land pays positive dividends infinitely often, the land price must be positive. We note this result as a lemma.

Lemma 3.1 (Positivity of land price). *If Assumption 2 holds, then in equilibrium $P_t > 0$ for all t.*

Proof. If $P_t = 0$, agents can take an arbitrarily large position in land x_t, which gives arbitrarily large dividend sometime in the future, violating optimality. □

Since there is no aggregate risk and the land and bonds can be held in positive or negative positions, in equilibrium these assets must yield the same return. We note this no-arbitrage condition as a lemma.

Lemma 3.2 (No arbitrage). *In equilibrium, the no-arbitrage condition*

$$\frac{P_{t+1} + D_{t+1}}{P_t} = R_t \tag{3.6}$$

holds.

Note that the left-hand side of (3.6), the gross return on land, is well defined because $P_t > 0$ by Lemma 3.1.

Individual optimization problem We next solve the individual optimization problem. To this end, it is convenient to define the beginning-of-period wealth w_t by the right-hand side of (3.2):

$$w_t := z_{t-1}k_{t-1} + (P_t + D_t)x_{t-1} + R_{t-1}b_{t-1}. \tag{3.7}$$

Define the fraction of post-consumption wealth invested in the production technology by $\theta_t = \frac{k_t}{w_t - c_t}$. Then the fraction of post-consumption wealth invested in the risk-free asset and land is $1 - \theta_t = \frac{P_t x_t + b_t}{w_t - c_t}$. Using these investment shares, the definition of wealth in (3.7), and the no-arbitrage condition (3.6), we obtain

$$\begin{aligned} w_{t+1} &= z_t k_t + (P_{t+1} + D_{t+1}) x_t + R_t b_t \\ &= (\theta_t z_t + (1 - \theta_t) R_t)(w_t - c_t). \end{aligned} \tag{3.8}$$

Using $1 = \frac{k_t + P_t x_t + b_t}{w_t - c_t}$ and the definition of θ_t, it follows from the leverage constraint (3.3) that

$$\theta_t = \frac{k_t}{w_t - c_t} = \frac{k_t}{k_t + P_t x_t + b_t} \leq \lambda_t. \tag{3.9}$$

Therefore using the utility function (3.1), the equation of motion for wealth (3.8), and the leverage constraint (3.9), letting $v_t(w, z)$ be the continuation value at time t given wealth w and productivity z, we can derive the Bellman equation

$$v_t(w, z) = \sup_{\substack{0 \leq c \leq w \\ 0 \leq \theta \leq \lambda_t}} \left[\log c + \beta \, \mathrm{E}_t[v_{t+1}(w', z')] \right], \tag{3.10}$$

where $w' = (\theta z + (1 - \theta) R_t)(w - c)$ and z' is drawn from F_{t+1}. The following proposition characterizes the solution to the Bellman equation (3.10).

Proposition 3.3 (Optimal consumption and investment)**.** *Suppose*

$$\sup_t |\mathrm{E}[\log(R_t + \lambda_t \max\{0, z - R_t\})]| < \infty.$$

Then the optimal consumption-investment problem (3.10) *has an essentially unique solution, which is given by*

$$c_t = (1 - \beta) w_t, \tag{3.11a}$$

$$\theta_t = \begin{cases} \lambda_t & \text{if } z_t > R_t, \\ \text{arbitrary} & \text{if } z_t = R_t, \\ 0 & \text{if } z_t < R_t. \end{cases} \tag{3.11b}$$

Proof. Immediate from Proposition B.2. □

The optimality of the myopic consumption rule for logarithmic utility (3.11a) is well known. The optimal investment rule (3.11b) merely states that agents choose maximal leverage if productivity exceeds the interest rate and otherwise do not invest in the production technology at all, which is obvious because productivity is known at the beginning of the period.

Equilibrium dynamics We now derive equilibrium conditions. In equilibrium, Lemmas 3.1 and 3.2 imply $R_t > 0$. Using the optimal investment rule (3.11b), we may compute the expected return on savings by

$$\begin{aligned} \mathrm{E}_t[\theta_t z + (1-\theta_t)R_t] &= \mathrm{E}_t[\theta_t(z-R) + R_t] \\ &= R_t + \lambda_t \int_0^\infty \max\{0, z - R_t\} \, \mathrm{d}F_t(z). \end{aligned} \tag{3.12}$$

Define the risk premium (expected excess return) on unlevered capital investment by

$$\pi_t(R) := \int_0^\infty \max\{0, z - R\} \, \mathrm{d}F_t(z). \tag{3.13}$$

Because $R \mapsto z - R$ is decreasing and affine (hence convex) and the max operator and integration preserve monotonicity and convexity, we obtain the following lemma.

Lemma 3.4 (Properties of risk premium). *Suppose Assumption 1 holds. Then $\pi_t : [0,\infty) \to \mathbb{R}$ defined by (3.13) is nonnegative, differentiable, convex, $\pi_t(\infty) = 0$, and $\pi_t'(R) = F_t(R) - 1 \le 0$, with strict inequality whenever $F_t(R) < 1$.*

Using the risk premium (3.13), the expected return in (3.12) becomes

$$\mathrm{E}[\theta_t z + (1-\theta_t)R_t] = \lambda_t \pi_t(R_t) + R_t.$$

Therefore integrating (3.8) and using the optimal consumption rule (3.11a), we obtain the law of motion for aggregate wealth $W_t = \int_I w_{it} \, \mathrm{d}i$:

$$W_{t+1} = \beta(\lambda_t \pi_t(R_t) + R_t) W_t. \tag{3.14}$$

For $t = 0$, letting $Y_0 = \int_I y_{i0} \, \mathrm{d}i$ be the aggregate endowment at $t = 0$, we obtain

$$W_0 = Y_0 + P_0 + D_0. \tag{3.15}$$

Integrating

$$P_t x_t + b_t = (1-\theta_t)(w_t - c_t) = \beta(1-\theta_t) w_t,$$

using market clearing conditions (3.4) and (3.5), and noting that z_{it} is IID across

i with an atomless cdf F_t, we obtain

$$\begin{aligned} P_t + B_t = \int_I (P_t x_{it} + b_{it}) \, \mathrm{d}i &= \beta W_t F_t(R_t) + \beta(1-\lambda_t) W_t (1 - F_t(R_t)) \\ &= \beta(\lambda_t F_t(R_t) + 1 - \lambda_t) W_t. \end{aligned} \tag{3.16}$$

To simplify the notation, introduce the variable

$$\alpha_t := \beta(\lambda_t F_t(R_t) + 1 - \lambda_t), \tag{3.17}$$

which is the fraction of aggregate wealth flowing into the asset market. Noting that F_t is a cdf and hence $F_t(R_t) \le 1$, we have $\alpha_t \le \beta$. Then (3.16) becomes $P_t = \alpha_t W_t - B_t$. Using the no-arbitrage condition (3.6) and (3.16), we obtain

$$R_{t-1} = \frac{P_t + D_t}{P_{t-1}} = \frac{\alpha_t W_t - B_t + D_t}{\alpha_{t-1} W_{t-1} - B_{t-1}}. \tag{3.18}$$

Using (3.14), the no-arbitrage condition (3.18) can be rewritten as

$$(\beta\alpha_t(\lambda_t \pi_t(R_t) + R_t) - R_{t-1}\alpha_{t-1}) W_{t-1} = B_t - R_{t-1} B_{t-1} - D_t. \tag{3.19}$$

We collect these observations in the following proposition.

Proposition 3.5 (Aggregate dynamics). *Suppose Assumptions 1 and 2 hold. Then the aggregate wealth W_t, land price P_t, and interest rate R_t in the rational expectations equilibrium are characterized by the following equations:*

$$\alpha_t = \beta(\lambda_t F_t(R_t) + 1 - \lambda_t), \tag{3.20a}$$

$$P_t = \alpha_t W_t - B_t, \tag{3.20b}$$

$$W_0 = \frac{Y_0 + D_0 - B_0}{1 - \alpha_0}, \tag{3.20c}$$

$$W_{t+1} = \beta(\lambda_t \pi_t(R_t) + R_t) W_t, \tag{3.20d}$$

$$\beta(\lambda_t \pi_t(R_t) + R_t)\alpha_t = R_{t-1}\alpha_{t-1} + \frac{B_t - R_{t-1}B_{t-1} - D_t}{W_{t-1}}. \tag{3.20e}$$

Interestingly, this model produces the financial accelerator: the real economy and the land price reinforce each other. To see this formally, suppose that the economy is closed ($B_t = 0$) and the interest rate is constant. Aggregating individual wealth (3.7), we obtain $W_t = Y_t + P_t + D_t$. Therefore an increase in the land price P_t raises aggregate wealth W_t. But an increase in the current aggregate wealth raises the next period's aggregate wealth $W_{t+1} = \beta(\lambda\pi(R) + R)W_t$ through investment and production: see (3.20d). Finally, this increased wealth feeds back into the land price through $P_{t+1} = \alpha W_{t+1}$: see (3.20b). Whether this positive feedback loop can sustain economic growth and high asset valuation depends on how high the leverage λ is.

3.3 Asset prices

We next study the asset pricing implications of the model. Rewriting the no-arbitrage condition (3.6), we obtain $P_t = (P_{t+1} + D_{t+1})/R_t$. Iterating this yields

$$P_t = \sum_{s=1}^{N} \left(\prod_{j=0}^{s-1} R_{t+j} \right)^{-1} D_{t+s} + \left(\prod_{j=0}^{N-1} R_{t+j} \right)^{-1} P_{t+N}. \tag{3.21}$$

As we let $N \to \infty$, the first term in (3.21) converges to the fundamental value of land defined by

$$V_t := \sum_{s=1}^{\infty} \left(\prod_{j=0}^{s-1} R_{t+j} \right)^{-1} D_{t+s}. \tag{3.22}$$

Since by Lemma 3.1 the second term in (3.21) is always positive, whether the land price P_t equals its fundamental value V_t depends on whether the transversality condition

$$\lim_{N\to\infty} \left(\prod_{j=0}^{N-1} R_{t+j} \right)^{-1} P_{t+N} = 0 \tag{3.23}$$

holds or not.

