COWLES COMMISSION
FOR RESEARCH IN ECONOMICS

REPORT OF THIRD ANNUAL RESEARCH CONFERENCE ON ECONOMICS AND STATISTICS

June 28 to July 23, 1937

COLORADO SPRINGS · COLORADO

THE COWLES COMMISSION FOR RESEARCH IN ECONOMICS

COLORADO SPRINGS, COLORADO

THE COWLES COMMISSION FOR RESEARCH IN ECONOMICS is a not-for-profit corporation, founded in 1932 for the purpose of conducting and encouraging investigations into economic problems. A function of the COMMISSION is to issue from time to time papers and significant monographs of an econometric or economic-statistical nature without, however, assuming responsibility for theories or opinions expressed therein. The COMMISSION is affiliated with the ECONOMETRIC SOCIETY, an international society for the advancement of economic theory in its relation to statistics and mathematics.

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INTRODUCTION

The Annual Research Conferences on Economics and Statistics, held at Colorado Springs under the auspices of the Cowles Commission for Research in Economics, originated in a series of informal meetings during the summer of 1935 following the sessions of the Econometric Society at Colorado College on June 22-24 of that year. At these gatherings a number of papers were presented and discussed informally by prominent economists who remained in the vicinity. These meetings were so successful that it was decided to continue them in subsequent years.

The Third Annual Research Conference was held at Colorado College from June 28 to July 23, 1937. Two lectures, followed by discussion periods, were scheduled each morning, the sessions beginning at 9:30 and closing about 12:30. Saturdays and Sundays were left free for expeditions. In addition to the daily technical lectures, three public evening lectures of popular interest were presented by prominent speakers at the Conference.

The scientific program was supplemented by an interesting series of social events. The visiting guests were welcomed on June 28 with a picnic supper at Palmer Park. Receptions were held at the residences of President and Mrs. Thurston J. Davies of Colorado College, Dr. and Mrs. Charles H. Boissevain of the Colorado Foundation for Research in Tuberculosis, Headmaster and Mrs. Francis Mitchell Froelicher of the Fountain Valley School of Colorado, and President and Mrs. Alfred Cowles 3rd of the Cowles Commission. Members of the Cowles Commission staff were entertained at a dinner at the Stratton Park Inn by the out-of-town guests, and a dinner was also held at Cheyenne Lodge, on the 9,500-foot summit of Cheyenne Mountain overlooking the city. In addition, numerous informal teas, dinners, and hikes were arranged. Many of the guests during the week ends visited various points of interest in the region such as Pikes Peak, the Royal Gorge, Mesa Verde cliff dwellings, Cripple Creek gold camp, Taos Indian pueblos, Santa Fe, and Cheyenne Round-Up.

Over 60 out-of-town guests registered at the Conference and attended part or all of the sessions. A list of them is given below. In attendance also were various people from Colorado Springs including the staff of the Cowles Commission and members of the Colorado College faculty.

OUT-OF-TOWN PARTICIPANTS

BATEN, PROF. AND MRS. WILLIAM DOWELL, Michigan State College, East Lansing, Mich.

BEAN, MR. LOUIS H., Agricultural Adjustment Administration, U. S. Department of Agriculture, Washington, D.C.

BRAY, PROF. HUBERT E., Rice Institute, Houston, Texas.

BROWN, DR. DOUGLAS V., Harvard Medical School, Boston, Mass.

BROWN, PROF. AND MRS. T. H., Harvard Business School, Boston, Mass.

DODD, PROF. AND MRS. E. L., University of Texas, Austin, Texas.

DUBRUL, MISS L. JANE, 1220 Edwards St., Cincinnati, Ohio.

EVANS, PROF. AND MRS. GRIFFITH C., University of California, Berkeley, Cal.

FICHTNER, DEAN AND MRS. C. C., University of Arkansas, Fayetteville, Ark.

FORD, PROF. L. R., Rice Institute, Houston, Texas.

FRIEDRICH, DR. WILLIAM G., Jungmannov nam 8a, Prague X, Czechoslovakia.

FRISCH, PROF. RAGNAR, University of Norway, Oslo, Norway.

FRY, DR. THORNTON C., Bell Telephone Laboratories, Inc., 463 West St., New York City.

GROVE, MR. R. N., Peru, Ind.

HARTKEMEIER, PROF. AND MRS. HARRY PELLE, University of Missouri, Columbia, Mo.

HEDRICK, PROVOST E. R., University of California at Los Angeles, Los Angeles, Cal.

HENKE, MISS FLORA E., St. Louis, Mo.

HENNEY, PROF. AND MRS. HOMER J., Kansas State College, Manhattan, Kan.

HIRSCH, MR. M. T., 1933 Dryden Road, Houston, Texas.

HUNTINGTON, PROF. EDWARD V., Harvard University, Cambridge, Mass.

KARVAS, PROF. IMRICH, Comenius University, Bratislava, Czechoslovakia.

KELLY, DEAN AND MRS. EDWARD J., University of Santa Clara, Santa Clara, Cal.

KING, MR. HAROLD J., Lincoln, Neb.
LEDERER, DR. WALTER, Hunter College, New York City.
LEHMER, PROF. AND MRS. D. H., Lehigh University, Bethlehem, Penn.
LEVY, MISS SOPHIA H., University of California, Berkeley, Cal.
LONGSTREET, MR. AND MRS. R. W., Purina Mills, St. Louis, Mo.
MACGREGOR, PROF. DONALD, University of Toronto, Toronto, Ont., Canada.
MCINTYRE, MR. AND MRS. FRANCIS, Stanford University, Palo Alto, Cal.
MARSCHAK, DR. J., University of Oxford, Oxford, England.
MENGER, PROF. KARL, University of Notre Dame, Notre Dame, Ind.
MEYER, DR. HEINRICH, Rice Institute, Houston, Texas.
OBERLÉ, MR. E. J., Rice Institute, Houston, Texas.
PAYNE, PROF. CHARLES K., New York University, New York City.
ROGERS, PROF. JAMES HARVEY, Yale University, New Haven, Conn.
ROOS, DR. AND MRS. CHARLES F., Mercer-Allied Corporation, 420 Lexington Ave., New York City.
RUSSELL, MISS GLADYS, 360 North Michigan Ave., Chicago, Ill.
SIMMONS, MRS. HARVEY Y., Cedar Rapids, Iowa.
SNYDER, MR. CARL, 33 Liberty St., New York City.
SPENCE, PROF. AND MRS. B. J., Northwestern University, Evanston, Ill.
STINE, DR. O. C., United States Department of Agriculture, Washington, D.C.
TAFT, MISS REBEKAH L., Andover, Mass.
VAN WINKLE, PROF. AND MRS. E. H., Van Rensselaer Polytechnic Institute, Troy, N.Y.
WEAVER, DR. AND MRS. WARREN, Rockefeller Foundation, 49 West 49th Street, New York City.
WILKS, PROF. AND MRS. S. S., Princeton University, Princeton, N. J.
WILSON, MISS ELIZABETH W., 1 Waterhouse St., Cambridge, Mass.
WORKING, PROF. AND MRS. E. J., University of Illinois, Urbana, Ill.
WORKING, PROF. AND MRS. HOLBROOK, Stanford University, Palo Alto, Cal.
YNTEMA, PROF. THEODORE O., University of Chicago, Chicago, Ill.

PLANS FOR 1938 CONFERENCE

Arrangements are now being made for the Fourth Annual Research Conference, to be held at Colorado College from Tuesday, July 5, to Friday, July 29, 1938. Rooms and board for those attending the Conference will be available in dormitories of Colorado College at approximately $40 for the four weeks, or $11 per week. Approximate summer railroad rates to Colorado Springs and return are: from New York, $82; from Chicago, $41; from Los Angeles, $58; from Dallas, $30.

There is no charge for attendance at the Conference. Those who are interested should notify the Cowles Commission in order that they may receive in the Spring a preliminary announcement of the program.

ABSTRACTS OF CONFERENCE LECTURES

Monday, June 28 — The Econometric Problem, HAROLD T. DAVIS, Professor of Mathematics, Indiana University (now Professor of Mathematics, Northwestern University) and Research Associate, Cowles Commission.

In this paper the econometric problem is regarded as consisting of two parts, one static and the other dynamic. The model to be followed is that of mechanics, which in its most fundamental aspect consists of an exploration on the one hand of potential functions and on the other of a minimizing principle such as that of least action. An isomorphism between the physical concept of potential and the economic concept of utility has long been established. The maximization of the profit function is analogous in many respects to the principle of least action in dynamics and leads to a virtual dynamics of time series.

The famous table of mechanical analogies published by Irving Fisher in 1892 furnishes a powerful guide to exploration and generalization in economics.

A number of problems can be discussed from the point of view of the dynamics of economics. The need for longer and more comprehensive series should be emphasized. The disparity between seven well-known indexes of common stock prices between the dates of July 6, 1932 and January 31, 1934 is an example of one of the major difficulties in the study of economic time series. One index in this period changed from a base of 100 to a level of 276, while a second index changed from 100 to 205. The random element in the movement of economic time series should be more thoroughly explored and the indeterminacy rectangle (time by erratic move) requires further definition.

The lag correlation function

$$r(t) = \int_{-\infty}^{\infty} x(s)\ x(s+t)\ dt$$

is important in econometrics. Its efficacy in the study of random series, that is to say, for a function $x(s)$ whose lag correlation has the property $r(0)=1$, $r(t)=0$, $t\neq 0$, may be emphasized. For such random series the following formula has been computed:

$$r_n(t)=\int_{-\infty}^{\infty}\Delta^n x(s)\ \Delta^n x(s+t)\,ds$$

$$=\frac{\cos\pi t\,\Gamma^2(n+1)}{\Gamma(n+t+1)\,\Gamma(n-t+1)}.$$

It is also possible to show that if

$$S[x(t)]=\int_0^t x(s)\,ds,\qquad S^{(2)}[x(t)]=S\{S[x(t)]\},$$

$$S^{(n)}[x(t)]=S\{S^{(n-1)}[x(t)]\},$$

then $S^{(n)}[x(t)]$ tends toward a cosine function of period L, equal to the length of the series, that is,

$$S^{(n)}[x(t)]=\int_0^t\frac{(t-s)^{n-1}}{(n-1)\,!}\,x(s)\,ds\backsim A\cos\frac{2\pi}{L}\,(t+a).$$

The harmonic analysis of time series is an example of the application of functionals to the problem of economics, and the relationship between the periodogram of a series and the periodogram of its lag correlation function is important.

The stability of the linear trend, $y=a+bt$, may be discussed in terms of the variations Δa and Δb, when such a trend is fitted to 21 overlapping periods of rail stock prices from 1830 to 1930.[1] Following are the values of the distribution of the parameters:

Δa: $A=3.3201$, $\sigma=5.4362$, $(\mu_3/2\sigma)=.0364$, $\beta_2=1.8992$,

Δb: $A=.008199$, $\sigma=.1435$, $(\mu_3/2\sigma)=-.0184$, $\beta_2=2.4644$.

The small values of β_2, which for normal distributions should equal 3, indicate an excessive disturbance in the trends.

[1]See "Abstracts of Papers Presented at the Research Conference on Economics and Statistics Held by the Cowles Commission . . . 1936, *Colorado College Publication*, General Series No. 208, Colorado Springs, 1936, p. 101.

The example throws considerable light upon the question of why "normal" lines of one period are not the normal lines of a second and, perhaps, contiguous period.

The paper concluded with a discussion of the importance of the dynamics of competition and the ideas collectively included in what is called the "macrodynamic theory of business cycles."

Monday, June 28—Index Numbers and Regression Coefficients as Means, Internal and External, EDWARD L. DODD, Professor of Actuarial Mathematics, University of Texas.

To form an index number I, suppose that quantities $q_1, q_2, \cdots, q_n$, of n commodities are *purchased* in the basic year at prices $p_1, p_2, \cdots, p_n$; and that in some designated year quantities $q'_1, q'_2, \cdots, q'_n$ are *purchased* at prices $p'_1, p'_2, \cdots, p'_n$. Then p'_i/p_i is the price ratio for the ith commodity. An index I can be formed as a weighted arithmetic mean of these price ratios, with weights $p_i q_i$, respectively. As commonly used, the p's and q's are positive numbers, and thus the weights are positive. And in this case, I is an *internal* mean.

But negative weights have been used in mechanics. Also, in statistics and finance.[1] Suppose direct account is taken of one unit of some commodity *disposed of* in the basic year at price a and in the designated year at price b. For these units, $q = q' = -1$. And suppose that in the two years the quantities *purchased* — the *positive* q's — are the same, that is: $q'_i = q_i$. Also suppose that $p'_i = p_i$. Then for quantities *purchased* each price ratio equals 1 (unity); while for the unit *disposed of* the price ratio is b/a. But now, if $b/a < 1$, then $I > 1$; and vice versa. Then I is *not intermediate* between price ratios b/a and 1. I is here an *external* mean. The Irving Fisher ideal index number based upon the above data is likewise an external mean.

[1]C. E. Bonferroni, "Sulle medie dedotte da funzioni concave," *Giornale di Matematica Finanziaria,* Vol. 9, 1926 or 1927, pp. 13-24.

For a finite[2] number of elements $x_1, x_2, \cdots, x_n$, I take $f(x_1, x_2, \cdots, x_n)$ as a mean if and only if for every value c which the x's can take on simultaneously $f(c, c, \cdots, c) = c$, assuming indeed that at least one such c exists. With such a definition external means are indeed possible.

In a recent paper,[3] I presented regression coefficients as means of certain slope ratios. I wish now to indicate a joint graphical representation of certain regression coefficients, using here only internal means. With variables t, x, and y measured from arithmetic means as origins—thinking perhaps of t as the time—consider the individual slopes or *trend indicators* $x_i/t_i = t_i x_i/t_i^2$ and $y_i/t_i = t_i y_i/t_i^2$. With weights t_i^2, the regression coefficients $\sum t_i x_i/\sum t_i^2$ and $\sum t_i y_i/\sum t_i^2$ are weighted arithmetic means of these slopes. Suppose, now, that x_i, y_i, and t_i are *homogeneous* co-ordinates of a point in a plane, with the usual co-ordinates $X_i = x_i/t_i$, $Y_i = y_i/t_i$. And suppose that, with weights t_i^2, $(\overline{X}, \overline{Y})$ is the *center of gravity* of the points (X_i, Y_i). Then $\overline{X}$ and $\overline{Y}$ are the two afore-mentioned regression coefficients respectively. Every convex polygon containing each (X_i, Y_i) will contain also $(\overline{X}, \overline{Y})$.

Moreover, on this same plot can be represented the coefficients of regression of y on x, and of x on y. For this purpose, let a radius vector be drawn from the origin through the point (X_i, Y_i) to some vertical line at the right margin of the plot. With a scale in which the horizontal distance of this vertical line from the origin is taken as unity, the ordinate of the point of intersection of the vector through (X_i, Y_i) is $Y_i/X_i = y_i/x_i = x_i y_i/x_i^2$. With weights x_i^2, the ordinate of the center of gravity of this system of weighted points is the regression coefficient $\sum x_i y_i/\sum x_i^2$. Similarly, on a horizontal line, we can locate $\sum y_i x_i/\sum y_i^2$.

[2] A more general definition, I gave a year ago: see *Colorado College Publication*, No. 208, p. 92.

[3] "Regression Coefficients as Means of Certain Ratios," *Amer. Math. Monthly*, Vol. 44, 1937, pp. 306-308.

Tuesday, June 29—Necessary and Sufficient Conditions for Means Arising from the Location and Scaling of Frequency Functions, PROFESSOR DODD.

The location and scaling of frequency functions was treated by R. A. Fisher[1] from the standpoint of maximum likelihood. With special consideration of the kind of means formed in the location and scaling process, I have presented two papers[2] to the American Mathematical Society, exhibitting *internal* means with *minimum* likelihood and *external* means with *maximum* likelihood.

In these papers, a rather free use was made of derivatives of functions to determine maximum or minimum likelihood. In the discussion of means now to be presented, derivatives are avoided.

But, as before, a function $f(x_1, x_2, \cdots, x_n)$ is regarded as a mean of $x_1, x_2, \cdots, x_n$, if at least one value of $f(c, c, \cdots, c) = c$.

The normal frequency function appears often in some such form as

$$\frac{1}{a\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{a}\right)^2}$$

with m as location, and a as scale—this a appearing in two places. For typographical simplicity, let us write $\text{Exp}(y) = e^y$. Then for a general frequency function, we may take $a > 0$, $t = x - m$, $\xi = t/a = (x-m)/a$, and write

$$f(x, m, a) = \text{Exp}[\varphi(\xi) - \log a],$$

where φ is largely arbitrary. As a particular case, we may note that if

$$\varphi(\xi) = -\xi^2/2 - \log\sqrt{2\pi},$$

there results the normal probability function just mentioned.

[1]*Philosophical Transactions*, Vol. 222, 1921-22, pp. 309-368.

[2]For abstracts, see *Bulletin of A. M. S.*, Vol. 43, 1937, p. 23-24; p. 42-43. The latter paper has appeared in *Annals of Mathematical Statistics*, Vol. 8, 1937, pp. 12-20.

DEFINITION. Let $\xi_i = (x_i - m)/a$, for $i = 1, 2, \cdots, n$. Assume that $\varphi(\xi)$ exists at least in some interval about $\xi = 0$. Then the likelihood L of m for given $x_1, x_2, \cdots, x_n$, and a, for which φ exists is here defined as the product of the n factors, $f(x_i, m, a)$:

$$L = F(m) = \Pi\, f(x_i, m, a), \quad i = 1, 2, \cdots, n.$$

THEOREM I. Suppose that $\varphi(0) > \varphi(\xi)$ if $\xi \neq 0$. Then for certain $x_1, x_2, \cdots, x_n, a$, there exists a function $\mu(x_1, x_2, \cdots, x_n, a)$ which taken as "location" m leads to the largest likelihood. That is:

$$F(\mu) > F(m) \quad \text{if} \quad m \neq \mu.$$

And this $\mu(x_1, x_2, \cdots, x_n, a)$, whenever it is defined, is a mean of $x_1, x_2, \cdots, x_n$.

THEOREM II. Suppose (1) that $\varphi(0)$ is not greater than every other $\varphi(\xi)$. And suppose (2) that with $x_1, x_2, \cdots, x_n, a$ for which $\varphi(\xi)$ is defined, there exists a single-valued function $M(x_1, x_2, \cdots, x_n, a)$ which taken as location leads to the largest likelihood; that is: $F(M) > F(m)$ if $m \neq M$. Then $M(x_1, x_2, \cdots, x_n, a)$ is not a mean of $x_1, x_2, \cdots, x_n$.

DEFINITION. With location m fixed in some way, take as a frequency function

$$f(x, m, a) = F(t, a) = \text{Exp}[\varphi(t/a) - \log a].$$

Assume that $\varphi(\xi)$ exists in some interval about $\xi = 1$; also about $\xi = -1$ if negative values of t are possible. With given $t_1, t_2, \cdots, t_n$ for which $\varphi(\xi)$ exists, the likelihood L of a will now be defined as the product of n factors:

$$L = \Phi(a) = \Pi\, F(t_i, a), \qquad i = 1, 2, \cdots, n.$$

THEOREM III. Suppose that, if $|t| \neq a$,

$$\varphi(t/|t|) - \log|t| > \varphi(t/a) - \log a.$$

Then, for certain values $t_1, t_2, \cdots, t_n$, there exists a function $a(t_1, t_2, \cdots, t_n)$ such that when a is taken as $|a|$, the likelihood L takes its largest value; that is:

$$\Phi(|a|) > \Phi(a), \qquad \text{if} \qquad a \neq |a|.$$

And $a(t_1, t_2, \cdots, t_n)$, when thus defined, is a mean of $t_1, t_2, \cdots, t_n$.

THEOREM III thus sets forth a *sufficient* condition for a mean from scaling. A certain converse brings out the *necessity* of this condition.

Tuesday, June 29—The Relation of Stock Prices to Earnings and Other Factors, CHARLES F. ROOS, Director of Research, Mercer-Allied Corporation.

For many years investors and speculators have been trying to price common stocks on the basis of earnings and dividends and other basic factors, and yet the fact is that stock prices are only in a rough way determined by earnings and dividends. Highest correlation is obtained by comparing earnings for the period for which prices are averaged and dividends for the next period ahead. For a quarterly reporting company, prices for the nth quarter correlate best with earnings for the nth quarter and dividends for the $(n-1)$th quarter. Nevertheless, prices during each period — a quarter, six months, a year, or two or more years, depending upon the company and the industry — fluctuate widely about the regression lines. In fact, the average of the maximum quarterly deviations for 100 stocks is about 20 per cent of the regression value of the stock.

Effects of monetary changes and other major political developments depend upon the position of stock prices in their characteristic ranges about the earnings-dividends regression values. Stocks in high parts of the ranges drop quickly when deflationary developments are on the horizon, whereas those prices that happen to be in the bottoms of their characteristic ranges are little affected. Similarly mildly inflationary situations will, in general, start upward prices of stocks which are in the bottoms of their characteristic ranges and only slightly, if at all, affect prices which are at the tops.

Stock prices tend to discount earnings as far ahead as can be seen even dimly. For instance, while the price of gold is increasing—the actual increase, not the nominal recognition of a change that has already taken place—prices of the stocks of large gold-mining corporations will discount earnings two or three years ahead. For some railroads, especially those serving agricultural areas, earnings are discounted with a fair degree of accuracy for a year ahead.

Forward discounting of earnings is shown also by the fact that average price-earnings ratios of vigorous companies are high relative to ratios of less aggressive concerns. When an upward trend for such companies is broken, the reaction which occurs is sharper than for stable companies which suffer losses in earnings. Thus, G. C. Evans' equation $y = ap + b + hdp/dt$ in the form $P = A_0\pi + B_0 d\pi/dt + C_0 D + D_0$, where P = price of stock, π = expected earnings, $d\pi/dt$ = derivative of earnings with respect to time, D = expected dividends, and A_0, B_0, C_0, and D_0 are constants, is remarkably well satisfied.

Writing the quantity D_0, which defines the fluctuation about the expected price, $A_0 + B_0 d\pi/dt + C_0 D$, as a constant, means that the percentage fluctuation in the price of a stock decreases as the price increases. Thus it is stated that a high-price stock selling at, say, $200 per share which fluctuates between $190 and $210 in a broad market fluctuates less percentagewise than does a stock which sells at $40 ± $10. This formulation is, however, in very close agreement with observation, but there are important exceptions.

Earnings as reported by companies can not always be used to obtain regression prices of stocks; corrections which make the accounting practices of the company correspond with accepted practices often must be made. In some instances such corrections transform what appear to be random scatter diagrams of price and earnings — no dividends paid — into well-defined correlation scatters with accurately defined variation ranges.

When the earnings are zero or negative, prices of the stock fluctuate characteristically around a value determined by the working capital of the company. Swings about such values appear to be associated with speculation and public participation in the market. When earnings appear, working capital seems to be relatively unimportant. It seems, in such instances, the market presumes that the management will be able to provide whatever working capital is needed.

In brief, earnings and dividends only vaguely determine security prices. During each quarter of the year prices of securities fluctuate widely about average prices based on earnings and dividends but, nevertheless, tend to remain within definite boundaries. Strike news, political developments, floods, and the like determine the timing and extent of deviations from correlation values.

Wednesday, June 30—What Determines Prices? CARL SNYDER, for many years with the Federal Reserve Bank of New York.

About the earliest authentic price records which we possess date from the time of Hammurabi, the great king of Babylonia, c. 2200 B.C. In later times for Greece, Egypt, and Rome, we have a considerable variety, sufficient to make clear that over a very long period of time, two or three thousand years or even more, there is little evidence of any long-sustained change in the general level of prices, at least as measured in some kind of common denominator like silver, which through all this period and up to the nineteenth century was the main medium of exchange. There were wide oscillations, as after the plunder of the East by Alexander, and later, as ingeniously portrayed by Heichelheim. But no clear secular trend.

That is up to the sixteenth century A.D. when the discovery of vast stores of silver and later of gold in the New World brought about the greatest sustained rise of prices of which we have any record. Then, from the end of the seventeenth century to date, again a prolonged level, with wide oscillations, as in the Napoleonic Wars, the World War, etc.; but without any definite secular trend.

Following this great "Price Revolution" came, a little later, what is known as the "Industrial Revolution" of the eighteenth and nineteenth centuries, resulting in a huge increase in production, both total and per capita, extending up to our own day. Therewith a problem; wheat is a striking example. It is fairly evident that the price of wheat in the days of Queen Anne could not have a great deal to do with the *production* of wheat in 1937; or vice versa. Yet though the production of wheat in this period has risen perhaps by fifteen or twenty times and now extends over both hemispheres, there has been no striking secular change (save in war times) in the long-time price of the commodity.

From a variety of similar evidence, it seems clear then that the main variations (which have been wide) in wheat prices, and other prices, have little or no consistent relation with the volume of the product, nor, for the most part, with their short-time variations. Thus, for example, the price of wheat, and to about the same degree the average prices of all basic commodities, in the United States and also in most other countries, fell pretty steadily from the close of our Civil War down to 1896. Therewith an abrupt change, not merely in the price of wheat but also in most other commodities, and in practically the same degree, rising steadily up to 1914 and then vertiginously in the World War, then falling with the accustomed crash thereafter. In all this long period there was very little variation in the steady growth of production, of all kinds, and remarkably slight variations at least in the total of world basic production.