The following theorem characterizes conditions under which land is priced at the fundamental value or asset price bubbles arise.

Theorem 3.6 (Characterization of bubbles)**.** *Suppose Assumptions 1 and 2 hold and a rational expectations equilibrium* $\{(P_t, R_t, B_t, (c_{it}, k_{it}, x_{it}, b_{it})_{i\in I})\}_{t=0}^{\infty}$ *exists with associated aggregate wealth* $\{W_t\}_{t=0}^{\infty}$*. Let* α_t *be defined in* (3.17) *and suppose that*

$$\limsup_{t\to\infty} D_t < \infty, \quad \liminf_{t\to\infty} R_t > 1, \quad \liminf_{t\to\infty} \alpha_t > 0. \tag{3.24}$$

Then the following statements are true.

(i) The fundamental value of land V_t *is finite and* $\limsup_{t\to\infty} V_t < \infty$*.*

(ii) If $\limsup_{t\to\infty} W_t < \infty$ *and* $\liminf_{t\to\infty} B_t > -\infty$*, then* $P_t = V_t$ *for all* t*, so the land price equals its fundamental value.*

(iii) If $\limsup_{t\to\infty} W_t = \infty$ *and* $\limsup_{t\to\infty} B_t/W_t \le 0$*, then* $P_t > V_t$ *for all* t*, so the land price exceeds its fundamental value.*

The first condition in (3.24) implies that the dividend stream $\{D_t\}_{t=0}^{\infty}$ is bounded, which may appear restrictive. However, it is straightforward to allow dividend growth as discussed in Appendix C.4.

According to statement (ii), if aggregate wealth W_t and external debt $\max\{0, -B_t\}$ are bounded in the long run, then the land price must always equal its fundamental value. According to statement (iii), if aggregate wealth W_t is unbounded and external savings $\max\{0, B_t\}$ is asymptotically negligible relative to aggregate wealth in the long run, then the land price must always exceed its fundamental value. In a closed economy, we have $B_t = 0$, so the conditions on B_t are necessarily satisfied. In this case, an asset price bubble occurs if and only if aggregate wealth is unbounded. Theorem 3.6 thus implies that in an economy with long run growth, an asset price bubble is inevitable.

4 Long run equilibria

Theorem 3.6 states that in any rational expectations equilibrium in which aggregate wealth is unbounded and the aggregate bond supply becomes asymptotically negligible (closed economy), the land price necessarily exhibits a bubble. However, the analysis is still incomplete because Theorem 3.6 involves assumptions on endogenous variables, namely the condition (3.24). To complete the analysis, in this section we construct robust examples of rational expectations equilibria in which the assumptions of Theorem 3.6 are satisfied.

4.1 Definition of equilibrium

Since time runs forever, studying the properties of general rational expectations equilibria is challenging. Therefore we first define the long run equilibrium concept in which aggregate variables or their growth rates converge as $t \to \infty$. Throughout this section we maintain the following assumption.

Assumption 3. *(i) The productivity distribution* $F_t = F$ *is constant and satisfies Assumption 1. (ii) The leverage limit* $\lambda_t = \lambda \ge 1$ *is constant. (iii) The dividend* $D_t = D > 0$ *is constant.*

Definition 2 (Long run equilibria)**.** We say that a rational expectations equilibrium $\{(P_t, R_t, B_t, (c_{it}, k_{it}, x_{it}, b_{it})_{i\in I})\}_{t=0}^{\infty}$ with associated aggregate wealth $\{W_t\}_{t=0}^{\infty}$ is a *long run equilibrium* if the following conditions hold.

(i) (Converging interest rate) $\lim_{t\to\infty} R_t = R > 0$ exists.

(ii) (Converging growth rate) $\lim_{t\to\infty} W_t/W_{t-1} = G > 0$ exists.

(iii) (Converging wealth if no growth) If $G \le 1$, then $\lim_{t\to\infty} W_t = W$ exists.

(iv) (Long run bond market clearing) $\lim_{t\to\infty} B_t/W_t = 0$.

We can interpret a long run equilibrium as a large open economy converging to a balanced growth path. Here by an "open" economy we mean that the agents can trade risk-free bonds with external agents so that the bond market need not clear exactly: $B_t \neq 0$ is possible. However, by a "large" economy we mean that the aggregate bond supply B_t must be asymptotically negligible relative to aggregate wealth (so $B_t/W_t \to 0$) and hence the bond market asymptotically clears.

For constructing closed-form examples, we define a special case of long run equilibria as follows.

Definition 3 (Trend stationary equilibria)**.** We say that a long run equilibrium $\{(P_t, R_t, B_t, (c_{it}, k_{it}, x_{it}, b_{it})_{i\in I})\}_{t=0}^{\infty}$ with associated aggregate wealth $\{W_t\}_{t=0}^{\infty}$ is a *trend stationary equilibrium* if $R_t = R$ and $G_t = G$ are constant.

By Theorem 3.6, in any long run equilibrium, the land price equals its fundamental value if and only if the economy does not grow. Therefore, in what follows we refer to an equilibrium with $G \le 1$ a *fundamental equilibrium* and an equilibrium with $G > 1$ a *bubbly equilibrium*.

Before studying the existence of equilibrium, we note the following simple restrictions. Suppose a long run equilibrium exists. Dividing both sides of (3.20b) by $W_t > 0$, letting $t \to \infty$, and using long run bond market clearing, we obtain

$$0 \le \lim_{t\to\infty} P_t/W_t = \beta(\lambda F(R) + 1 - \lambda) \implies 1 - F(R) \le \frac{1}{\lambda}.$$

Dividing both sides of (3.20d) by $W_t > 0$ and letting $t \to \infty$, the aggregate wealth growth rate must satisfy

$$G = \beta(\lambda\pi(R) + R). \tag{4.1}$$

4.2 Existence of fundamental equilibrium

Suppose that there exists a long run equilibrium with $G \le 1$. Then by definition $W = \lim_{t\to\infty} W_t$ exists. Letting $t \to \infty$ in the no-arbitrage condition (3.20e) and using long run bond market clearing, we obtain

$$(\lambda F(R) + 1 - \lambda)(\lambda\pi(R) + R - R/\beta) = -\frac{D}{\beta^2 W}. \tag{4.2}$$

Since the left-hand side of (4.2) is finite, it must be $W > 0$. If $G < 1$, then $W = 0$, a contradiction. Therefore it must be $G = 1$, and (4.1) implies the equilibrium condition

$$\lambda\pi(R) + R = \frac{1}{\beta}. \tag{4.3}$$

Furthermore, substituting (4.3) into (4.2), we obtain

$$(\lambda F(R) + 1 - \lambda)(1/\beta - 1)R = \frac{D}{\beta^2 W}.$$

Since the right-hand side is positive, it must be $\lambda F(R) + 1 - \lambda > 0$. Finally, if $R \le 1$, then the fundamental value of the asset is infinite and an equilibrium does not exist. Therefore a necessary condition for R to be a long run equilibrium interest rate is

$$R \in \mathcal{R}_f := \{R > 1 : 1 - F(R) < 1/\lambda, \lambda\pi(R) + R = 1/\beta\}. \tag{4.4}$$

The following lemma provides a necessary and sufficient condition for $\mathcal{R}_f$ to be nonempty.

Lemma 4.1. *Define*

$$\mathcal{R}'_f := \{R > 1 : 1 - F(R) < 1/\lambda, \lambda\pi(R) + R \le 1/\beta\}, \tag{4.5}$$

which is a convex subset of $(1, 1/\beta]$*. Then* $\mathcal{R}_f$ *in* (4.4) *is nonempty if and only if* $\mathcal{R}'_f \neq \emptyset$*.*

The following theorem provides necessary and sufficient conditions for the existence of a fundamental trend stationary equilibrium.

Theorem 4.2 (Existence of fundamental equilibrium)**.** *Suppose Assumption 3 holds. Then a fundamental trend stationary equilibrium exists if and only if* $\mathcal{R}'_f \neq$

$\emptyset$. *Under this condition, the variables must satisfy the following restrictions:*

$$\begin{aligned} &G = 1, && R \in \mathcal{R}_f, && B_t = 0, \\ &W_t = \frac{D}{(R-1)\alpha}, && P_t = \frac{D}{R-1}, && Y_0 = \frac{1 - R\alpha}{(R-1)\alpha} D. \end{aligned}$$

Note that when $R \in \mathcal{R}_f$, we have $R \leq 1/\beta$, so $R\alpha \leq R\beta \leq 1$. Therefore $Y_0 \geq 0$.

4.3 Existence and determinacy of bubbly equilibrium

Suppose that there exists a long run equilibrium with $G > 1$. Then $W_t \to \infty$. Letting $t \to \infty$ in the no-arbitrage condition (3.20e) and using long run bond market clearing, we obtain

$$(\lambda F(R) + 1 - \lambda)(\lambda \pi(R) + R - R/\beta) = 0.$$

If $1 - F(R) = 1/\lambda$, then (3.20b) implies $P_t = -B_t$, so the land price is entirely determined by the exogenous bond supply, which is uninteresting. Thus we focus on the case $1 - F(R) < 1/\lambda$, which implies the equilibrium condition

$$\lambda \frac{\pi(R)}{R} = \frac{1}{\beta} - 1. \tag{4.6}$$

Under this condition, (4.1) implies $1 < G = R$. Therefore a necessary condition for R to be a long run equilibrium interest rate is

$$R \in \mathcal{R}_b := \left\{ R > 1 : 1 - F(R) < 1/\lambda, \lambda \frac{\pi(R)}{R} = 1/\beta - 1 \right\}. \tag{4.7}$$

Note that since by Lemma 3.4 π is strictly decreasing whenever $\pi > 0$, which is the case when $\lambda \pi(R)/R = 1/\beta - 1$ (because $\beta < 1$), there exists at most one such R. Therefore the set $\mathcal{R}_b$ in (4.7) is either empty or a singleton.