Very clear then that the prices of commodities, and of everything else bought and sold, are simply *relative* prices, that is, relative to the general level of all kinds of prices, which tend, so far as our records extend, to rise and fall very much together. Therewith it seems clear that the attempt to derive the price of any particular set of commodities, as farm prices and the like, from the product or from variations of the product, inevitably fail. There are fairly definite relations of the annual or short-time variations of certain products, with the volume produced, but even these relations differ rather widely from one type of commodity to another, so that the conditions of plenty or scarcity, for example, in farm prices act in an almost inverse manner as upon such commodities as iron or copper and the like.

All this we may sum up by saying that the actual or absolute prices of commodities, and in general of all prices, are in the main determined by factors outside of the commodities themselves. Notable exceptions at times, of course. In other words all prices are *relative* prices; and this relation it is now clear is to the average purchasing power of the monetary unit, which in turn is related to the quantity of the metals in circulation as, for example, in the sixteenth and seventeenth centuries; and in our own time to the volume of credit in use, built upon this monetary base. Extended researches seem to have confirmed this conclusion to the hilt.

It has further been shown that, curiously enough, the variations or oscillations of prices in the United States and Great Britain are not, as widely supposed, closely related to the state of trade. Thus, for example, we know now that we may have a severe depression, as in 1907, with little change in the general price level, and in the volume of bank credit in use; and vice versa. It has been found that the velocity or rate of turnover of this bank credit is closely related to the variation in the volume of trade, relative to its long-time trend. In the famous "Equation of Exchange" of Simon Newcomb, the factors of V and the short-time variations of Trade thus tend mutually to cancel, so that the equation reduces to some form where $M \times$ a constant k always equals P. That is, the general price level is always in "equilibrium" with the volume of credit, the factor of k representing the rate of increase in the long-time volume of trade.

Thus it would appear that the problem of what determines price reduces practically to the thesis of Jean Bodin, set forth in 1568. The statistical proofs for this result have been given in a series of articles by the present writer.[1]

[1]See summary in "The Problem of Monetary and Economic Stability," *Quarterly Journal of Economics*, Feb. 1935, p. 173.

Wednesday, June 30, Thursday, July 1—The Measurement of Markets, T. H. BROWN, Professor of Business Statistics, Harvard Business School.

The purpose of these lectures is to present an outline of certain problems of business to which technical statistical methods may be applied. A logical development requires that the application in each case shall originate with actual business situations rather than with hypotheses derived from economic theory. The acid test of the results which may be obtained is the value of the findings to the business world.

The measurement of markets in business arises as a part of the broader group of problems known as the making of plans. The full range of company controls includes not only the making of plans, but also the continuing problems of operating in relation to those plans or to standards. Although the measurement of markets involves many qualitative facts, the term implies that numerical facts ultimately must be interpreted through the use of statistical analyses.

The approach to the problem of the measurement of markets may be indicated by three questions: (1) What does the market demand? (2) How much will the market absorb? (3) What price can be obtained for the finished goods? The first of these commonly is termed market analysis; the second, the determination of potential markets; and the third, price-setting policy.

The answers to these questions may be obtained through the application of mathematics in the form of statistical analysis, but this must be deferred until certain groundwork has been laid. This general statement does not imply that methods which have been found satisfactory in another group of problems necessarily can be applied in this case. It may happen that procedures which are suitable in the natural sciences, in engineering, in the social sciences, or even in theoretical economics may not apply here. In fact, the fundamental characters of various kinds of business show distinct differences.

Another thing which must be noted is the manner in which the numerical information may arise. In the first place, data may originate as a result of the handiwork of someone who has a penchant for collecting figures. In such a case the collector, or another person, may study the accumulated information with the hope of extracting from it some truths which will be interesting or possibly useful. In the second place, data may be gathered because of their bearing upon certain definitely recognized problems. The gathering of data pertaining to the stock market, employment, or other similar questions illustrates this type of procedure. Finally, numerical information may be accumulated as a direct result of an intent to discover definitely recognized truths relative to definitely stated problems. In this connection we may use R. A. Fisher's term "Design of Experiment."

To these foregoing fundamentals there must be added a definition as well as a statement of two general principles. The definition is that of a statistical summary. This analogy to the definition of an accounting statement is: "A statistical summary is a statistical statement made for a given purpose and complete for that purpose."[1] Each of the two parts of the definition is important. The first indicates that statistical work in business must originate from definite business problems. The second part indicates that the statistical work must be carried to a point such that the executive will understand directly the simplified facts which exist in a given situation. The principles involved are those of standards and of controls. The application of statistical procedures to definite problems inevitably involves a comparison of the results with a statistically determined standard. Without such a comparison judgments are hopeless. The second principle is that of operating with the objective of obtaining control over a given situation. Shewhart, in the conference last year, presented many fundamental questions which are involved. The principles which he set up, however, are of wider application than in the limited field of manufacture.

With the essential background for studying business problems in mind, it is now reasonable to attack the first of the three questions, "What does the market want?"

[1]T. H. Brown, *Problems in Business Statistics*, New York, 1931, page 15.

The problem is that of polling wishes. This may be done either by census or by means of a sample. The *Literary Digest* poll of the 1936 presidential election is a classic example of how a sample may give an erroneous result. On the other hand, the very serious expense involved requires that a corporation make use of the sampling method rather than attempting to take a census. The usual procedure in approaching such problems is: first, to design a questionnaire, which must be checked repeatedly in order to eliminate ambiguities or leading questions; second, to gather the necessary figures through the help of a corps of investigators who will make random calls throughout a market; and third, to tabulate and interpret the results.

It is now appropriate to proceed with the techniques involved. The first question is that of recognizing the difference between the statistics of attributes and the statistics of variables, as defined by Yule in his *Theory of Statistics*. Thus, it is possible in market investigations to count the number of people who own a radio, an automobile, or any other attribute. On the other hand, we may determine the ages of typewriters actually in use in various offices in a business community. These ages are classified as variables.

From the sample count arises the problem of the error in a percentage, and the problem of distinguishing between two percentages in such a way that we may be able to say definitely that the sample shows a significant difference between two cases. A similar problem in the case of variables arises in the determination of the error of a mean value. This question of sampling errors has a significance in cost estimates when the inverse form of the problem is stated: How large should the sample be in order that the error shall not be greater than some given amount? An answer to this question enables the executive to obtain an estimate of the expense involved.

The problem of determining how many units of a given commodity a market will absorb commonly is known as that of determining the potential market. This term may be used in different senses, referring to total possible sales regardless of time, to possible sales in a given year, or to a controller's estimate in connection with financial planning.

The determination of the ultimate potentials in several markets may be accomplished through the use of an index number reflecting the willingness to buy and the ability to buy of the people in one market relative to those in another. Such an index number of per capita buying power is multiplied by the relative number of buying units in the several markets. For this purpose the market data published by the government are of great use.

In the case of one product in a single market area the method of attack is the more direct one of estimating the number of units which can be sold as judged from the sale of other similar or associated products. In some cases the method of multiple correlation has been used to secure relative ratings for the various series involved; the potentials thus obtained actually are relative sales potentials rather than absolute potentials.

The problem of determining the price at which a product should be sold has been only partially solved. For a given product in a competitive market it is certain that the economic relationship of price, supply, and demand is not usable. The reason is that presumably the manufacturer can determine the number of units which he will supply and at the same time can set his price within wide limits. The price, especially in the case of monoply products or of products which have certain inherent special qualities giving them a quasi-monopoly position, is not affected as in the case of other classes of goods. At the present moment one approach to this fundamental problem is that of reasoning from the prices at which other similar products are actually sold.

In discussing these problems of business perhaps the most important factor is that of determining the approach which is to be made. It is certain that, unless the fundamental considerations involved are carefully observed, the results are likely to be grossly at variance with observed facts. The application of statistical methods must be made with the expectation of obtaining the fundamental truths which are involved in each case. These are obtained only if the data have been condensed to a form which reduces the volume of original data to a minimum and eliminates the extraneous elements present. Business problems as a whole present a group of questions which apparently require a characteristic set of methods which will be useful in their solution. It is expected that when these have been more fully discovered and their application taught, business will be able to operate more effectively and more efficiently.

Thursday, July 1—The Dynamics of the Security Market, DR. ROOS.

A satisfactory theory of security price behavior must take into consideration the facts which the author discussed in his first lecture and certain others. First, there is the evidence uncovered by Harold T. Davis, Ragnar Frisch, and others which indicate nearly-periodic movements of 3½ years and 5 years in stock prices. Also in long series is a cycle of 16-20 years. Whether each is a separate and distinct cycle or whether two combine to give the harmonic which equals the third is a matter for debate, but, nevertheless, we are confronted with these cycles.

Of fundamental importance is the fact that earnings for all companies do not necessarily, or even usually, move in the same direction nor change by the same percentage. Thus, the general directions in which individual stock prices move, as determined by correlations of prices with earnings and dividends, are subject to frequent change. Moreover, contrary to popular belief, not all stocks are bargains when the stock indexes are low, nor are they overpriced when the indexes are high. The facts are that successive lows in the market averages are caused by an increasing proportion of stocks moving towards the lower limits of their ranges; and new highs in the averages are made because increasing proportions of the total stock list are moving towards the upper boundaries of their ranges.

When a sudden monetary change occurs such as the inflationary change of 1933 nearly all stock prices touch their upper boundaries. Moves in such instances seem very large, because prior to the move a series of deflationary occurences has, as a rule, caused nearly all stocks to reach the lower boundaries of their characteristic ranges. Thus, the averages move from a minimum for the earnings-dividend conditions existing to a maximum. Factors which can cause these sympathetic moves between upper and lower price bounds are (1) important change in the price of gold by the United States or by a major country such as England, (2) change in fundamental banking processes such as a raising of the rediscount rate, (3) general strikes, (4) serious consideration of major social-reform legislation. Minor news items usually affect directly only a few leading stocks and have only very short-lived influences.

Earnings and dividends which influence major moves in stock prices are highly correlated with clearings of banks outside New York; that is, they depend upon the volume of bank credit times its velocity of turnover. Consequently, stock prices are fundamentally determined by the same factors that make business good or bad. Also there is some evidence that stock-price changes themselves influence business confidence; hence the velocity of bank deposits and in turn stock prices. Causation is thus in both directions.

A mathematical theory useful in organizing the facts presented here and in the first lecture can be formulated as follows:

$$(1) \qquad P(t) = A_0\pi(t, \alpha) + B_0\, d\pi/dt + C_0 D(t, \beta) + F(t) \qquad \text{for } \pi > 0, D \geq 0$$
$$= F(t) \qquad \text{for } \pi \leq 0$$

where $P(t)$ is the price of an individual stock; $\pi(t, \alpha)$ is the value of earnings for the period of time $(\alpha—t)$ for which the stock price customarily discounts earnings; and $D(t, \beta)$ is the value of dividends for the period $(\beta—t)$ in the future.

The function $F(t)$ accounts for the residual variation and, therefore, must be of the form

$$(2) \qquad F(t) = f[V, dP/dt, t] + (A_1 + B_1 W) W_0,$$

where W_0 is the average of the working capital $W(t)$ over a preceding period of time, the length of which depends upon the particular company and industry to which (2) applies; $f[V, dP/dt, t]$ is a functional of volume of transactions V, corresponding to particular price changes dP/dt and news items represented implicitly by t. The quantity F is a functional at least to the extent that it represents the summation of random causes as they influence price and volume to change cyclically.

The mathematics of such cyclical movements induced by random causes has been studied by E. Slutsky, R. Frisch, H. T. Davis, Herbert E. Jones, and others. In particular, Mr. Jones has derived the formula

$$r_n(t) = \int_{-\infty}^{\infty} \Delta^n x(s)\, \Delta^n x(s+t)\, ds$$

$$= \cos \pi t\, \Gamma^2(n+1) - [\Gamma(n+t+1)\; \Gamma(n-t+1)],$$

where $x(s)$ is a random time series. Also, as Professor Davis has shown,

$$S^n[x(t)] = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} x(s)\, ds \backsim A \cos \frac{2\pi}{L}(t+a),$$

where L is the length of the series and a is a phase constant. Some such cumulation of random events conceivably leads to the short-time cyclical movements in the functional

$$f[V_1, dP/dt, t].$$

The longer cycles of 3½ years and 5 years previously mentioned are associated with changes in earnings or dividends or both. The author, in a paper entitled "A Mathematical Theory of Price and Production Fluctuations and Economic Crises," presented to the American Mathematical Society in 1928 and published in the *Journal of Political Economy* for October, 1930, first considered this problem. For our present purposes it is merely necessary to introduce the theory developed there and to relate it to our quantity $\pi(t, a)$ of formula (1).

On page 504 of the *Journal* the author formulated earnings in two ways:

(1) $$\pi(t, a) = \int_t^a [\gamma pu - Q(u, u', p, p', t)]\, E(t_1, t)\, dt,$$

where $E(t_1, t) = \text{exponential}\left(-\int_{t_1}^t \delta(r)\, dr\right)$, where $\delta(r)$ is the force of interest $= dS/S$ for an invested sum S.

(2) $$\pi(t, a) = \int_t^a [\gamma pu - Q(u(t-\mu))]\, E(t_1, t)\, dt,$$

where μ is a lag factor.

The derivatives u' and p' can be regarded as measures of labor costs in the future and of expansion requirements. Thus, they are used to express the well-known fact that when the production is increasing and prices are increasing, labor is demanding higher wages and shorter hours. In the same situation new capital equipment will be purchased at increasingly higher prices. Essentially the same picture is obtained by using either (1) or (2) to represent these phenomena; but the mathematics for (1) is simpler and therefore the author has chosen it.

The coefficient γ is a factor, sometimes psychological (see *Dynamic Economics*, pp. 171-173) which may relate either to expected demand, i.e., γu, or to expected price, γp. It is, in other words, a coefficient of entrepreneurial expectations based upon past factors.

Our entrepreneurs will be operating subject to certain demand functions which in some cases will be of the type proposed by Evans:

(3) $$u(t) = ap(t) + b + h\,dp/dt;$$

and in other cases, particularly when capital goods are in-involved, the author's functional demand:

(4) $$u(t) = ap(t) + b + \int_0^t k(s, t)\, p(s)\, ds.$$

An entrepreneur choosing production $u(t)$ or price $p(t)$ satisfying either (3) or (4) and maximizing (1), say, will find his solution among the solutions of the fourth-order differential equation:

(5) $$L_4 d^4p/dt^4 + L_3 d^3p/dt^3 + L_2 d^2p/dt^2 + L_1 dp/dt + L_0 p - p_0 = 0,$$

where the definitions of the constants $L_4, \cdots L_0$ and p_0 are given on page 508 of the author's article on crises. As is well known to mathematicians, the solutions of this equation depend upon the character of the roots of the corresponding fourth-degree algebraic equation, and these may be all real or imaginary or a combination of two real and two imaginary. The real roots may be either positive or negative. Consequently, as the author has demonstrated before, a variety of double- and single-period solutions are possible. Where only a single period is indicated, the differential equation degenerates into an equation of the second order. But it should be recalled that definite evidence exists for at least two periods and hence a fourth-order differential equation is to be expected.

Whether a single- or multiple-period solution or combination with exponentials is obtained depends upon the interest rate, which we have represented by the force of interest $\delta(r)$. Clearly, if p and u fluctuate as is indicated here, profits will also fluctuate, unless by some miracle wages and other costs of production fluctuate with the same period and are in phase. New inventions, redistributions of income, wage changes, and a variety of other impacts will be sufficient to change the character of the solutions. Thus, one should expect the price or production extremals to be fixed only for relatively short periods of time, say, a few months or a year, depending upon the nature of the industry. Moreover, because of frequent impacts the series representing them will have some characteristics of random series.

In considering this phase of forecasting, the author in an article entitled "Some Problems of Business Forecasting," published in the *Proceedings of the National Academy of Sciences* for 1929, considered the influence of special factors on the character of the extremals and proposed that normal times of stability are characterized by the averaging out of peaks and troughs of price and production in different industries. Crises, on the other hand, arise when major peaks and troughs coincide and by their amplifying or depressing effect influence bank deposits or other monetary media. As a matter of fact, deposits and velocity, both of which depend obviously upon $\Sigma p_i u_i$, do show characteristic cycles of 3½ and 5 years which are common to stock prices. The longer cycles of 16-20 years may be beats arising from a combination of 3½- and 5-year cycles or may be caused by a building cycle of 16-20 years. How such a building cycle is self-generating and perpetuating has been shown by the author in his book, *Dynamic Economics.*

Finally, both $\Sigma p_i u_i$ and building activity have important direct influence on deposits and their velocity. The former is associated with commercial loans and the latter with loans on real estate. Commercial loans may be represented by a formula of the type:

$$L = g_0 \frac{B(t)}{W(t)} + \int_{t-t_0}^{t} (B - h_0 w - k_0 M)\, \psi_3(x, t)\, dx,$$

where $B(t)$ = volume of business = $\Sigma p_i u_i$, W = dollar value of working capital, w = wages including salaries, M = cost of materials, $\psi(x, t)$ is a weight function, and g_0, h_0, and k_0 are constants.

Also of importance of late in determining the volume of bank deposits have been holdings of government securities. Brokers' loans and loans on securities have been negligible but could easily assume importance. Formulas interrelating prices and production of goods and services and prices of capital goods or securities and volume of transactions with deposits and other monetary factors have been presented so as to exhibit cyclical characteristics in the last chapter of the author's *Dynamic Economics.*

Friday, July 2—Drawings from Rectangular Distributions, DICKSON H. LEAVENS, Research Associate, Cowles Commission.

Data of numbers drawn in the Chinese National State Lottery suggested an investigation to determine whether the drawings had been fairly conducted. The numbers had been obtained by drawing from numbered wooden balls mixed up in large brass drums. The data were tested by comparing the mean, variance, and standard deviation of the frequency distribution of the units, tens, $\cdots$ digits of the numbers in the sample with the expected values for the rectangular distribution from which they were drawn; and by applying the chi-square test of goodness of fit. The results indicated that on the whole the shuffling was adequate, but that in some of the drawings it might have been desirable to use more care in pouring the balls into the drums so that all those of each thousand would not go in together.

The use of wooden balls, with suitable equipment and routine, therefore seems an excellent method for drawings. For comparatively small universes, a ball can be provided for each number, and drawings made without replacements. For larger universes, two or more sets of balls can be used: for example, 6 sets of 10 balls each from which to draw the successive digits of a 6-digit number. Drawings with replacements here are equivalent to drawings with replacement from a series of 1,000,000 numbers.

Similar tests were applied to numerous series of American bond numbers drawn for redemption, but were inconclusive because in most cases previous drawings had been made, and there was no assurance that the advertised numbers were actually drawn from a rectangular distribution. Satisfactory tests could be made only by using the full data available in trust-company files.

Since the chance of repetitions in drawings with replacements is considerable when a large proportion of the total numbers is to be drawn, this method is unsuitable for bond drawings. For this purpose drawings without replacements are indicated. Various methods are used in practice: (1) paper slips, as used in some lotteries, are undesirable, because of the difficulty of proper mixing; (2) cardboard, metal, or celluloid slips are better, but require care in mixing; (3) punched tabulating cards facilitate recording and checking, but an automatic mechanical mixing device for them is not yet available.

Friday, July 2 — The Theory of Runs as Applied to Time Series, HERBERT E. JONES, Research Associate, Cowles Commission.

The theory of runs in its application to time series is divided into two parts: (1) the study of the ratio of sequences to reversals, and (2) the study of the distribution of runs. The analysis is confined to the investigation of the *direction* of changes in time series, that is, the signs of the first differences, which form a series of plus signs, minus signs, and zeros. A *run* is defined as a sequence of like signs, the length of the run being the number of like signs in a group. A *reversal* occurs when a plus sign follows a minus sign or vice versa. A *sequence* occurs when a plus sign follows a plus sign or a minus sign a minus sign. In this study a zero is considered the same as the preceding sign, but may be treated by more general formulae if necessary.

Economic time series often have been compared with random series, cumulated random series, or different forms of periodic series. It is of interest, therefore, to determine the expected number of reversals and sequences and the distribution of runs for these types of time series.

With the following notation:

n = number of observations,
n' = number of first differences = $n-1$,
m = number of class intervals in the frequency distribution of values,
f_i = frequency of values in class interval i,
f_+ = frequency of plus signs,
f_- = frequency of minus signs,
f_0 = frequency of zeros,
$E(R)$ = expected number of reversals,
$E(S)$ = expected number of sequences,
ϱ = ratio of sequences to reversals,
${}_jP_k$ = permutation of j things k at a time,

we can derive the following formulae.

For a *random* time series, with a frequency distribution defined by f_i, $i = 1, 2, 3, \cdots, m$, where f_1 is the frequency of the smallest value, f_m the frequency of the largest value, we have,

$$(1) \qquad E(R) = 2\sum_{i=1}^{m-1} \sum_{q=1}^{f_i} {}_{f_i}P_q(n-\varphi_i)(n-\varphi_i-1)/{}_nP_{q+1},$$

$$\text{where} \quad \varphi_i = f_1 + f_2 + \cdots f_i.$$

For a rectangular distribution with constant frequency f, Equation (1) reduces to

$$(2) \qquad E(R) = (2n-f-3)/3,$$

and for a normal distribution, to

$$(3) \qquad E(R) = 2(n-2)/3.$$

Equation 2 reduces to Equation (3) if $f = 1$, and is approximately true for all distributions if $m > 10$.

For a *cumulated random* series, that is, a series whose first differences are random, we have,

$$(4) \qquad E(R) = \frac{2f_+ f_-}{n'} \left[\frac{(n'-f_0)(n'-1+f_0) + f_0(f_0-1)}{(n'-1)(n'-f_0)} \right].$$

If $f_+ = f_-$, and $f_0 = 0$, we have,

$$(5) \qquad E(R) = n'/2.$$

The above formula, however, applies only when the number of plus signs and number of minus signs in the series is given.

If, on the other hand, we consider all possible permutations of plus and minus signs, and furthermore, if the probability of obtaining a plus is equal to the probability of obtaining a minus, we have,

$$(6) \qquad E(R) = (n'-1)/2.$$

If the series is some *periodic function,* defined by

$$(7) \qquad y = f(t),$$

the expected number of reversals equals the number of real roots of the difference equations

$$(8) \qquad \Delta y = 0$$

between t_0 and t_n.

From the relation

$$(9) \qquad E(R) + E(S) = n - 2$$

we can quickly determine the number of sequences.

The standard errors of $E(R)$ are,

$$(10) \qquad \sigma_R = (2n - 4)^{\frac{1}{2}}/3, \qquad \text{Random series,}$$

$$\sigma_R = (n - 2)^{\frac{1}{2}}/2, \qquad \text{Cumulated Random series.}$$

Since the above formulae depend on n it is more convenient to have a criterion independent of the number of observations. We use, therefore, the ratio of sequences to reversals as a criterion which measures the randomness of the series. We then have,

$$(11) \qquad \varrho = \frac{E(S)}{E(R)} = \tfrac{1}{2}; \qquad \text{Random Series,}$$

$$(12) \qquad \varrho = 1, \qquad \text{Cumulated Random Series.}$$

This single ratio is not sufficient, however, since in a sine curve the ratio may vary from infinity to zero, depending on the interval between observations. It can be shown that for random series and cumulated random series the ratio is constant; that is, independent of the interval between observations.

If the ratios are significantly different from the expected values we are led to believe that there is some type of structure within the series. On the other hand, if the ratios vary with different intervals between observations it is of importance to determine whether the variation is a chance phenomenon or due to periodicity. We can determine this by investigating the distribution of the runs. If we let:

n = number of observations,
v = length of runs,
p = probability of obtaining a plus sign,
q = probability of obtaining a minus sign,

we have

$$(13) \qquad E(v) = \frac{2(v^2+3v+1)(n-v-2)}{(v+4)!}, \qquad \text{Random Series,}$$

(14) $E(v) = (n-1)p^2q^2r_{v-2}$, Cumulated Random Series,

where $r_k = p^k + q^k$, and $v < (n-2)$.

For cumulated random series, Bortkiewicz[1] gives the formulae for the standard errors of the expected number of runs.

For a pure sine curve the distribution will have a single class interval with a length equal to one-half the period. If a time series has a relatively stable period the distribution will tend towards a bell-shaped distribution.

One must be cautioned against the use of the χ^2 test for testing the significance of these distributions since the *total* frequency of the observed and expected number of runs is not necessarily the same.

Monday, July 5 — Some Unsettled Questions in Commodity Price Theory, HOLBROOK WORKING, Economist, Food Research Institute, Stanford University.

Statistical and economic analyses of price series are concerned largely with problems of economic fluctuations — problems of dynamics rather than of statics. The conclusions bear obviously on questions of the distribution of wealth and income. They bear also on questions of production, both because production is affected by economic fluctuations, and because they illuminate problems of marketing, which is a part of production. The contributions of commodity price analysis to economic theory must not be viewed wholly from the standpoint of an economic theory developed in advance of analysis of the statistics. In economics, as in other sciences, some of the most profitable research is that which develops facts not taken into account by existing theories — sometimes facts which require development of new theories which may differ substantially from the old ones.