We now present two existence and uniqueness/determinacy results on bubbly long run equilibria. The first result, Theorem 4.3, provides necessary and sufficient conditions for the existence and uniqueness of a trend stationary equilibrium, where the exogenous bond supply B_t is appropriately chosen. The second result, Theorem 4.4, proves the existence and local determinacy of long run equilibrium.

Theorem 4.3 (Existence and uniqueness of bubbly equilibrium)**.** *Suppose Assumption 3 holds and let* $\underline{R} = \max\{1, F^{-1}(1/\lambda - 1)\}$. *Then a bubbly trend sta-*

tionary equilibrium with $1 - F(R) \neq 1/\lambda$ *exists if and only if* $\lambda\pi(\underline{R})/\underline{R} > 1/\beta - 1$. *Under this condition, the equilibrium is unique and the variables must satisfy the following restrictions:*

$$G = R, \qquad R \in \mathcal{R}_b, \qquad B_t = -\frac{D}{R-1},$$
$$W_t = R^t \frac{(R-1)Y_0 + RD}{(1-\alpha)(R-1)}, \qquad P_t = \alpha W_t + \frac{D}{R-1}.$$

As discussed in the introduction, a common criticism to pure bubble models is that they suffer from equilibrium indeterminacy. The following theorem shows that in our model, bubbly long run equilibria are locally determinate.

Theorem 4.4 (Local determinacy of bubbly long run equilibria)**.** *Let everything be as in Theorem 4.3,* $B_t = B$ *(constant), and suppose that* F *is differentiable. Then for large enough initial aggregate endowment* $Y_0 > 0$*, there exists a unique bubbly long run equilibrium, i.e., the bubbly long run equilibrium is locally determinate.*

The value added of Theorem 4.4 is as follows. Although Theorem 4.3 establishes the existence and uniqueness of a bubbly trend stationary equilibrium, it does not rule out indeterminacy in long run equilibria that are not trend stationary. Theorem 4.4 does rule out this possibility, which is in sharp contrast to the pure bubble literature.

Since by Lemma 3.4 π is decreasing, by condition (4.6), in order for a bubbly long run equilibrium to exist, it is necessary that

$$\lambda \operatorname{E}[\max\{0, z-1\}] = \lambda\pi(1)/1 > \lambda\pi(R)/R = 1/\beta - 1. \tag{4.8}$$

The intuition for the necessary condition (4.8) is relatively simple. Because the economy features two sectors (constant-returns-to-scale production and land), in order for aggregate wealth to grow, the production sector must grow. This is the case if agents are patient (β is large, making the right-hand side of (4.8) small), leverage is lax (λ is large), or agents are productive ($\operatorname{E}[\max\{0, z-1\}]$ is large). Scheinkman (2014, p. 22) highlights the importance of the relationship between technological progress and asset price bubbles, noting "asset price bubbles tend to appear in periods of excitement about innovations". Our result is consistent with this stylized fact if we interpret that agents become productive with the arrival of new technologies. Moreover, Scheinkman (2014) also points out that bubbles may have positive effects on innovative investments and economic growth by facilitating finance. Even in our model, bubbles raise economic growth by financing productive investments, which in turn sustains growing bubbles.

By comparing Theorems 4.2 and 4.3, it is clear that in the fundamental regime, economic growth equals dividend growth even if the technology is linear, and it is independent of the leverage constraint or other parameter values. In contrast, we have $G = R > 1$ in the bubbly regime. This implies that once the economy enters the bubbly regime, it behaves like an endogenous growth model.

4.4 Financial conditions and emergence of bubbles

As discussed in the introduction, rational bubble models rests on financial constraints, and in all existing papers asset price bubbles are more likely to arise under tight financial conditions. The following theorem, although almost obvious, shows that bubbles inevitably emerge under loose financial conditions, consistent with stylized facts (Kindleberger, 2000).

Theorem 4.5 (Inevitability of bubbles with lax leverage)**.** *Suppose Assumption 3 holds and* $\Pr(z > 1/\beta) > 0$*. Then there exists a leverage threshold* $\bar{\lambda}$ *such that all long run equilibria are bubbly if* $\lambda \geq \bar{\lambda}$*.*

Proof. Suppose a fundamental long run equilibrium exists. Then the equilibrium condition (4.3), Lemma 4.1, and the monotonicity of π in Lemma 3.4 imply that

$$\frac{1}{\beta} = \lambda\pi(R) + R > \lambda\pi(1/\beta) + 1.$$

Since $\pi(1/\beta) > 0$ because $\Pr(z > 1/\beta) > 0$, it follows that $\lambda < \bar{\lambda} := \frac{1-\beta}{\beta\pi(1/\beta)}$. Therefore if $\lambda \geq \bar{\lambda}$, there exist no fundamental long run equilibria. ☐

The intuition for this result is as follows. As long as the leverage limit λ is tight enough, the interest rate $R > 1$ can adjust such that the aggregate wealth growth rate G in (4.1) remains 1 and there are no bubbles. However, as the leverage limit is relaxed, $G = 1$ can no longer be supported with any interest rate $R > 1$ that makes the land value finite. At this point the only possibility to restore the equilibrium is for the economy to grow with capital investment financed by the asset price bubble.

Because the definition of the sets of possible long run interest rates $\mathcal{R}_f, \mathcal{R}_b$ in (4.4) and (4.7) are relatively complicated, we seek to simplify the descriptions. To this end, note that the fundamental and bubbly equilibrium conditions (4.3) and

(4.6) are equivalent to

$$\phi_f(R) := \frac{\beta}{1-\beta R}\pi(R) = \frac{1}{\lambda}, \tag{4.9a}$$

$$\phi_b(R) := \frac{\beta}{1-\beta}\frac{\pi(R)}{R} = \frac{1}{\lambda}, \tag{4.9b}$$

respectively. Since $\lambda \geq 1$ is the leverage limit, the number $1/\lambda \leq 1$ can be interpreted as the minimum equity requirement or minimum down payment for borrowing. Note that since equilibrium requires $R > 1$, it follows that

$$\frac{\phi_b(R)}{\phi_f(R)} = \frac{1/R - \beta}{1-\beta} < 1,$$

so $\phi_f(R) > \phi_b(R)$ for $R > 1$. Furthermore, $\phi_f(1) = \phi_b(1) = \frac{\beta}{1-\beta}\pi(1)$. Under an additional assumption, we obtain the following simple characterization of long run equilibrium interest rates.

Proposition 4.6. *If* $\Pr(z > 1/\beta) > 0$ *and* $\mathrm{E}[z \mid z \geq 1] > 1/\beta$*, then*

(i) ϕ_f *is strictly increasing for* $R \in [1, 1/\beta)$ *and* $\phi_f(1/\beta) = \infty$*,*

(ii) ϕ_b *is strictly decreasing whenever* $\phi_b > 0$ *and* $\phi_b(\infty) = 0$*.*

Consequently,

(i) if $1/\lambda > \frac{\beta}{1-\beta}\pi(1)$*, then* $\mathcal{R}_f$ *is a singleton and* $\mathcal{R}_b = \emptyset$*, and*

(ii) if $1/\lambda < \frac{\beta}{1-\beta}\pi(1)$*, then* $\mathcal{R}_b$ *is a singleton and* $\mathcal{R}_f = \emptyset$*.*

The assumptions of Proposition 4.6 are quite weak: indeed they hold for β sufficiently close to 1 by Assumption 1. Under this assumption, as the equity requirement decreases, there is a phase transition from the fundamental equilibrium to the bubbly equilibrium.

We provide a simple numerical example to illustrate Proposition 4.6.

Example 1. Suppose $1 - F(z) = \kappa \mathrm{e}^{-z/\bar{z}}$ so that an agent has positive productivity with probability $\kappa \in (0, 1]$, and conditional on positive productivity, z is exponentially distributed with mean $\bar{z} > 0$. Figure 1 shows the graphs of ϕ_f, ϕ_b when $\beta = 0.95$, $\kappa = 0.02$ (2% probability of positive productivity), and $\bar{z} = 1.5$ (50% expected return when productivity is positive); see Appendix D for details. Given the equity requirement $1/\lambda$, the equilibrium interest rate is determined as the intersection of the horizontal line at level $1/\lambda$ and the graphs of ϕ_f, ϕ_b. A phase transition from the fundamental regime to the bubbly regime occurs at equity requirement around 30%.

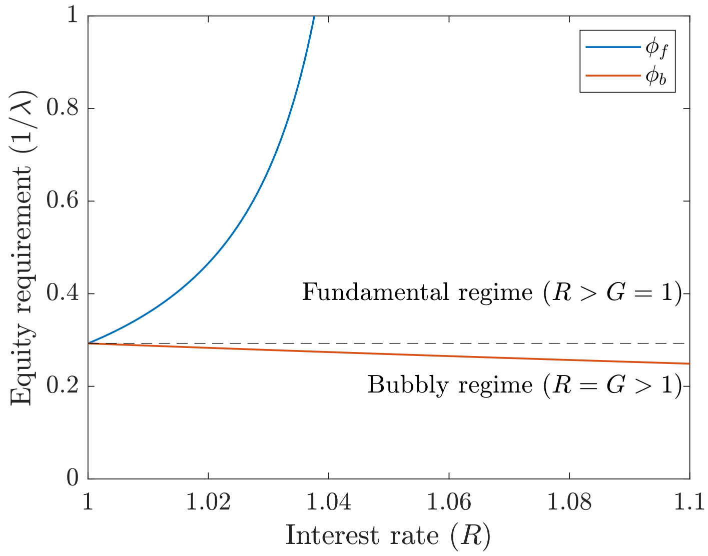

Figure 1: Determination of long run interest rate.

Note: The figure shows how the equity requirement determines the long run interest rate. ϕ_f, ϕ_b denote the functions in (4.9).