[1] L. Bortkiewicz, *Die Iterationen*, Berlin, 1917, p. 83.

Three general questions deserve consideration in connection with commodity price research: (1) the consequences of pricing method; (2) the theory of the general price level; and (3) the relations of commodity prices to business cycles.

Pricing methods are subject to control, and it is necessary frequently to choose between alternatives which may include maintenance of free markets, institution of various regulations, establishment of collective bargaining, or institution of government regulation of prices. Reasoning regarding the relative merits of these different pricing methods is futile without knowledge of the facts from which the reasoning should proceed. Commodity price analysis may supply much information on the consequences of different pricing methods. An especially interesting question is that of the effects of speculative trading. A more general problem is that of the extent to which prices of different commodities are affected by errors of price judgments by traders, and by swings of sentiment involving group psychology.

The prevalent theory that general price movements are attributable to some *general price influences* is open to question as regards movements covering fairly short periods, and perhaps periods as long as two or three years. There is considerable evidence that such general price movements may arise from special influences, each of which bears directly only on the price of some single commodity or small group of commodities, but which has a broad ultimate effect owing to certain "sympathetic" price responses. Changes in the general wholesale price level over short periods are perhaps commonly best explained by a theory of the *propagation of special price influences.*

Cyclical fluctuations in business appear to react on prices of some commodities through effects on consumers' purchasing power. For wheat and perhaps for other commodities, fluctuations in consumers' income have only negligible effects, while other phenomena associated with the business cycle are of great importance. These other phenomena may be dominant in shaping the cyclical fluctuations of commodity prices in general.

Monday, July 5—Factors to be Taken into Account in Analyzing the Prices of Farm Products, O. C. STINE, Chief of Division of Statistical and Historical Research, Bureau of Agricultural Economics, United States Department of Agriculture.

This subject is so stated as to provide a wide-open door for a discussion of the problems in analyzing prices of farm products. One may distinguish three objectives of analysis:

(1) To develop the economic theory of prices,

(2) To verify or illustrate theories that are held,

(3) To determine a basis for judging what prices to expect tomorrow.

The primary interest of the author in analysis is to determine causal relations that may provide a basis for estimating the prices of tomorrow or judging what will be the effect of conditions that will prevail tomorrow upon the price of a product that is to be brought into the market under those conditions. The assumption is *that price-making forces produce consistent results and are sufficiently universal to make it possible to evaluate tomorrow on the basis of yesterday.*

The first important step in price analysis is the development of significant hypotheses as to causal relations; and the problem is to measure the significance of each of the several forces, or combinations of forces, in determining a price under the given or expected conditions. As a rule, it seems, the analyst approaches his problem without sufficient attention to the hypotheses that may be significant in dealing with the problem. The economists who have written most of our textbooks have cultivated the price field too lightly to produce many useful suggestions as to the factors or conditions that may be significant in dealing with a specific price problem. The analyst must be prepared to delve more deeply than the general economist. The well-trained economist sometimes fails as an analyst because he is too much inclined to generalize from a few points and to produce results which have slight or no substantial base.

The mathematical analyst, on the other hand, is too much inclined to a technical treatment of data, and to assume causal relationships wherever correlation exists. One of the strongest tendencies against which we must struggle in applying mathematics to economics is the working for a perfect fit or a high degree of correlation without due regard to causal relationships.

Between the economist who may be merely trying to prove his theory by reference to a few figures, and the mathematician or statistician who derives his theory from a few figures, there is not much basis for preference. If economics is a science, we must seek diligently for causal relationships, and determine the character and strength of the economic forces, not only in general but with respect to specific commodities and circumstances.

Three approaches to the analysis of prices of farm products may be distinguished as:

(1) Historical review,
(2) Economic hypotheses, and
(3) Statistical analysis.

All three are required for the success of the analyst in understanding the factors that determine the price of a product and developing dependable measures of the forces in price making. I would place the historical background in the position of first importance. One must know his commodity in terms of prices and associated conditions. This is necessary in selecting significant hypotheses and in deciding whether or not the available data are such that they may be depended upon to provide measures of relationships.

Among the conditions to be observed for their effect upon prices are degrees of competition in production and in buying, and custom or monopolistic limitations upon competition. Significant ties or relationships among commodities must be looked for over a period of time, and unusual variations among these relationships must be carefully scrutinized.

Since the prices of farm products are very responsive to conditions that cause changes in the general price level, the greatest need at the present time, as a basis for making real progress in farm-product price analysis, is the development of an understanding and a measurement of the forces determining the general price level so that significant changes in that level may be anticipated. The practical value of this last proposition is obvious in the challenging question of today: Are we now in the process of inflation in the United States?

Tuesday, July 6—Parity Price Ratios: Their Validity and Limitations, DR. STINE.

The validity of parity price ratios depends upon the consistency of relationships among factors in the production of commodities and services, on the one hand, and, on the other hand, among factors in determining the demand for the products. The several agricultural products are not isolated in their conditions of production, but compete directly for land, labor, and capital. As the price of one goes up, resources are shifted from the others to it; and conversely. The same is true with reference to the demand for the staple products of agriculture. Each has established a place among the others which remains fairly stable through a period of years. The result of these production and demand relationships over a long period of years has been that the prices of wheat, corn, and oats have fairly well-established relationships. The production and prices of livestock products have established what might be called normal relationships with the supplies and prices of feedstuffs.

The consistency of relationships and the tendency to establish normals exist among nonagricultural as well as agricultural products and as between services and commodities. This is to be noted in the tendency for prices of nonagricultural and agricultural products to swing together in long periods, which can be observed in the movement of wholesale prices disregarding short-time fluctuations. The simplest illustration of the relationship between services and commodities may be noted in comparing the wages paid to hired men on farms and the prices of agricultural products. A balancing of exchanges among producers of commodities and services results in a tendency to maintain price ratios among the staple commodities and services. Weather, wars, monopolies, and government activities may temporarily upset the balance, but all violent disturbances are followed by tendencies toward reestablishing the balance.

In the administration of the Agricultural Adjustment Act, parity price ratios were used as yardsticks for the determination of government policy. The validity of these yardsticks depended upon the assumptions stated above.

Among the limitations upon the validity of price ratios, one may list:

(1) The difficulty of determining a period in which the balance is sufficiently stable to use as a base for measuring at any given time the degree of parity or disparity.

(2) The conditions of supply of any particular commodity are subject to change with significant changes in the technique of production or the development of new producing areas.

(3) The demand for a particular commodity may be significantly changed by changes in the conditions of living or by the development of substitutes for that commodity.

(4) The price relations of a commodity in one area may be significantly modified by changes in the relation of that producing area to consumers of the product of that area. These limitations are particularly significant with reference to nonstaple commodities, but they have some significance in relation to the staple commodities. It follows from these observations that as a rule, although parity ratios may be reasonably valid as between agricultural and nonagricultural products and between related agricultural products, they may have little practical value for individual commodities whose position in production or in the diet of people is more or less unstable.

Another point to observe is that through a long period of years there are significant shifts in the needs or demands of consumers. Progress as a rule means change in living habits and a modification of the relation of goods and services to the efforts required to supply needs. It follows from this that parity ratios between goods and services or costs ought not to be fixed for long upon any one base.

In the discussions of parity between the agricultural and nonagricultural sections of the population of the United States, the real consideration is, how to maintain a parity of opportunity between those persons living on farms and those not living on farms. As we make progress, the beneficial results should be shared equally among the several producers of goods and services. This, the real parity, can not be measured so accurately in prices or in dollar income as in availability of opportunities for education and well-being in living. Price ratios may be used as the handiest statistical device to measure for short periods degrees of parity or disparity, but they can not be applied rigidly to individual commodities or even groups of commodities for a long period of years, without regard to how the prices are affecting the distribution of goods and services among the people with whom we are concerned.

Tuesday, July 6—Characteristics of Agricultural Supply Curves, LOUIS H. BEAN, Economic Advisor, Agricultural Adjustment Administration, United States Department of Agriculture.

Considerable progress has been made in recent years in the quantitative determination of demand curves, but very little has been done on supply curves. The purpose of this paper is to show that a great deal of progress could be made in the analysis of supply curves particularly for agricultural commodities if we approach the problem in a very simple and logical manner without the confusion of unnecessary assumptions, and if we resort to the graphic method of correlation.

Just as the demand curve is defined as quantities taken, or to be taken, by purchasers at different prices, so we may define the supply curve as quantities sold or to be sold by producers. It is necessary to differentiate between commodities produced once a season, such as crops, and commodities produced fairly continuously during the year, such as livestock, and their products. In the case of annual crops a series of supply schedules is possible. They may deal with any part of a season, such as a week, month, quarter, or the season as a whole. If, for example, we desire the farmers' supply schedule for the first quarter of the potato season, all we need is annual data on production, sales during the quarter, and price received during, or perhaps just preceding, the quarter, the latter appropriately adjusted for changes in the general level of prices paid by farmers. A multiple correlation treating sales or marketings as dependent on production (or total supply) and price will reveal the net relation of price to sales, and this is the supply schedule desired. This procedure may be followed for every quarter, dealing in each case with the supply available at the beginning of each quarter, the quantities sold, and the appropriate price.

Similarly a supply schedule for the season as a whole may be determined by correlating the amount sold during the year with total supply and price.

These supply curves, relating to the season during which a crop, the supply of which is fixed, is marketed, should not be confused with the supply schedule which pertains to the production of the following year or years. The latter, in the case of annual crops, can be developed from prices of one year and acreage the following year or years, as shown in the author's second paper.

The quantities sold by producers and the prices received by them, of course, give the basis for a demand schedule for dealers at farm prices. Similarly the quantities not sold, but used instead in the farm homes for feed, or for carry-over, form the basis for the farmer's reservation demand. These two schedules may be summed into a total-supply-price curve.

Price determination for a fixed supply may be shown in one of two ways, either as the intersection of the dealer-demand curve with the farmer-supply curve, the latter obtained by deducting the farmer-reservation schedule from the fixed supply; or as the intersection of a fixed supply with the total-supply-price curve.

These facts may be illustrated by developing the supply curves for three different commodities, potatoes, wheat, and cotton. In the case of potatoes and wheat the reservation schedules are derived from annual quantities fed to livestock, used as food in the farm home, or retained as carry-over (wheat). In the case of cotton the reservation schedule is determined by the relation of price to carry-over at the end of the season. It is clear from these illustrations where competition is freely at work that "instantaneous" supply and demand curves can be developed and that they differ in elasticity both as between commodities and as between periods within the marketing season.

Wednesday, July 7 — Scientific Bases for Forecasting Wheat Prices, PROFESSOR HOLBROOK WORKING.

Viewed broadly, the forecasting of prices includes "conditional prediction": the appraisal of price consequences to be expected in the event of various possible developments which are not themselves predictable. The course of prices is always subject to influence by unpredictable developments, but specific developments which need to be contemplated as possible vary from time to time; and the price consequences to be expected from any particular (unpredictable) development differ with differences in attendant circumstances, which may be known. The term "appraisal of the price outlook" may be used to suggest a broader consideration of prospects than is commonly expected in a "price forecast."

Changes in indicated or expected "world" supply of wheat are major causes of price change, but are largely unpredictable. The volume and geographical distribution of existing supplies, however, are important circumstances conditioning the price effects of future changes in supply. Certain supply situations are commonly associated also with particular subsequent price movements, more or less independent of subsequent changes in production. For example, the existence of a wheat surplus is commonly associated with subsequent decline in prices of wheat futures, even though the size of the surplus is not increased.

Changes in the general wholesale price level over intervals of 12 months are commonly accompanied by considerably larger changes in wheat prices. Changes in general business activity, however, have no importance for wheat prices except as they may be accompanied by changes in wholesale prices in general.

The occurrence of a wheat price increase of 14 cents or more (on a 1913 price-level basis) within 5 weeks or less raises a strong presumption of an early decline of about the same magnitude. The promptness and rapidity of such price reaction depend upon the season of the year and other circumstances. Under certain special conditions, only a small and temporary reaction is to be expected after such a price advance.

Price prospects are often best indicated by reference to the course of prices in an earlier period in which the essential conditions were broadly similar. In such comparisons, similarity in price movements over immediately antecedent intervals may be an important aspect of the similarity of conditions. Much of the value of such comparisons of different periods lies in their convenience for exposition.

Since prices rest in large measure on human judgments, prospects for prices need to be appraised in the light of present attitudes of traders. Study of characteristics of recent price developments is often illuminating in this connection. It is useful to take account of peculiarities in relationships among different wheat prices (differential price behavior) and of any evidence on what markets have taken the leadership in recent price movements.