In our model, the interest rate R would be less than 1 without bubbles when leverage is above the critical value defined by $\bar{\lambda} = \frac{1-\beta}{\beta\pi(1)}$. In other words, as λ increases and approaches $\bar{\lambda}$, R decreases and approaches 1. Obviously, $R \leq 1$ cannot be an equilibrium because the land price would explode. This is why bubbles are necessary for the existence of equilibrium when leverage exceeds the critical value $\bar{\lambda}$.

4.5 Wealth distribution

In any trend stationary equilibrium, the optimal consumption-investment rule in Proposition 3.3 implies that individual wealth evolves according to

$$w_{i,t+1} = \beta(\lambda \max\{0, z_{it} - R\} + R)w_{it}. \tag{4.10}$$

Since (4.10) is a random multiplicative process (logarithmic random walk), it does not admit a stationary distribution if agents are infinitely lived. To obtain a stationary wealth distribution, we consider a Yaari (1965)-type perpetual youth model in which agents survive with probability $\upsilon < 1$ every period, and deceased agents are replaced with newborn agents. If we assume that the discount factor β already includes the survival probability and that the wealth of deceased agents is equally redistributed to newborn agents,[6] the aggregate dynamics remains identical to the infinitely-lived case. We discuss each case $G = 1$ and $G > 1$ separately.

If $G = 1$, then $W_t = W > 0$ is constant. Define the relative wealth $s_t := w_{t+1}/W_{t+1}$, where we have suppressed the individual subscript and shifted the time subscript because w_{t+1} is determined at time t. Then dividing the equation of motion for wealth (4.10) by $W_{t+1} = W_t$ and using the equilibrium condition (4.3) to eliminate λ, we obtain

$$s_t = \begin{cases} (1 + (1 - \beta R) g(z_t)) s_{t-1} & \text{with probability } \upsilon, \\ 1, & \text{with probability } 1 - \upsilon, \end{cases} \tag{4.11}$$

where

$$g(z) := \frac{\max\{0, z - R\}}{\pi(R)} - 1. \tag{4.12}$$

If $G > 1$, then $W_{t+1} = RW_t$. Dividing the equation of motion for wealth (4.10) by $W_{t+1} = RW_t$ and using the equilibrium condition (4.6) to eliminate λ, we obtain

$$s_t = \begin{cases} (1 + (1 - \beta) g(z_t)) s_{t-1} & \text{with probability } \upsilon, \\ 1, & \text{with probability } 1 - \upsilon. \end{cases} \tag{4.13}$$

According to the definition in Beare and Toda (2022, §2), the stochastic processes (4.11) and (4.13) are both Markov multiplicative process with reset probability $1 - \upsilon$, which admit unique stationary distributions. To characterize the tail behavior of the wealth distribution, we introduce the following assumption.

Assumption 4. *The productivity distribution is thin-tailed, i.e., for all $j \in \mathbb{N}$ the productivity distribution has a finite j-th moment: $\mathrm{E}[z^j] = \int_0^\infty z^j \, \mathrm{d}F(z) < \infty$.*

Assumption 4 is sufficient (but not necessary) for (4.14) below to have a solution. See Beare and Toda (2022, Fig. 2(c)) for why this type of assumption is needed. The following theorem establishes the uniqueness of the stationary relative wealth distribution and characterizes its tail behavior.

Theorem 4.7 (Wealth distribution). *Suppose Assumptions 3 and 4 hold, agents survive with probability $\upsilon < 1$, and a trend stationary equilibrium with interest*

[6] It is straightforward to consider settings where there are life insurance companies that offer annuities to agents, there are estate taxes, or newborn agents start with wealth drawn from some initial distribution.

rate R and wealth growth rate G exists and $\Pr(z > R) > 0$*. Then the following statements are true.*

(i) There exists a unique stationary distribution of relative wealth $s_t = w_{t+1}/W_{t+1}$*.*

(ii) The stationary distribution has a Pareto upper tail with exponent $\zeta > 1$ *in the sense that* $\lim_{s\to\infty} s^{\zeta} \Pr(s_t > s) \in (0, \infty)$ *exists.*

(iii) The Pareto exponent ζ *is uniquely determined by the equation*

$$1 = \rho(\zeta) := \begin{cases} \upsilon \, \mathrm{E}[(1 + (1 - \beta R) g(z))^{\zeta}] & \text{if } G = 1, \\ \upsilon \, \mathrm{E}[(1 + (1 - \beta) g(z))^{\zeta}] & \text{if } G > 1, \end{cases} \tag{4.14}$$

where $g(z)$ *is defined by* (4.12)*.*

(iv) Letting $\zeta_f(R), \zeta_b(R) > 1$ *be the Pareto exponents in the fundamental and bubbly regime determined by* (4.14) *given the equilibrium interest rate* $R > 1$*, we have* $\zeta_f(R) > \zeta_b(R)$*.*

As shown by Proposition 1 of Beare and Toda (2022), $\rho(\zeta)$ in (4.14) is convex is ζ and $\rho(0) = \upsilon < 1 < \infty = \rho(\infty)$, which explains the uniqueness of ζ. Noting that $\mathrm{E}[g(z)] = 0$ by the definitions of $\pi(R)$ in (3.13) and $g(z)$ in (4.12), we obtain $\rho(1) = \upsilon < 1$, which explains $\zeta > 1$. Intuitively, $\zeta > 1$ follows from the fact that in equilibrium, the wealth distribution must have a finite mean (otherwise market clearing is not well defined). As $\upsilon \to 0$, we obtain $\zeta \to 1$, which is known as Zipf's law. The fact that the Pareto exponent is lower (wealth inequality is higher) in the bubbly regime than in the fundamental regime corresponding to the same equilibrium interest rate is that the "growth shock" $g(z)$ in (4.12) is multiplied by $1 - \beta$ in the bubbly regime (see (4.13)), whereas it is multiplied by $1 - \beta R$ in the fundamental regime (see (4.11), and we have $1 - \beta > 1 - \beta R$ because $R > 1$.

Figure 2 shows the Pareto exponent ζ that solves (4.14) with survival probability $\upsilon = 0.99$ for the equilibrium interest rate R in Example 1. As the equity requirement $1/\lambda$ is relaxed in the fundamental regime, the interest rate and the Pareto exponent go down. The intuition for this result is that because $g(z)$ in (4.12) is not so sensitive to R, the decrease in R associated with relaxing leverage (see Figure 1) amplifies the growth shock $g(z)$ through the relative wealth dynamics (4.11). However, once in the bubbly regime, the Pareto exponent is relatively flat. The intuition is that because $g(z)$ is not so sensitive to R, the relative wealth dynamics (4.13) becomes insensitive to the interest rate. This result implies that in the bubbly regime, the presence of bubbles generates an equalizing force. Although high productive agents can choose high leverage, the associated increase in the interest rate allows low productive agents to catch up, and wealth inequality becomes insensitive to leverage. Thus bubbles provide an equal opportunity for everyone to produce more.

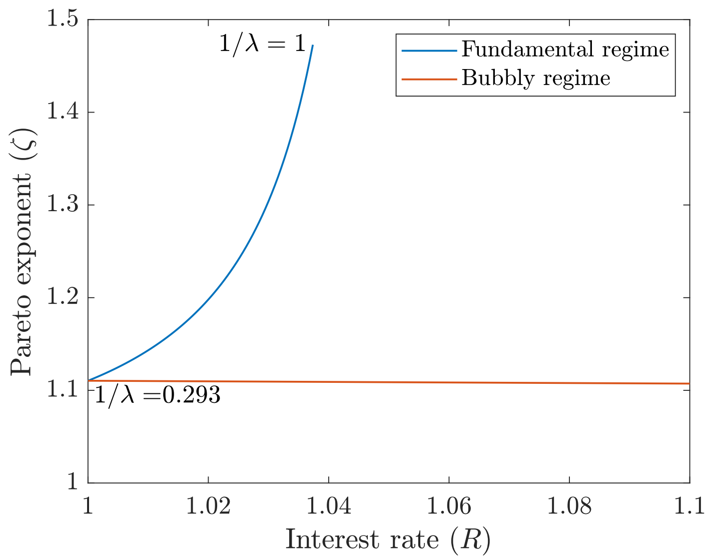

Figure 2: Determination of wealth Pareto exponent.

Note: The figure shows the Pareto exponent ζ that solves (4.14) for the equilibrium interest rate R determined in Figure 1.

5 Related literature

It is well known since Samuelson (1958) that OLG models can support rational asset price bubbles. Although it is harder to generate bubbles in models with infinitely-lived agents, Bewley (1980) constructed a model in which fiat money has a positive value by introducing borrowing constraints. Kocherlakota (1992) showed that in a deterministic model with infinitely-lived agents, sequential trading, and shortsales constraints, an asset in positive net supply cannot exhibit a bubble unless shortsales constraints bind and the present value of endowment is infinite for at least one agent. As discussed in the introduction, Santos and Woodford (1997) proved the impossibility of bubbles under borrowing constraints if the asset is in positive net supply and the present value of the aggregate endowment is finite.

Proposition 1(b) of Tirole (1985) shows that when an OLG economy is dynamically inefficient, introducing a bubble asset restores efficiency; see Blanchard and

Fischer (1989, §5.2) for a textbook treatment. However, McCallum (1987) shows that if land is also used in production, its presence rules out dynamically inefficient equilibria and hence bubbles. Thus generating bubbles in an economy with a dividend-paying asset is considered difficult, which is one reason why the existing literature on rational bubbles has almost exclusively focused on pure bubbles (see Footnote 3).

As noted in the introduction, while pure bubble models generally feature equilibrium indeterminacy, Proposition 1(c) of Tirole (1985) shows that in an OLG model with positive population growth and constant rents, if the interest rate without bubbles is negative, bubbles occur as the unique equilibrium outcome. Although one of our results shares a similarity in that bubbles are necessary for equilibrium existence and equilibrium is determinate, there are substantial differences between his model and ours. Most importantly, while Tirole (1985) studies a stylized OLG model with exogenous population growth and hence the financial sector plays no role, in our model the growth rate of the economy and the existence condition of bubbles are endogenously determined by the interaction between the production and endowment sectors through leverage.

Our paper is also related to the large macro-finance literature, which includes Greenwald and Stiglitz (1993), Kiyotaki and Moore (1997), Bernanke et al. (1999), He and Krishnamurthy (2013), and Brunnermeier and Sannikov (2014), among others. These papers show that even a small shock to the economy can have large effects through the "financial accelerator"—a feedback loop between asset prices and macroeconomic activities amplifying the effects. Like these papers, in our model the interaction between asset prices and real economic activities plays an important role in shaping the equilibrium. However, these papers all consider one-sector models, in which aggregate wealth and dividends grow at the same rate and thus cannot generate bubbles. In contrast, our model features two sectors and hence aggregate wealth and dividends could be decoupled. The growth rate of the economy is endogenously determined through the leverage constraint and the balance of the two sectors. Once the interaction between the two sectors becomes strong enough with the lax leverage constraint, asset price bubbles are inevitable, i.e., there is a phase transition to the bubble economy.

6 Concluding remarks

Since the Santos and Woodford (1997) Bubble Impossibility Theorem, it has been recognized that there are fundamental difficulties in generating asset price bubbles in rational equilibrium models with dividend-paying assets in positive net supply. As a result, the rational bubble literature has almost exclusively focused on "pure bubbles", i.e., assets that pay no dividends and hence are intrinsically useless, although these models are subject to several criticisms including equilibrium indeterminacy.

As discussed in the introduction, this paper advances the literature on asset price bubbles in two respects. First, we provided simple example economies with a bubbly equilibrium but without any fundamental equilibria, which shows the necessity of asset price bubbles in some economies. Second, we presented a two-sector endogenous growth model with a rational asset price bubble that circumvents all criticisms to the pure bubble models: in our model the bubble is attached to a dividend-paying asset, the equilibrium is determinate, and asset price bubbles necessarily emerge as the financial constraints are relaxed.

Our model has two crucial features to render the bubble possibility and necessity results. The first is incomplete markets. Market incompleteness allows the present value of aggregate endowment to be infinite when discounted by the risk-free rate (thus circumventing the Santos-Woodford impossibility result), while making the present value of individual endowments finite when discounted by individual marginal rates of substitution so that the equilibrium is well defined. The second is that the economy consists of two sectors with different output elasticities. In our example economy, we supposed that land produces dividends inelastically and the production technology is linear. This feature allows the economy to either converge to the steady state or grow exponentially depending on patience, productivity, and leverage limit by decoupling economic growth from dividend growth.

Because the purpose of our paper is to theoretically establish the possibility and necessity of asset price bubbles in rational equilibrium models, we focused on providing theorems and abstracted from applications. We leave detailed investigations of applied models for future research.

A Proofs

Proof of Proposition 2.2. We divide the proof into several steps.

Step 1. $\limsup_{t\to\infty} G^{-t}P_t > 0$ *holds.*

Suppose $\lim_{t\to\infty} G^{-t}P_t = 0$. Then by (2.6b) we obtain

$$\begin{aligned}R_t &= \frac{1}{\beta}\left(\frac{BG^{t+1} - \frac{1}{G-1}D + P_{t+1}}{AG^t + \frac{1}{G-1}D - P_t}\right)^\gamma \\ &= \frac{1}{\beta}\left(G\frac{B - \frac{1}{G-1}DG^{-t-1} + G^{-t-1}P_{t+1}}{A + \frac{1}{G-1}DG^{-t} - G^{-t}P_t}\right)^\gamma \to \frac{1}{\beta}(BG/A)^\gamma < 1, \end{aligned} \tag{A.1}$$

where the last inequality follows from the assumption $A > \beta^{-1/\gamma}GB$. By (A.1), we can take $\bar{R} < 1$ such that $R_t < \bar{R}$ for large enough t. But then the present value of dividends and hence the asset price is infinite, which is a contradiction.

Step 2. If a fundamental equilibrium exists, then $\liminf_{t\to\infty} R_t \geq G$.

If $\liminf_{t\to\infty} R_t < G$, then we can take large $T > 0$ such that $R_t \leq G$ for $t \geq T$. Letting $q_t > 0$ be the Arrow-Debreu price, it follows that

$$q_t P_t = \left(q_T \prod_{s=T}^{t-1} \frac{1}{R_s}\right) P_t \geq q_T G^{T-t} P_t = q_T G^T G^{-t} P_t.$$

Letting $t \to \infty$, we obtain $\limsup_{t\to\infty} q_t P_t > 0$, so the transversality condition (2.1) fails, which is a contradiction.

Step 3. There exist no fundamental equilibria.

If there exist a fundamental equilibrium, it must be $\liminf_{t\to\infty} R_t \geq G$. Since $G > 1$, the argument before Proposition 2.2 implies $R_t \to (BG/A)^\gamma/\beta < 1$, which is a contradiction. □

Proof of Lemma 3.2. If $(P_{t+1} + D_{t+1})/P_t > R_t$, increasing x_t by Δ and reducing b_t by $P_t\Delta$, the leverage constraint (3.3) is unaffected but the right-hand side of the budget constraint (3.2) (where $t-1$ is replaced with t) increases by $(P_{t+1} + D_{t+1} - R_t P_t)\Delta > 0$, which enables to increase consumption c_{t+1}. Therefore an optimal consumption does not exist. A similar argument applies if $(P_{t+1} + D_{t+1})/P_t < R_t$. Therefore in equilibrium the no-arbitrage condition (3.6) must hold. □

Proof of Lemma 3.4. We suppress the t subscript to simplify the notation. Non-negativity, monotonicity, and convexity of π are obvious because the function $R \mapsto \max\{0, z - R\}$ is nonnegative, decreasing, and convex, and integration preserves these properties. Since $\max\{0, z - R\} \leq \max\{0, z\} = z$, $\max\{0, z - R\} \to 0$ as $R \to \infty$, and Assumption 1 implies $\int_0^\infty z \,\mathrm{d}F(z) < \infty$, an application of the dominated convergence theorem yields the continuity of π and $\pi(\infty) = 0$. Finally, we show the strict monotonicity of π. Since F is continuous, the function $R \mapsto \max\{0, z - R\}$ is almost everywhere differentiable with derivative 0 if $z < R$ and -1 if $z > R$. Therefore an application of the dominated convergence theorem implies that π is differentiable and

$$\pi'(R) = -\int_0^\infty 1(z > R)\,\mathrm{d}F(z) = F(R) - 1 \leq 0,$$

with strict inequality if $F(R) < 1$. □

Proof of Theorem 3.6. We prove the more general Theorem C.1, which allows dividend growth. We divide the proof into several steps.

Step 4. The fundamental value V_t in (3.22) is finite and $\limsup_{t\to\infty} V_t \mathrm{e}^{-dt} < \infty$.

The first condition in (C.2) implies that there exists $\bar{D} > 0$ such that $D_t \leq \bar{D}\mathrm{e}^{dt}$ for all t. The second condition in (C.2) implies that there exists $\bar{R} > \mathrm{e}^d$ and $T \in \mathbb{N}$ such that $R_t \geq \bar{R}$ for $t \geq T$. Then $\left(\prod_{j=0}^{s-1} R_{t+j}\right)^{-1} \leq \bar{R}^{-s}$ for $t \geq T$, so

$$V_t = \sum_{s=1}^{\infty} \left(\prod_{j=0}^{s-1} R_{t+j}\right)^{-1} D_{t+s} \leq \sum_{s=1}^{\infty} \bar{R}^{-s} \bar{D}\mathrm{e}^{d(t+s)} = \frac{\bar{D}\mathrm{e}^{dt}}{\bar{R}\mathrm{e}^{-d} - 1} < \infty.$$

This uniform upper bound implies $\limsup_{t\to\infty} V_t \mathrm{e}^{-dt} < \infty$. By the definition of the fundamental value (3.22), we have $V_t = (V_{t+1} + D_{t+1})/R_t$. Iterating this yields

$$V_t = \sum_{s=1}^{N} \left(\prod_{j=0}^{s-1} R_{t+j}\right)^{-1} D_{t+s} + \left(\prod_{j=0}^{N-1} R_{t+j}\right)^{-1} V_{t+N}. \tag{A.2}$$

Since $V_{t+N} < \infty$ for large enough N, (A.2) implies $V_t < \infty$ for all t.

Step 5. If $\limsup_{t\to\infty} W_t \mathrm{e}^{-dt} < \infty$ and $\liminf_{t\to\infty} B_t \mathrm{e}^{-dt} > -\infty$, then $P_t = V_t$.

The first term in (3.21) converges to V_t as $N \to \infty$. Letting $t \to \infty$ in (3.16) and noting that $\alpha_t \leq \beta$, it follows from (3.20b) that

$$\limsup_{t\to\infty} P_t \mathrm{e}^{-dt} \leq \beta \limsup_{t\to\infty} W_t \mathrm{e}^{-dt} - \liminf_{t\to\infty} B_t \mathrm{e}^{-dt} < \infty.$$

Therefore for large enough N, we have

$$\left(\prod_{j=0}^{N-1} R_{t+j}\right)^{-1} P_{t+N} \leq \bar{R}^{-N} P_{t+N} = \mathrm{e}^{dt}(\mathrm{e}^d/\bar{R})^N P_{t+N}\mathrm{e}^{-d(t+N)} \to 0$$

as $N \to \infty$ because $\mathrm{e}^d/\bar{R} < 1$. Hence the second term in (3.21) converges to 0 as $N \to \infty$, implying $P_t = V_t$.