Wednesday, July 7—Monopolistic Elements in Pricing: the Domestic-Export Spread, FRANCIS MCINTYRE, Instructor in Statistics, Stanford University (now Research Associate, Cowles Commission).

When the producer of a single commodity engages actively in the sale of his product at different prices depending solely upon whether his consumption is to take place at home or abroad, we have the phenomenon called dumping. Such price discrimination may be carried on for purposes of building up a market, for retaliation against foreign competition, because the foreign market was weaker than expected and goods shipped on consignment were disposed of at less than the expected price, or simply to maximize short-run profits. Only this last motive is considered here and the discussion is restricted to two markets for convenience.

If perfect competition prevails in each market — perfect competition being defined as existing in any market in which the producer faces a horizontal average-revenue (demand) curve — all of the producer's output will be sold in whichever market displays the higher prevailing price. If perfect competition prevails in one market and not in the other, price discrimination will occur and will be in favor of the perfectly competitive market except in the case in which all of the output is sold in the imperfectly competitive market. If neither of the two markets is perfectly competitive there is no a priori reason why the price in one market need always be above that in the other. F.o.b. refinery price quotations for refined copper are available for sales to domestic and foreign consumers which quotations reveal for recent months persistent dumping. If we subtract the export quotations from the domestic, we obtain a price spread which is sometimes positive, sometimes negative. The positive spread may be sustained by transportation costs and tariff barriers but the negative may not. For a positive spread, monopoly is not necessary and imperfect competition is sufficient, but a negative value for the spread requires a much greater degree of control on the part of producers of the commodity over price and the ultimate disposition of the product.

Thursday, July 8—Apportionment of Representatives in Congress, EDWARD V. HUNTINGTON, Professor of Mechanics, Harvard University.

The United States Constitution provides that "Representatives shall be apportioned among the several states according to their respective numbers, counting the whole number of persons in each state, excluding Indians not taxed; . . . but each state shall have at least one representative." The Constitution also provides that a reapportionment shall be made within every period of ten years, so as to keep pace with the relative changes in population among the several states.

Suppose now that the size of the House has been determined upon (at present 435), and suppose the population of each state is known; how many representatives shall be assigned to each state? At first sight it would seem that the answer to this question could be obtained by any high-school boy by solving a simple problem in proportion. The number of representatives belonging to any state should be to the total number of representatives as the population of that state is to the total population of the country. As a matter of fact, however, the problem is not as simple as it looks, because the answer obtained by simple arithmetic usually will involve a fractional number.

This difficulty has been the source of great annoyance and bitter debate in Congress every ten years since the founding of the country. Daniel Webster thought he had a solution in 1832. His plan was to compute the "exact mathematical part or portion" due to each state, and then use the nearest whole number (neglecting any fraction less than one-half, and adding one for each fraction greater than one-half). It was soon discovered, however, that this Webster plan was unworkable in practice, because it could not be relied upon to yield the right total. A congressman named Vinton thought he had a solution in 1850. This plan had to be discarded, because in some cases it worked out that a state which had gained in relative population might actually lose in the number of its representatives. This was the case with Alabama in 1881, and the "Alabama paradox," while extremely perplexing to the congressmen of that day, was immediately recognized as a fatal defect of the Vinton plan. Professor W. F. Willcox thought he had a solution in 1910. The Willcox plan, known as the "method of major fractions," is an ingenious but rather complicated modification of the Webster and Vinton plans and, although lacking in mathematical justification, is still in use today.

It was not until 1920 that the problem received a satisfactory mathematical analysis. In that year the writer pointed out that the details of the numerical computation, which had occupied the center of the stage up to that time, were not so important as the fairness of the resulting apportionment. A "fair" apportionment is one in which each state is as nearly as possible on a par with every other state in the matter of representation. Some inequalities will inevitably occur, since fractional congressmen are not available; but if these inequalities cannot be reduced by any transfer from any state to any other state, then the apportionment is as fair as it can be made. There is just one method of computation which will completely satisfy this requirement. This is known as the "method of equal proportions." If this method of equal proportions had been used in 1921, instead of the method of major fractions, the apportionment for that year would have been improved in the case of six states.

The writer's proposed new method of equal proportions immediately aroused much opposition among the politicians, chiefly on the ground that the problem of apportionment was "primarily a political and not a mathematical" problem. The matter was referred to the Joint Committee of the American Statistical Association and the American Economic Association to Advise the Director of the Census, which body concluded in 1921 that "the method of equal proportions, consistent as it is with the literal meaning of the words of the Constitution, is logically superior to the method of major fractions." The question of method was not settled at that time, however, since Congress, in spite of the clear constitutional mandate, failed to make any reapportionment for the decade.

In 1929 the question came up again, and a very clear exposition of the mathematical facts was given by Joseph A. Hill, then Director of the Census, in hearings before the House

Committee on the Census. A special committee of the National Academy of Sciences reported that the method of equal proportions was to be preferred for two reasons: first, "because it satisfies the test (of a desirable apportionment) when applied either to sizes of congressional districts or to numbers of representatives per person"; and secondly, "because it occupies mathematically a neutral position with reference to emphasis on larger and smaller states."

In spite of these authoritative reports, however, the Senate voted against the method of equal proportions, the vote being based on a totally erroneous description of that method. To avoid the danger of a second failure to make a reapportionment, Senator Vandenberg secured the passage of a bill, signed by President Hoover on June 18, 1929, which provides that in case Congress fails to reapportion promptly after any decennial census, then a reapportionment shall take place automatically, the size of the House remaining unchanged, and the method of computation being the method last used. The President is to inform Congress what the apportionment would be under the old method of major fractions and also under the new method of equal proportions; but the method of equal proportions will not be put into effect unless Congress takes affirmative action to that end.

In 1930, by an odd coincidence, both methods led to the same apportionment of 435 members, so that the question of method did not arise. In 1940, it remains to be seen what action, if any, Congress will take. The mathematical theory of the problem is thoroughly established.

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Thursday, July 8—Agricultural Supply Curves and Industrial Stability, MR. BEAN.

The farmer's response to price has a very definite bearing on industrial fluctuations. This may be revealed by developing the nature of the supply schedules derived from prices of one season and acreage or livestock numbers of the next season, and studying the impact of these responses to price fluctuations on the industries affected. The supply curves based on price-acreage relationships that have been developed thus far reveal certain important facts. They indicate the prices for which acreage or subsequent supply tends to be held constant. They reveal the effects of relatively high and of relatively low prices. They reveal how the cost level has fluctuated, cost in this case being represented by prices associated with acreage stability.

The nature of the relation of price to subsequent acreage differs a great deal by commodities.[1] There are several major commodities, however, that reveal similar characteristics. One of these characteristics is that acreage expansion following very high prices is not much greater than that following modrately high prices. Another characteristic is that the contraction in acreage following low prices is greater than the increase in acreage following high prices. This is true for such commodities as cotton, wheat, potatoes, and other field crops. It is also true of the number of hogs on farms. From this tendency it follows that industries supplying farmers with production goods, such as fertilizer, feed, machinery, are subjected to fluctuations in their markets growing out of the instability represented by the price-acreage curves. Other things constant, it appears that a year of relatively high prices followed by a year of relatively low prices involves a smaller acreage planted than the sum of acreages in two years following a period of prices relatively stable.

The characteristics of the subsequent supply curves may also be related to the general business cycle. At certain criti-

[1]See L. H. Bean, "Farmers' Response to Price," *Journal of Farm Economics*, July, 1929.

cal points, such as at the bottom of the depression in 1932-33, one of the primary forces making for the beginning of revival may be traced to the cotton price-acreage curve. For example, nearly a record crop was produced in 1931 and prices went well below the point of price-acreage stability. The subsequent reduction in acreage induced by this fact, together with poor growing conditions, created a crop prospect situation in the summer of 1932 where rising prices could be anticipated. The 1932 crop turned out to be about a third less than in 1931 and this became a potent factor in rising textile prices, a very sharp advance in textile production and employment. Other industries did not share in this advance during the fall months of 1932, except in the sense that their downward drift may have been retarded. Subsequently, however, the improvement in the textile situation in New England and the South must have meant improvement also in many associated lines of economic activity.

The fluctuations in supplies and prices represented by the characteristics of the agricultural supply curves may also be shown to have a direct bearing on industrial profits. There seems to be a tendency, for example, for cotton manufacturers to lose more in years of falling cotton prices than they gain in years of rising prices, volume remaining constant. Similarly, they tend to lose more in years of low activity than they gain in years of high activity, price being constant. Other things equal, cotton manufacturers have a basic interest in greater price-production stability of about the same order as that of farmers. The same conclusions seem to be warranted for the meat packers and probably for other segments of industrial activity dependent on agricultural supplies and prices, which in the aggregate may constitute about half of our manufacturing production.

Friday, July 9 — Some Difficulties in the Analysis of Stock-Market Phenomena, THEODORE O. YNTEMA, Professor of Statistics, University of Chicago.

Stock-market phenomena constitute an important field for economic research. The study of them will extend our knowledge of speculative markets, now pitifully small, will contribute to better social control of securities exchanges, will make quack practices more difficult, and may throw some light on the business cycle. Progress in stock-market research has been slow because the phenomena to be studied are so complex, because so few investigators have been competent, and because the interchange of ideas and the publication of results have been restricted.

While the gross aspects of the market have been subjected to exhaustive study, the differential behavior of various phenomena has received little attention. Illustrative of the latter are differences in price behavior of stocks in various industry groups, in price and volume of trading of investment and of speculative issues, in volume of round lots and of odd lots, in volume of sales and purchases and in long or short position of members of the exchanges and of the public, in credit for speculative purposes obtained by various groups and from various sources, and in the movements of indexes reflecting the dispersion and skewness of price changes. Such phenomena are interesting not merely because relatively little explored but because there are logical grounds for expecting many such differentials to be related to fluctuations of stock prices.

In analyzing the differential behavior of such series numerous difficulties are encountered. In some cases the data are available only for a short time and in others can be derived only at the expense of great labor. Most series relating to the stock market are characterized by trend, business-cycle, intermediate, and very-short-run fluctuations. For study of intermediate fluctuations (extending over one to several weeks) it is necessary to eliminate the trend and business-cycle components. Usually the intercorrelation between series whose differential behavior is to be analyzed is very high and this behavior cannot be satisfactorily judged except by actual calculation and plotting of differences or residuals. Furthermore many of the correlations are curvilinear, some extremely so. If two variables are related by a curvilinear function, the correlation of their first differences or deviations from trend will not be satisfactory because the regression between the differences or deviations will vary with the slope of the underlying regression between the original series. Entirely aside from the effects of curvilinearity, regressions between short-run differences or deviations will sometimes exhibit changes of slope as the business cycle moves from trough to peak and peak to trough. Finally there are problems of the approximate scale of measurement of the variables. For example, intermediate fluctuations of stock prices have larger amplitude at high prices than at low prices but the reverse is true on the logarithmic scale. Bond prices show larger intermediate fluctuations at low prices even on the absolute scale. This question of scale is of course interrelated with those of curvilinearity and cyclical shifts of regressions. What would be desirable would be to discover such transformed scales, if they exist, as would yield linear regressions between variables, invariant with respect to cyclical swings. In any case, transformation of scale, even though roughly approximate, is a powerful tool in studying residual differences between time series characterized by curvilinear correlation.

(Illustrations of these difficulties were then described. It was also shown that the graphic method of multiple-correlation analysis may be extended to some cases in which the relation of the independent variables to the dependent variable is not additive over the independent variables but is of the multiplicative type.)

Friday, July 9 — Is Commodity Price Inflation Imminent? E. J. WORKING, Associate Professor of Agricultural Economics, University of Illinois.

Since different people mean different things by the term "inflation" it is necessary in my discussion to indicate what meaning is intended. In some respects it would be desirable to define a commodity price inflation as any rise of commodity prices *caused* by an increase in the volume of money or credit. However, it is not possible to determine conclusively whether or not most small or slow price rises constitute inflation in this sense. Consequently, only large and rapid rises of the price level constitute prima facie evidence of inflation. In order to be entirely objective, this paper is concerned with the prospects for a rise of as much as fifty per cent in the Bureau of Labor Statistics "all commodity" price index within a period of two years.

Inflation is considered imminent in the United States by many people. These may be classified in two groups according as they hold to the "quantity theory" of money and prices or to what may be termed the "value of gold" theory. Those who hold to the former theory consider that the tremendous increase in the credit base and the increase in credit resulting from the unbalanced budget indicate that inflation is likely. The other group considers that supplies of gold are sufficient to maintain a price level in terms of gold somewhat above the level in the years immediately preceding the World War. This would involve a rise of about 30 to 50 per cent and, in terms of the present devalued dollar, a price level in the United States of about 85 to 100 per cent above the prewar level and 25 to 40 per cent above the 1926 level.