Step 6. If $\limsup_{t\to\infty} W_t \mathrm{e}^{-dt} = \infty$ *and* $\limsup_{t\to\infty} B_t/W_t \le 0$*, then* $P_t > V_t$ *for all* t*.*

Since the first term in (3.21) converges to V_t as $N \to \infty$ and

$$\liminf_{N\to\infty} \left(\prod_{j=0}^{N-1} R_{t+j}\right)^{-1} P_{t+N} \ge 0,$$

we obtain $P_t \ge V_t$ for all t. Dividing both sides of (3.20b) by W_t and letting $t \to \infty$, since $\limsup_{t\to\infty} B_t/W_t \le 0$, it follows from the third condition in (C.2) that

$$\liminf_{t\to\infty} P_t/W_t \ge \liminf_{t\to\infty} \alpha_t - \limsup_{t\to\infty} B_t/W_t \ge \underline{\alpha}$$

for some $\underline{\alpha} > 0$. Therefore $\limsup_{t\to\infty} P_t \mathrm{e}^{-dt} \ge \underline{\alpha} \limsup_{t\to\infty} W_t \mathrm{e}^{-dt} = \infty$. Since $\limsup_{t\to\infty} V_t \mathrm{e}^{-dt} < \infty$, we have $P_t > V_t$ infinitely often. Therefore for any t, we can take N such that $P_{t+N} > V_{t+N}$. Subtracting (A.2) from (3.21), we obtain

$$P_t - V_t = \left(\prod_{j=0}^{N-1} R_{t+j}\right)^{-1} (P_{t+N} - V_{t+N}) > 0. \qquad \square$$

Proof of Lemma 4.1. We first show the convexity of $\mathcal{R}_f'$. By definition we have $\mathcal{R}_f' = \mathcal{R}_1 \cap \mathcal{R}_2$, where

$$\begin{aligned}
\mathcal{R}_1 &= \{R > 1 : 1 - F(R) < 1/\lambda\}, \\
\mathcal{R}_2 &= \{R > 1 : \lambda\pi(R) + R \le 1/\beta\}.
\end{aligned}$$

Since F is a cdf and hence monotonic, $\mathcal{R}_1$ is convex. $\mathcal{R}_2$ is convex because π is convex by Lemma 3.4 and the sum of convex functions is convex. Therefore $\mathcal{R}_f'$ is convex. Furthermore, if $R \in \mathcal{R}_f'$, it follows from $\pi \ge 0$ that $1/\beta \ge \lambda\pi(R) + R \ge R$. Therefore $\mathcal{R}_f' \subset (1, 1/\beta]$.

Suppose $\mathcal{R}_f \ne \emptyset$. Since clearly $\mathcal{R}_f \subset \mathcal{R}_f'$, we have $\mathcal{R}_f' \ne \emptyset$. Conversely, suppose $\mathcal{R}_f' \ne \emptyset$ and take $\bar{R} \in \mathcal{R}_f'$. Then $\bar{R} \le 1/\beta$. Define $g(R) := \lambda\pi(R) + R$. Since $\bar{R} \in \mathcal{R}_f'$, we have $g(\bar{R}) \le 1/\beta$. Since $\pi \ge 0$, we have $g(1/\beta) \ge 1/\beta$. Since π is continuous, so is g. Therefore by the intermediate value theorem, there exists $R \in [\bar{R}, 1/\beta]$ that satisfies $g(\bar{R}) = 1/\beta$. Since $\bar{R} \le R$ and F is a cdf, we have $F(\bar{R}) \le F(R)$, so $1 - F(R) \le 1 - F(\bar{R}) < 1/\lambda$. Therefore $R \in \mathcal{R}_f$ and hence $\mathcal{R}_f \ne \emptyset$. $\square$

Proof of Theorem 4.2. $G = 1$ and $R \in \mathcal{R}_f \subset \mathcal{R}'_f$ are necessary for equilibrium by the discussion leading to Theorem 4.2. In this case $W_t = W_0$ for all t. Multiplying both sides of (3.20e) by W_0 and using the equilibrium condition (4.3), we obtain

$$\alpha(1-R)W_0 = B_t - RB_{t-1} - D \iff B_t = RB_{t-1} + D - (R-1)\alpha W_0.$$

Since $R > 1$, the solution B_t to the difference equation diverges unless it is constant. Therefore for long run bond market clearing $B_t/W_t \to 0$ to hold, it is necessary that B_t is constant, which implies

$$B_t = \alpha W_0 - \frac{D}{R-1}.$$

But then B_t/W_t is constant, so $B_t/W_t \to 0$ implies $B_t = 0$, which holds if and only if $W_0 = \frac{D}{(R-1)\alpha}$. Using the initial condition (3.20c), we obtain $Y_0 = \frac{1-R\alpha}{(R-1)\alpha}D$. Conversely, it is obvious that these quantities define a trend stationary equilibrium with $G = 1$ and interest rate R. $\square$

Proof of Theorem 4.3. Let $\underline{R} = \max\{1, F^{-1}(1/\lambda - 1)\}$. If an equilibrium with $1 - F(R) \neq 1/\lambda$ exists, then the interest rate must satisfy $R \in \mathcal{R}_b$, where $\mathcal{R}_b$ is defined in (4.7). Then clearly $R > \underline{R}$, and since $\pi(R)$ is strictly decreasing by Lemma 3.4, we have $\lambda\pi(\underline{R})/\underline{R} > 1/\beta - 1$. Conversely, if $\underline{R}$ satisfies this inequality, by the intermediate value theorem $R \in \mathcal{R}_b$ exists.

Since $R \in \mathcal{R}_b$ satisfies (4.6), we obtain $G = R > 1$. Then $W_t = R^t W_0$. Furthermore, the no-arbitrage condition (3.20e) implies $B_t = RB_{t-1} + D$. Solving this difference equation, we obtain

$$B_t = -\frac{D}{R-1} + R^t\left(B_0 + \frac{D}{R-1}\right).$$

Therefore the long run bond market clearing implies

$$0 = \lim_{t\to\infty}\frac{B_t}{W_t} = \frac{B_0 + \frac{D}{R-1}}{W_0},$$

so $B_0 = -\frac{D}{R-1}$ and hence $B_t = -\frac{D}{R-1}$ for all t. Then (3.20c) and $W_t = R^t W_0$ imply

$$W_t = R^t\frac{(R-1)Y_0 + RD}{(1-\alpha)(R-1)}.$$

Finally, (3.20b) implies $P_t = \alpha W_t + \frac{D}{R-1}$. Conversely, it is obvious that these quantities define a trend stationary equilibrium with $G = R > 1$. $\square$

Proof of Theorem 4.4. Under the maintained assumptions, the equilibrium dynamics (3.20) reduces to

$$\beta(\lambda\pi(R_t)+R_t)(\lambda F(R_t)+1-\lambda)=R_{t-1}(\lambda F(R_{t-1})+1-\lambda)-\frac{(R_{t-1}-1)B+D}{\beta W_{t-1}},$$
$$W_t=\beta(\lambda\pi(R_{t-1})+R_{t-1})W_{t-1}.$$

Letting $x_t=(R_t,1/W_t)$, we can rewrite the dynamics as

$$\begin{aligned}
&\beta(\lambda\pi(x_{1t})+x_{1t})(\lambda F(x_{1t})+1-\lambda)\\
&=x_{1,t-1}(\lambda F(x_{1,t-1})+1-\lambda)-\frac{(x_{1,t-1}-1)B+D}{\beta}x_{2,t-1},\\
&x_{2,t}=\frac{1}{\beta(\lambda\pi(x_{1t})+x_{1t})}x_{2,t-1}.
\end{aligned}$$

A straightforward application of the implicit function theorem shows that, around the steady state $\bar{x}=(R,0)$, the system of nonlinear difference equations can be expressed as $x_t=f(x_{t-1})$, where f is a C^1 function defined on a neighborhood of $\bar{x}$ taking values in $\mathbb{R}^2$ and satisfying $\bar{x}=f(\bar{x})$. Furthermore, the Jacobian of f at $\bar{x}$ is

$$Df=\begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix},$$

where

$$\begin{aligned}
f_{11}&=\frac{\lambda F(R)+1-\lambda+R\lambda F'(R)}{\beta(\lambda\pi'(R)+1)(\lambda F(R)+1-\lambda)+\beta(\lambda\pi(R)+R)\lambda F'(R)},\\
f_{12}&=-\frac{(R-1)B+D}{\beta^2(\lambda\pi'(R)+1)(\lambda F(R)+1-\lambda)+\beta^2(\lambda\pi(R)+R)\lambda F'(R)},\\
f_{21}&=0,\\
f_{22}&=\frac{1}{\beta(\lambda\pi(R)+R)}.
\end{aligned}$$

Since the equilibrium is bubbly, the equilibrium condition (4.6) implies

$$\beta(\lambda\pi(R)+R)=R. \tag{A.3}$$

Therefore $f_{22}=1/R<1$. Furthermore,

$$f_{11}=\frac{1+k}{\beta(\lambda\pi'(R)+1)+k},$$

where $k = \frac{R\lambda F'(R)}{\lambda F(R)+1-\lambda} \geq 0$ because $F'(R) \geq 0$ and $\lambda F(R) + 1 - \lambda > 0$ since $R \in \mathcal{R}_b$ in (4.7). We claim that $\beta(\lambda\pi'(R) + 1) \in (0, 1)$ and hence $f_{11} > 1$. To see this, by Lemma 3.4 we have $\lambda\pi'(R) + 1 = \lambda F(R) + 1 - \lambda > 0$. Define the function $g(x) := \beta(\lambda\pi(x)+x)-x$. Then $g(0) = \beta\lambda\pi(0) > 0$, and by Lemma 3.4, g is convex and $g(\infty) = -\infty$ because $\pi(\infty) = 0$ and $\beta < 1$. Since (A.3) implies $g(R) = 0$, it must be $g'(R) < 0$, implying $\beta(\lambda\pi'(R) + 1) < 1$.

Putting all pieces together, the Jacobian of f at $\bar{x}$ is

$$Df = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ 0 & 1/R \end{bmatrix},$$

where $f_{11} > 1 > 1/R$. Therefore Df has one stable and one unstable eigenvalue. The local stable manifold theorem (see Irwin (1980, Theorems 6.5 and 6.9) and Guckenheimer and Holmes (1983, Theorem 1.4.2)) implies that for any sufficiently small $x_{2,0} > 0$ (hence sufficiently large $W_0 > 0$), there exists a unique orbit $\{x_t\}_{t=0}^{\infty}$ converging to $\bar{x}$. Since $W_0 = \frac{Y_0+D}{1-\alpha_0}$ by (3.20c), it follows that for large enough $Y_0 > 0$, there exists a unique bubbly long run equilibrium. □

Proof of Proposition 4.6. Since $\Pr(z > 1/\beta) > 0$, it follows from the definition of π in (3.13) that $\pi(1/\beta) > 0$. Then $\phi_f(R) > 0$ for $R \in [1, 1/\beta)$ and $\phi_f(1/\beta) = \infty$. Since $\pi'(R) = F(R) - 1$, for $R \in [1, 1/\beta)$ we obtain

$$\begin{aligned}
\phi_f'(R) &= \frac{\beta}{(1-\beta R)^2}(\pi'(R)(1-\beta R) + \beta\pi(R)) \\
&= \frac{-\beta\pi'(R)}{(1-\beta R)^2}\left(-\beta\frac{\pi(R)}{\pi'(R)} - 1 + \beta R\right) \\
&= \frac{\beta(1-F(R))}{(1-\beta R)^2}(\beta(\mathrm{E}[z \mid z \geq R] - R) - 1 + \beta R) \\
&= \frac{\beta(1-F(R))}{(1-\beta R)^2}(\beta\,\mathrm{E}[z \mid z \geq R] - 1) > 0,
\end{aligned}$$

where the last line follows from $1 - F(R) > 1 - F(1/\beta) > 0$ and $\mathrm{E}[z \mid z \geq R] \geq \mathrm{E}[z \mid z \geq 1] > 1/\beta$. Therefore ϕ_f is strictly increasing.

Since by Lemma 3.4 π is strictly decreasing whenever $F(R) < 1$ (and hence $\pi(R) > 0$), ϕ_b is strictly decreasing whenever $\phi_b > 0$. Furthermore, $\pi(\infty) = 0$ implies $\phi_b(\infty) = 0$.

Noting that $\phi_f(1) = \phi_b(1) = \frac{\beta}{1-\beta}\pi(1)$, if $1/\lambda > \phi_f(1)$, there exists a unique $R > 1$ with $\phi_f(R) = 1/\lambda$ and $\phi_b(R) < \phi_b(1) < 1/\lambda$ for all $R > 1$. Hence $\mathcal{R}_f$ in (4.4) contains at most one point and $\mathcal{R}_b = \emptyset$. Using the definition of π and

$\mathrm{E}[z \mid z \geq 1] > 1/\beta$, we obtain

$$\begin{aligned}\frac{1}{\lambda} > \frac{\beta}{1-\beta}\pi(1) &= \frac{\beta}{1-\beta}(1-F(1))(\mathrm{E}[z \mid z \geq 1]-1)\\ &= \frac{1-F(1)}{1-\beta}(\beta\,\mathrm{E}[z \mid z \geq 1]-\beta)\\ &\geq 1-F(1) \geq 1-F(R), \qquad \text{(A.4)}\end{aligned}$$

so $\mathcal{R}_f$ is nonempty. Thus $\mathcal{R}_f$ is a singleton.

If $1/\lambda < \phi_b(1)$, since ϕ_b is strictly decreasing and $\phi_b(\infty) = 0$, there exists a unique $R > 1$ with $\phi_b(R) = 1/\lambda$. By (A.4), $\mathcal{R}_b$ is a singleton. Since ϕ_f is strictly increasing, $\mathcal{R}_f$ is empty. □

Proof of Theorem 4.7. The uniqueness of the stationary relative wealth distribution follows from Proposition 3 of Beare and Toda (2022). To show the Pareto tail result, define $\rho(\zeta)$ by (4.14) for $\zeta \geq 0$. By Assumption 4, we have $\rho(\zeta) \in (0, \infty)$ for all $\zeta \geq 0$, and clearly ρ is continuous. Since by assumption $z > R$ (and hence $g(z) > 0$) with positive probability, we have $\rho(\infty) = \infty$. Noting that $\mathrm{E}[g(z)] = 0$ by the definitions of $\pi(R)$ in (3.13) and $g(z)$ in (4.12), we obtain $\rho(1) = \upsilon < 1$. Therefore by the intermediate value theorem, there exists $\zeta \in (1, \infty)$ such that $\rho(\zeta) = 1$. By Proposition 1 of Beare and Toda (2022), ζ is unique.

By Assumption 1, the cdf F is atomless. Therefore by Theorem 2 of Beare and Toda (2022), the stationary distribution of relative wealth has a Pareto upper tail with exponent $\zeta > 1$ in the sense that $\lim_{s \to \infty} s^{\zeta} \Pr(s_t > s) \in (0, \infty)$ exists.

Since $\mathrm{E}[g(z)] = 0$ and $R > 1$ implies $1 - \beta R < 1 - \beta$, by Proposition 5 of Beare and Toda (2022) (where $1 - \beta R$ and $1 - \beta$ correspond to $\sigma_{nn'}$ in their paper), it follows that $\zeta_b(R) < \zeta_f(R)$. □

B Optimal consumption in nonstationary environment

In this appendix we solve the optimal consumption-investment problem with CRRA utility

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \qquad \text{(B.1)}$$

where the case $\gamma = 1$ is interpreted as log utility (3.1). Because the productivity is known at time t, the optimal investment rule (3.11b) is obvious. Define the return on wealth at time t conditional on productivity z by

$$G_t(z) \coloneqq R_t + \lambda_t \max\{0, z - R_t\}.$$

Then the Bellman equation (3.10) with CRRA utility (B.1) becomes

$$v_t(w,z) = \sup_{0 \le c \le w} \left[\frac{c^{1-\gamma}}{1-\gamma} + \beta \operatorname{E}_t[v_{t+1}(G_t(z)(w-c), z')] \right]. \tag{B.2}$$

The following proposition characterizes the optimal consumption rule.

Proposition B.1 (Optimal consumption with CRRA utility). *Suppose the utility function is given by* (B.1) *and*

$$\sum_{n=0}^{\infty} \prod_{s=0}^{n} (\beta \operatorname{E}[G_s(z)^{1-\gamma}]^{1/\gamma} < \infty. \tag{B.3}$$

Then the optimal consumption rule is $c_t = w_t/a_t$, where

$$a_t = 1 + \sum_{n=0}^{\infty} \prod_{s=0}^{n} (\beta \operatorname{E}[G_{t+s}(z)^{1-\gamma}]^{1/\gamma}. \tag{B.4}$$

Proof. To solve the Bellman equation (B.2), following the idea of Ma et al. (2022), it is convenient to define $g_t(w) \coloneqq \operatorname{E}_t[v_t(w,z)]$. Applying the law of iterated expectations, the Bellman equation (B.2) can be transformed as

$$g_t(w) = \sup_{0 \le c \le w} \left[\frac{c^{1-\gamma}}{1-\gamma} + \beta \operatorname{E}_t[g_{t+1}(G_t(z)(w-c))] \right]. \tag{B.5}$$

Let us guess that $g_t(w) = \frac{a_t^\gamma}{1-\gamma} w^{1-\gamma}$ satisfies the transformed Bellman equation for some $a_t > 0$. Substituting this guess into (B.5), the objective function in the right-hand side becomes

$$\frac{c^{1-\gamma}}{1-\gamma} + \beta \frac{a_{t+1}^\gamma}{1-\gamma} \operatorname{E}[G_t(z)^{1-\gamma}](w-c)^{1-\gamma}.$$

Clearly this function is strictly concave in c, and setting the derivative to 0 yields the optimal consumption

$$c = [1 + (\beta \operatorname{E}[G_t(z)^{1-\gamma}])^{1/\gamma} a_{t+1}]^{-1} w. \tag{B.6}$$

Substituting this consumption into (B.5), under the guess of $g_t(w)$, we obtain

$$\begin{aligned}\frac{a_t^\gamma}{1-\gamma}w^{1-\gamma} &= \frac{c^{1-\gamma}}{1-\gamma} + \frac{c^{-\gamma}}{1-\gamma}(w-c) = \frac{1}{1-\gamma}c^{-\gamma}w \\ &= \frac{1}{1-\gamma}[1+(\beta\,\mathrm{E}[G_t(z)^{1-\gamma}])^{1/\gamma}a_{t+1}]^\gamma w^{1-\gamma}.\end{aligned}$$

Dividing both sides by $\frac{w^{1-\gamma}}{1-\gamma}$ and taking the $1/\gamma$-th power, we obtain

$$a_t = 1+(\beta\,\mathrm{E}[G_t(z)^{1-\gamma}])^{1/\gamma}a_{t+1}. \tag{B.7}$$

Under condition (B.3), if we define a_t by (B.4), then (B.7) trivially holds. Therefore the guess $g_t(w) = \frac{a_t^\gamma}{1-\gamma}w^{1-\gamma}$ indeed satisfies the transformed Bellman equation (B.5). Furthermore, (B.6) and (B.7) imply the consumption rule $c_t = w_t/a_t$.

Finally, we verify the transversality condition $\mathrm{E}_0[\beta^t u'(c_t)w_t] \to 0$. Using $c_t = w_t/a_t$, we obtain $\beta^t u'(c_t)w_t = \beta^t a_t(w_t/a_t)^{1-\gamma}$. Combining the budget constraint, $c_t = w_t/a_t$, and (B.7), we obtain

$$\frac{w_{t+1}}{a_{t+1}} = G_t(z)\frac{a_t-1}{a_t}\frac{w_t}{a_{t+1}} = G_t(z)(\beta\,\mathrm{E}[G_t(z)^{1-\gamma}]^{1/\gamma}\frac{w_t}{a_t}.$$

Taking the $(1-\gamma)$-th power, multiplying by β^{t+1}, and taking the time t conditional expectation, we obtain

$$\mathrm{E}_t[\beta^{t+1}(w_{t+1}/a_{t+1})^{1-\gamma}] = (\beta\,\mathrm{E}[G_t(z)^{1-\gamma}]^{1/\gamma}\beta^t(w_t/a_t)^{1-\gamma}.$$

Iterating this equation, we obtain

$$\mathrm{E}_0[\beta^t u'(c_t)w_t] = \mathrm{E}_0[\beta_t a_t(w_t/a_t)^{1-\gamma}] = (w_0/a_0)^{1-\gamma}a_t\prod_{s=0}^{t-1}(\beta\,\mathrm{E}[G_s(z)^{1-\gamma}]^{1/\gamma} \to 0$$

using (B.3), (B.4), and (B.7). □

Remark 1. In a stationary environment, the left-hand side of (B.3) becomes a geometric series, and the condition (B.3) reduces to the classical condition $\beta\,\mathrm{E}[G(z)^{1-\gamma}] < 1$. See Ma and Toda (2021, p. 8) for an extensive discussion of this condition.