The conclusions of the "value of gold" theorists are based on the assumption that the demand for gold will return to "normal," yet almost all of the price-level changes (in terms of gold) during the past quarter century are admittedly due to changes in the demand for gold. Events of the past quarter century also discredit the idea that an increase in the amount of credit which a banking system can extend will be followed in the course of a few years by a corresponding increase in the amount of credit extended. Even more erroneous is the idea that any marked increase in credit which may occur will necessarily be accompanied by a rising level of commodity prices.

In view of the inadequacies of the above theories, a new approach to an explanation of commodity-price-level changes and their prospects is necessary. This is found in what may be termed the "price structure" approach. This approach considers the changes in the price level as the average of changes of individual prices. It considers the forces affecting both international and internal price relationships and how these changes may affect the average price level. A study of actual price movements in different countries shows that prices of "international" commodities in these countries (when expressed in terms of gold or any other common measure) tend to move up and down together in spite of numerous exceptions due to price controls and trade restrictions. Furthermore, the average commodity price levels in the various countries tend to move with the prices of the international commodities.

In view of these facts how might inflation occur in the United States? It could happen only if (1) there should be a major and world-wide rise of prices in terms of gold or (2) a depreciation in the value of our currency relative to gold.

A rapid and large world-wide rise of commodity prices measured in terms of gold is not likely except in the event of a major war. Although gold production has increased, monetary stocks are increasing very slowly. Furthermore there remain evidences of continued maladjustments of international trade relationships which tend to cause a scarcity of gold and restrictions of credit in some countries. In the absence of a major war such a tendency toward maldistribution of gold is more likely to result in a decline than in an important rise of world commodity prices measured in terms of gold.

An inflation of prices in the United States which would involve the depreciation of our currency in terms of gold would be altogether possible as far as economic interrelationships are concerned. It might take place as a result of further dollar devaluation such as that of four years ago. On the other hand an unbalanced budget or credit expansion through other means could result in monetary instability and depreciation of our exchange. However, at a time of improving business activity it would probably be "political suicide" for the administration in power to allow an extreme depreciation of the value of the dollar such as would be necessary for any great degree of inflation. In view of the means of monetary control available to the government it is altogether possible to prevent any further depreciation of the dollar. Hence commodity price inflation involving dollar depreciation seems unlikely in the near future.

There is an important tendency for the level of commodity prices to be influenced largely over short periods by price rigidities combined with what may be termed the "internal stresses of the price structure." Because of this there is, over moderately long periods of time, more tendency for the volume of credit to be adjusted to the needs of the business situation and the commodity price level than for the price level to increase because there is an available base for credit expansion. A marked distinction is to be drawn between commodity prices and security prices in this respect. Possibilities of credit expansion are much more likely to result in security price inflation than in commodity price inflation.

Monday, July 12—Derivation of the Simplified Economic System; *Tuesday, July 13*—Applications to Taxation, Unemployment, and Interest; *Wednesday, July 14*—Indices and the Simplified System; GRIFFITH C. EVANS, Professor of Mathematics, University of California.

A simplified or reduced system of economics may serve as a skeleton on which to hang such words as capital, labor, and consumption goods, with the idea that such a system will supply at least a long-term analysis of what may be called "moving equilibrium." If one regards capital production per unit time as made up of two parts: x, that which goes back into capital production, and y, that which goes into production of consumption goods, the technical arrangement for the manufacture of production goods may be described roughly by the equation

$$x + y = \varphi(x, z),$$

z being the amount of labor employed per unit time in that manufacture. Similarly, the manufacture of consumption goods is represented by the equation

$$U = \vartheta(y, l - z),$$

admitting for the present that the only situations of the system which are to be investigated are those in which the total amount of labor employed is l.

A typical problem to be investigated is the description of the situation which will make U a maximum. With x and z as arbitrary variables, this situation is given by the equations $\varphi_1 = 1$, $\vartheta_1\varphi_2 = \vartheta_2$, with $x + y = \varphi(x, z)$, where ϑ_1, ϑ_2, etc., represent derivatives of ϑ, etc., with respect to the first or second arguments. Thus

$$\vartheta_2 = \partial\vartheta / \partial(l - z).$$

Second-order conditions must also be stated if one wishes to discuss the nature of the functions φ_1, ϑ with respect to practical considerations. The first derivatives of these functions will naturally be taken as positive.

A second problem will be the description of pure competition. Prices p_1 for capital, p_2 for labor (with other prime factors if desired), p_3 for consumption commodity, are introduced, so that cost functions may be evaluated. The statement that the costs are to be minimized in each of the various classes of production, with prices regarded as outside the choice of the manufacturers, yields relations from which the prices may be eliminated. The resulting equations in x, y, z turn out to be the three already obtained. That is, U is again a maximum.[1] This result was obtained in the article on the subject in *Econometrica* (1934).

[1] G. C. Evans, "Maximum Production Studied in a Simplified Economic System," *Econometrica*, Vol. 2, 1934, pp. 37-50.

A closer description of the price relations is obtained by making extremal the values of the profits

$$\pi_1 = p_1(x+y) - p_1x - p_2z; \quad \pi_3 = p_3u - p_1y - p_2(l-z).$$

This device was used by O. Lange in his theory of interest in such a system,[2] and independently by the speaker and his associates in the University of California seminar. In fact, the device yields the ratios of the prices, by means of the equations

$$p_2/p_1 = \vartheta_2, \quad p_1/p_3 = \vartheta_1, \quad p_2/p_3 = \vartheta_2,$$

—relations which have an evident interpretation in classical economics.

But before going further to make applications of the skeleton system to particular problems of taxation, etc., it is desirable to see that the above relations have more than a mere shorthand significance. For this purpose, an analysis was carried out in the University of California seminar by Francis Dresch. His discussion involves the determination of trade and price indices, in accordance with the scheme of Divisia, for the various classes of economic goods, allowing an arbitrary number of goods of each class, and an arbitrary number of entrepreneurs for each good. The indices Q_1, Q_2, Q_3 thus represent rates of production of capital goods, prime factors, and consumption goods, with corresponding price indices P_1, P_2, P_3. By introducing also partial differential quotients of indices, it turns out that in the case of strict competition the above equations of the skeleton situation are exactly reproduced, remembering of course that price indices are not prices. It transpires also that Q_3 is again made extremal.

With this guarantee of meaningfulness it is safe to turn again to the applications of the skeleton system. Thus, if the cost function $c_1 = p_1x + p_2z$ is replaced by $c_1 = p_1x + p_2z + \xi p_1x$ and $c_3 = p_1y + p_2(l-z)$ is replaced by $p_1y(1+\xi) + p_2(l-z)$, one obtains with Ronald Shephard a new equilibrium situation corresponding to the introduction of a tax ξ per dollar of capital services. Regarding ξ as a relatively small

[2]O. Lange, "The Place of Interest in the Theory of Production," *Review of Economic Studies*, Vol. 3, 1936, pp. 159-192.

fraction, the values of the changes δx, δy, δz, from the old to the new values may be found by means of linear equations. But what is more important in Shephard's analysis is that also the changes in the standards of living may be found. Thus, if we assume that in a steady state the π_1 and π_3 and the p_2l are spent on consumption goods, equations

$$\pi_1 = ap_3U, \quad p_2l = bp_3U, \quad \pi_3 = cp_3U,$$

define numbers a, b, c which represent the standards of living of the three classes. The changes δa, δb, δc can thus also be calculated, and furnish a large number of theoretical results depending on what assumptions are made with respect to various marginal coefficients of elasticity involved in the technical relations of capital and labor. Still simpler results were obtained by Mr. Shephard in the case of the sales tax, where a, b, c turned out to be all diminished in the same proportion.

This is one type of application. Another type is obtained by the introduction of an extra variable to indicate some quantity of special interest, such as the total amount of currency in the study of interest, inaugurated by O. Lange in the *Review of Economic Studies*, and the unemployment benefits in terms of which Professor Tingey, in the University of California Seminar, commenced the study of unemployment in the skeleton system. In the latter case it is necessary to replace $l—z$ by a new variable w, and introduce some relation for the offer of labor, e.g.,

$$w + z = f(p_2/p_3).$$

If now unemployment benefits are introduced, of amounts kp_2z and kp_2w to be paid by the employers, with similar payments by the employees, the difference between the situations, where $k = 0$ and where k is a small positive quantity, may be analyzed by the same method of linear relations between small quantities, in terms again of the same marginal coefficients of elasticity as in the problem of taxation.

It is hoped that economists will find such skeleton methods of analysis desirable in a first approach to the study of long-run questions. But also more dynamic questions may be treated by means of the introduction of times and time intervals into Dr. Dresch's equations. And here even regimes of monopoly and restricted competition have been set up, and compared with those of free competition.

Monday, July 12 — The Pareto Curve and Its Significance for Our Time, MR. SNYDER.

In the world in which we live we discover everywhere the most extraordinary variety and, it may be added, disparity. From the billions of suns that blaze in the sky to the ultra-microscopic particles that float in the air. So in the world of life; from the giant sequoias and the monstrous whales, to microbes and the ultra-viruses that bring us plagues and epidemics. So also in human affairs: some few are very successful, highly talented, endowed with capacities and abilities far beyond the mass of their fellows. The rule throughout all nature: *in*-equality.

Does this almost infinite variety and disparity present merely a random or haphazard distribution or is it, everywhere, dominated by a remarkable law? This is a statistical problem, difficult at present to answer because curiously enough we have not measures of distribution amid a wide variety of different kinds of populations. In reality hardly more than one wide field is open for investigation.

This is the distribution of income, and to a certain extent of wealth, in highly developed commercial states like our own country or the leading nations of Europe; largely derived from income-tax data. Long ago it was suspicioned that this distribution did follow a definite law, but the first to give the problem a careful, laborious investigation was Vilfredo Pareto, born in Paris of an Italian father, an emigré, and a French mother; educated as an engineer, who later became director of state railways in Italy; later Professor of Economics in Lausanne.

What, very simply, Pareto found was that from the highest to the lowest brackets of income there was a steady progression in numbers and if the numbers in these different brackets were plotted they followed a very remarkable and almost uniform curve. Plotted on double-log paper and the data cumulated, this approached very closely to a straight line. It was first presented about 1896 and thereafter subjected to a recurrent gunfire of criticism and objection. Largely these objections arise from meticulous observations, that under a microscope at least there are some variations detectable in the slope of the line. Considering the nature of the data and especially the obvious inadequacy of the reported incomes in the lower brackets, this is hardly surprising.

But there is a larger interest in this very interesting field and that is whether the Pareto Curve is not a general expression for practically every kind of highly specialized ability, from that of the billiard player or baseball pitcher to the spectacular performance of the chess player, or the lightning calculator in mathematics, or any kind of special talent. Clearly there are not in this world many Newtons or Shakespeares or Faradays, any more than there is a large population of Dizzy Deans or Willie Hoppes. Any more than there are thousands of John D. Rockefellers or Henry Fords or the Croesuses of old. Everywhere a vast variety of talent, and disparity in ability. Such at least was the conception to which the writer had given a good deal of attention for a number of years without being able to gather a great deal of factual data in support of the idea. It was very interesting to find last year that Professor Harold T. Davis had hit upon the same concept and made the same effort towards statistical proof of its validity. Not a great deal of available data. Enough, however, it has seemed to me, to indicate pretty clearly that the thesis is valid, and further that the Pareto Curve is destined to take its place as one of the great generalizations of human knowledge.

I believe we may go much further and say that it has a profound significance in the field of economics and still more broadly what we may call in a large sense the political economy. It is clear that we can not by legislation or any fanciful or fantastic law create a Newton, a Shakespeare, or a Faraday. No more than we can legislate the number or variety of very excellent poets or mathematicians or inventors, who follow, in considerable numbers, in their wake. One thing at least in this difficult field seems clear; we can not plan for or regiment, or in any artificial way bring forth great genius. And all of high value and interest, and understanding, and color, in this rather perplexing and sometimes difficult world, has been derived from the especially talented few.

But it seems not so clear that the wealth, capital, and broad diffusion of well-being in the more highly developed civilizations like those of the United States and Europe, are likewise due to an extraordinarily small number of men: the inventors, the discoverers, the contrivers and innovators, and the singularly gifted, with the very rare talent for organization and likewise the accumulation of wealth. These are the true creators and, as I believe, will prove the true saviours of our modern world.

Tuesday, July 13—The "Equally Likely" Concept in the Definition of Probability, THORNTON C. FRY, Bell Telephone Laboratories, Inc.