We next consider the case of log utility (3.1).

Proposition B.2 (Optimal consumption with log utility)**.** *Suppose the utility function is given by* (3.1) *and* $\sum_{n=0}^\infty \beta^n\,\mathrm{E}[\log G_n(z)]$ *is finite. Then the optimal consumption rule is* $c_t = (1-\beta)w_t$.

Proof. We start with the transformed Bellman equation

$$g_t(w) = \sup_{0 \le c \le w} \left[\log c + \beta \operatorname{E}_t[g_{t+1}(G_t(z)(w-c))]\right]. \tag{B.8}$$

Let us guess that $g_t(w) = a_t + \frac{1}{1-\beta} \log w$ satisfies this equation for some $a_t \in \mathbb{R}$. Substituting this guess into (B.8), the objective function in the right-hand side becomes

$$\log c + \beta \left[a_{t+1} + \frac{1}{1-\beta} \operatorname{E}[\log G_t(z)] + \frac{1}{1-\beta} \log(w-c)\right].$$

Clearly this function is strictly concave in c, and setting the derivative to 0 yields the optimal consumption $c = (1-\beta)w$. Substituting this consumption into (B.8), under the guess of $g_t(w)$, we obtain

$$a_t = \log(1-\beta) + \frac{\beta}{1-\beta} \log \beta + \frac{\beta}{1-\beta} \operatorname{E}[\log G_t(z)] + \beta a_{t+1}.$$

Iterating this equation, we obtain a finite value for a_t if $\sum_{n=0}^{\infty} \beta^n \operatorname{E}[\log G_n(z)]$ is finite. Since $\beta^t u'(c_t) w_t = \beta^t w_t / c_t = \frac{\beta^t}{1-\beta} \to 0$, the transversality condition holds. □

C Extensions

In the main text, we have presented a minimal example of a rational expectations equilibrium with bubbles. To show the robustness of our results, we discuss how each assumption can be relaxed.

C.1 Relaxing log utility

In the main text, we assumed log utility only for making the optimal consumption rule simple. Suppose instead that agents have constant relative risk aversion (CRRA) utility (B.1) with relative risk aversion $0 < \gamma \neq 1$. In this case, the optimal consumption rule (3.11a) becomes $c_t = m_t w_t$, where $m_t \in (0,1)$ is the marginal propensity to consume determined by the recursion

$$\frac{1}{m_t} = 1 + (\beta \operatorname{E}[(R_t + \lambda_t \max\{0, z - R_t\})^{1-\gamma}])^{1/\gamma} \frac{1}{m_{t+1}}. \tag{C.1}$$

See Proposition B.1 for details. In this case the equilibrium dynamics (3.20) should be modified such that β is everywhere replaced with $1 - m_t$ and (C.1) needs to be included. The resulting dynamical system is no longer recursive but is a system of forward-backward difference equations.

If we are interested only in trend stationary equilibria, then setting $m_t = m$, $\lambda_t = \lambda$, and $R_t = R$ in (C.1), we obtain

$$m = 1 - (\beta \operatorname{E}[(R + \lambda \max\{0, z - R\})^{1-\gamma}])^{1/\gamma}.$$

The analysis in Section 4 remains valid by replacing β with $1 - m$.

C.2 Relaxing iid productivity

In the main text, we assumed that productivity is IID across agents and time, which is a strong assumption. However, it is straightforward to allow Markov dependence in our model. For instance, suppose that there are finitely many productivity states indexed by $n \in \{1, \dots, N\}$, and let $P = (p_{nn'})$ be the transition probability matrix for the productivity state. Suppose that an agent in state n draws productivity from some distribution with cdf F_n and let

$$\pi_n(R) := \int_0^\infty \max\{0, z - R\} \, \mathrm{d}F_n(z)$$

be the risk premium conditional on being in state n. Let $W_{n,t}$ be the aggregate wealth held by agents in state n at time t. Then the law of motion for aggregate wealth (3.14) needs to be modified to

$$W_{n',t+1} = \beta \sum_{n=1}^{N} p_{nn'}(\lambda_t \pi_n(R_t) + R_t) W_{n,t}.$$

Similarly, the market clearing condition (3.16) needs to be modified to

$$P_t + B_t = \beta \sum_{n=1}^{N} (\lambda_t F_n(R_t) + 1 - \lambda_t) W_{n,t}.$$

Thus the analysis remains largely the same except that the dimension of the dynamical system (3.20) is higher. The wealth Pareto exponent can still be characterized by applying the results of Beare and Toda (2022).

C.3 Relaxing atomless F_t

If the productivity distribution has atoms, then F_t is discontinuous. Since F_t is increasing, there are at most countably many points of discontinuity. In this case the properties of π_t in Lemma 3.4 continue to hold except that π_t is now differentiable only at continuity points of F_t. At discontinuity points, $F_t(R_t)$ in (3.17) needs to be replaced with some $q_t \in [F_t(R_t-), F_t(R_t)]$. Because the long run equilibrium conditions (4.3) and (4.6) do not involve α, the analysis in Section 4 remains valid.

C.4 Relaxing bounded dividends

In the main text, we assumed that the dividend stream $\{D_t\}_{t=0}^{\infty}$ is bounded. However, the following theorem shows that it is straightforward to allow dividend growth.

Theorem C.1 (Characterization of bubbles with dividend growth)**.** *Let everything be as in Theorem 3.6 except that (3.24) is replaced with*

$$\limsup_{t\to\infty} D_t \mathrm{e}^{-dt} < \infty, \quad \liminf_{t\to\infty} R_t > \mathrm{e}^d, \quad \liminf_{t\to\infty} \alpha_t > 0 \tag{C.2}$$

for some $d \in \mathbb{R}$. Then the following statements are true.

(i) The fundamental value of land V_t is finite and $\limsup_{t\to\infty} V_t \mathrm{e}^{-dt} < \infty$.

(ii) If $\limsup_{t\to\infty} W_t \mathrm{e}^{-dt} < \infty$ and $\liminf_{t\to\infty} B_t \mathrm{e}^{-dt} > -\infty$, then $P_t = V_t$ for all t, so the land price equals its fundamental value.

(iii) If $\limsup_{t\to\infty} W_t \mathrm{e}^{-dt} = \infty$ and $\limsup_{t\to\infty} B_t/W_t \leq 0$, then $P_t > V_t$ for all t, so the land price exceeds its fundamental value (bubble).

As is clear from Theorem C.1, what is important for obtaining an asset price bubble is that the interest rate exceeds the dividend growth rate (so that the asset price is finite) and that the aggregate wealth growth rate exceeds the dividend growth rate.

Dividend growth, however, slightly changes the equilibrium conditions. For concreteness, consider the long run setting in Section 4 and suppose that $D_t = D_0 \mathrm{e}^{dt}$ so that the dividends grow at rate e^d. Then there are two types of long run equilibria: one in which aggregate wealth grows at the same rate as dividends ($G = \mathrm{e}^d$), and another in which aggregate wealth grows faster than dividends

$(G > \mathrm{e}^d)$. Both cases can be handled in a way analogous to the analysis of Section 4. For instance, when $G = \mathrm{e}^d$, the equilibrium condition (4.3) becomes

$$\lambda\pi(R) + R = \frac{\mathrm{e}^d}{\beta},$$

and the condition $R > 1$ in (4.4) needs to be replaced with $R > \mathrm{e}^d$. The case $G > \mathrm{e}^d$ is similar.

D Details on Example 1

In this appendix we provide the details of computing Example 1. Suppose $1 - F(z) = \kappa\mathrm{e}^{-z/\bar{z}}$. Using the definition of π in (3.13) and integration by parts, we obtain

$$\begin{aligned}
\pi(R) &= \int_R^\infty (z - R)\,\mathrm{d}F(z) = -\int_R^\infty (z - R)(1 - F(z))'\,\mathrm{d}z \\
&= -\left[(z - R)(1 - F(z))\right]_R^\infty + \int_R^\infty (1 - F(z))\,\mathrm{d}z \\
&= \int_R^\infty \kappa\mathrm{e}^{-z/\bar{z}}\,\mathrm{d}z = \kappa\bar{z}\mathrm{e}^{-R/\bar{z}}.
\end{aligned}$$

Thus ϕ_f, ϕ_b in (4.9) can be computed analytically. To compute the Pareto exponent, we need to evaluate the expectations in (4.14). For $G = 1$, we obtain

$$\begin{aligned}
\mathrm{E}[(\lambda\max\{z - R, 0\} + R)^\zeta] &= \int_0^R R^\zeta\,\mathrm{d}F(z) + \int_R^\infty (\lambda(z - R) + R)^\zeta\,\mathrm{d}F(z) \\
&= R^\zeta(1 - \kappa\mathrm{e}^{-R/\bar{z}}) + \int_R^\infty (\lambda(z - R) + R)^\zeta\frac{\kappa}{\bar{z}}\mathrm{e}^{-z/\bar{z}}\,\mathrm{d}z.
\end{aligned}$$

Using the change of variable $z = \bar{z}x + R$, the last integral becomes

$$\kappa\mathrm{e}^{-R/\bar{z}}\int_0^\infty (\lambda\bar{z}x + R)^\zeta\mathrm{e}^{-x}\,\mathrm{d}x.$$

We use the 15-point Gauss-Laguerre quadrature to evaluate this integral. The case $G > 1$ is similar.

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