In the introductory chapter in *Probability and its Engineering Uses* an attempt was made to distinguish between two uses of the word probability: the first as a name for an abstract concept, the other for a number which measures this probability. Attention was also called to the fact that the so-called definitions of *probability* defined the second use of the term and not the first. Later, in a paper presented before the Mathematical Association of America in 1933, it was noted that all of the important definitions of probability make use of a concept serving the same logical purpose as the "equally likely cases" in the Laplacian definition. It was also pointed out that no one of these definitions can be applied in all the various fields in which the theory of probability is used, so that the practical question is not which definition is right and which wrong, but which happens to be most appropriate to a particular need.

The present paper, which is intended to be read in the light of these earlier discussions, presents the following thesis regarding the first use of the term:

that it is a class concept somehow derived from our limited experience by processes of abstraction and generalization;

that in this respect it is logically isomorphic with our concepts of other measurable attributes such as length;

that the attempt to associate it with any particular type of experience, such as experiences with heterogeneous populations or with statistical sequences, rather than with the totality of all our experiences with uncertain events, does not remove the basic mystery, which really has to do with the epistemology of class concepts in general, rather than with probability in particular;

and that since the attribute which defines the class is common to all its members, this attribute can be spoken of with equal validity as a property of the class as a whole, or of any single member in it. In other words, it is proper to call the expected success ratio in an infinite sequence of trials "the probability of success"; but it is also proper to speak of "the probability of success" in a single try divorced from the sequence.

As regards the other use of the term, it is pointed out:

that we do not define the abstract class concept of probability but a method of measuring it;

that in so doing the idea of "equally probable events" never has been avoided and presumably never can be;

that this does not constitute redundancy in the definitions since the proof of equality does not demand measurement;

finally that in all these respects the problem of setting up a system of measurement for probability is logically isomorphic with the similar problem for length, or for any other physical quantity for which a relation of the form $a > b$ is recognizable.

The paper concludes with a discussion of the relationship between the numbers obtainable from physical experiments and the highly idealized concepts with which the mathematical theory of probability deals. It is pointed out that the logical difficulties met in this connection also are of exactly the same kind as occur in the application of mathematics to any of the physical sciences.

Wednesday, July 14, Wednesday, July 21, Thursday, July 22 — General Choice-Field Theory, RAGNAR FRISCH, University of Norway.

The theory of choice is very broad in scope, applying for instance to the behavior of the entrepreneur in a capitalistic society as well as to the chief of production in a planned economy and to the housewife who decides how much to use on food, clothing, shelter, etc. It is concerned with certain principles that are *general in their application* within all these fields. Frequently one is led to consider notions like vectors, potentials, and the like. Hence the term choice-field theory.

A general choice-field theory would contain the following parts:

1. The postulational construction of choice fields.
2. Quantity fields and their classification.
3. Relation between want, demand, and budget composition in a quantity field.
4. The money flexibility and its significance in the quantity field.
5. Methods of statistical measurements in a quantity field.
6. Price-level comparisons between different quantity fields.

Some points regarding 1, 2, and 4 will be discussed in the three lectures on general choice-field theory, while a special lecture will be devoted to 6.

Only by a strict postulational method can a logically satisfactory foundation for general choice-field theory be built up. Two types of approach are conceivable: (I) a pure *choice* postulational method; and (II) a *function-* (or vector-) assuming method. In (I) nothing but elementary postulates regarding the choice acts are admitted (like those listed below), and the existence of certain functions (or vectors) with a choice meaning is proved. In (II) such functions (or vectors) are *assumed to exist.* The construction of choice experiments which will reveal their nature is then comparatively easy. All the analyses by Pareto and his followers, on the measurability of utility, are of this sort (Pareto's analyses, however, containing a number of mistakes as will appear from the next lecture). Method (I) is more fundamental and will be followed.[1]

Let us first discuss the concrete situation that suggests the nature of the logical elements to be combined. We consider an individual (or family or other consumption group) to which a number of questions are put regarding its preferences. The *decision point* is the entity of all the conditions that prevail at the moment when the questions are asked. The *reception points* (one, two, or several) are the situations that are assumed to prevail when the goods or services involved in the questions are to be received. They will be denoted by capital letters $X, Y, Z, \ldots$ The *choice objects* (at least two) are the complexes of goods or services involved in the question. They will be denoted by small letters, $a, b, c, \ldots$ A choice object is something subject to choice; a reception point is not, but is a datum. This distinction is fundamental. A *choice alternative* is an association between a choice object and a reception point (a simple alternative): "the object a to be received in the situation X"; or several such associations (a collectional alternative): "a_1 in X_1 *and* a_2 in X_2, etc." A *choice question* is a question regarding which one of two alternatives is preferred. It may be *local*: "a in X or b in X?" or *interlocal*: "a in X or b in Y?" or *collectional*: "a_1 in X_1 *and* a_2 in X_2, etc., or b_1 in Y_1 *and* b_2 in Y_2, etc.?" The determinateness postulates below express that one and only one of the three possibilities exists: (1) The individual prefers the first alternative; (2) He prefers the second; (3) The choice is indifferent ("you may give me either").

[1]As far as I know the first approach of this sort was given in my pamphlet, "Sur un problème d'économie pure," *Norsk Matematisk Forenings Skrifter*, Series 1, No. 16, 1926.

LOGICAL BASE ("Universe of Discourse")

	I	II
	Fundamental (undefined) concepts, operations, and relations.	Definitions and basic postulates (expressed in terms of the logical elements in Column I).
1	Elements of K_1 (class 1): $X, Y, Z, \ldots$	
2	Elements of K_2 (class 2): $a, b, c, \ldots$	
3	Elements of K_3 (class 3): $\Phi, \Psi, \Theta, \ldots$. Also special symbols used.	
4		$\alpha, \beta, \gamma, \ldots$, elements of a subclass of K_2 to be called K_2' (class two prime).
5	The binary* operator to be used between any element of K_1 and any element of K_2 (taken in this order) yielding an element of K_3.	$a\|X$, $b\|Y$, etc. (pronounce "a stroke X") are elements of K_3.
6	The binary operator $+$ to be used between any two elements of K_2 yielding a new element of K_2: $a + b = c$.	*Postulate* 6_1: The sum of elements of K_2 is associative: $(a + b) + c = a + (b + c)$. *Postulate* 6_2: The sum of elements of K_2' is commutative: $\alpha + \beta = \beta + \alpha$.
7		*Postulate*: There exists one and only one element of K_2, to be denoted by 0, such that $a + 0 = a$ and $0 + a = a$ for any a in K_2 ("The zero element in K_2").

LOGICAL BASE ("Universe of Discourse") (continued)

	I	II
8	The binary operator seq to be used between any element of K_1 and any element of K_2 (taken in this order) yielding an element of K_1: X seq $a = Y$.	*Note*: If X and Y are given, there may exist one, none, or several a such that X seq $a = Y$.
9		*Definition*: Let a be any element of K_2 and X any element of K_1. Then any element b of K_2 such that (X seq a) seq $b = X$ is called a reverse of a with respect to X. (Such elements may or may not exist.)
10	The binary operator & (pronounce "and" or "ampersand") to be used between any elements of K_3 yielding a new element of K_3.	*Definition*: Elements of K_3 obtained by 5, i.e., elements of the form $a\|X$, are called *simple;* those of the form $a_1\|X_1$ & $a_2\|X_2$, are called *collectional.* *Postulate*: The operator & is associative and commutative.
11	The dyadic relation pref (pronounce "preferred to") to be used between any two elements of K_3. For instance: $a\|X$ pref $b\|Y$.	*Definition*: The relation Φ pref Ψ may also be written Ψ depref Φ (pronounce "Ψ depreferred to Φ").
12	The dyadic relation equiv (pronounce "equivalent to") to be used between any two elements of K_3. For instance: $a\|X$ equiv $b\|Y$.	*Postulate*: If Φ equiv Ψ, then Ψ equiv Φ.

*More generally we could have considered the case where the elements of K_2 were classified with respect to those of K_1, so that the operator | could only be used if a belongs to the subclass of K_2 associated with X; but for our present purpose this is not necessary.

[Note: If all the subsequent choice postulates are admitted, the logical base and the whole reasoning can no doubt be simplified. But if it is wanted to bring out clearly how far it is possible to get with a given minimum of choice postulates, the above logical base is about as simple as it is possible to make it.]

CHOICE POSTULATES

	Local	Interlocal	Collectional
Determinateness	Let X be any element of K_1, a and b any two elements of K_2. Then one and only one of the following three relations holds good: pref $a\vert X$ depref $b\vert X$. equiv	Let X and Y be any two elements of K_1, a and b, any two elements of K_2. Then one and only one of the following three relations holds good: pref $a\vert X$ depref $b\vert Y$. equiv	Let $X_1 X_2, \cdots, Y_1, Y_2, \cdots$ be two sets of elements of K_1 (not necessarily the same number of elements in both sets); $a_1, a_2, \cdots b_1, b_2, \cdots$, two sets of elements of K_2. Then one and only one of the following three relations holds good: pref $a_1\vert X_1$ & $a_2\vert X_2, \ldots$ depref $b_1\vert Y_1$ & $b_2\vert Y_2, \ldots$ equiv
Transitivity*	If $a\vert X$ pref $b\vert X$ and $b\vert X$ pref $c\vert X$, then $a\vert X$ pref $c\vert X$.	If $a\vert X$ pref $b\vert Y$ and $b\vert Y$ pref $c\vert Z$, then $a\vert X$ pref $c\vert Z$.	If $a_1\vert X_1$ & $a_2\vert X_2, \cdots$ pref $b_1\vert Y_1$ & $b_2\vert Y_2, \cdots$ and $b_1\vert Y_1$ & $b_2\vert Y_2, \cdots$ pref $c_1\vert Z_1$ & $c_2\vert Z_2$, then $a_1\vert X_1$ & $a_2\vert X_2, \cdots$ pref $c_1\vert Z_1$ & $c_2\vert Z_2, \cdots$
Additivity*	If $\alpha\vert X$ pref $\beta\vert X$ and $\gamma\vert X$ pref $\delta\vert X$ then $\alpha + \gamma\vert X$ pref $\beta + \delta\vert X$.	If $\alpha\vert X$ pref $\beta\vert Y$ and $\gamma\vert X$ pref $\delta\vert Y$, then $\alpha + \gamma\vert X$ pref $\beta + \delta\vert Y$.	If $a_1\vert X_1$ & $a_2\vert X_2, \cdots$ pref $b_1\vert Y_1$ & $b_2\vert Y_2, \cdots$ and $c_1\vert Z_1$ & $c_2\vert Z_2, \cdots$ pref $d_1\vert V_1$ & $d_2\vert V_2, \cdots$, then $a_1\vert X_1$ & $a_2\vert X_2, \cdots$, $c_1\vert Z_1$ & $c_2\vert Z_2, \cdots$ pref $b_1\vert Y_1$ & $b_2\vert Y_2, \cdots$, $d_1\vert V_1$ & $d_2\vert V_2, \cdots$
Continuity*	Let X be an element of K_1 and α and β any two elements of K_2' such that $\alpha\vert X$ pref $\beta\vert X$. There exists an element ε of K_2' such that $\varepsilon\vert X$ depref $0\vert X$ and $\alpha + \varepsilon\vert X$ pref $\beta\vert X$.	Let X and Y be any two elements of K_1, and α and β any two elements of K_2' such that $\alpha\vert X$ pref $\beta\vert Y$. There exist elements ε of K_2' such that $\varepsilon\vert X$ depref $0\vert Y$ and $\alpha + \varepsilon\vert X$ pref $\beta\vert Y$.	
Connectivity*	If $a\vert X$ pref $b\vert Y$ and $c\vert X$ pref $d\vert Y$ seq b, then $a + c\vert X$ pref $b + d\vert Y$.		
Reversitility	If b is a reverse of a with respect to X, then $a + b\vert X$ equiv $0\vert X$.		
Contractivity	$\alpha\vert X$ & $\beta\vert X$ equiv $\alpha + \beta\vert X$.		

For certain purposes it is convenient to consider choice objects that are "small." For these, and these only, the local and interlocal additivity postulates are admitted. Such small objects will be denoted by $\alpha, \beta, \gamma, \ldots$ The sum of two objects a and b is defined as the object which consists of receiving (starting from a given reception point) first a and then b ("first" and "then" only meaning order, but not implying that the rapidity of receipt over time is relevant).

We shall assume that the accepting of a in X implies that a new reception point is realized, uniquely defined by a and X (the point "X followed by a").

Now let us pass to an exact postulational specification.[2] In the following the equality sign $=$ will be used (as part of the language) to express the fact that the two logical elements thus compared may be substituted for each other anywhere.

By adopting various combinations of these postulates it was proved that certain vectors and integrals exist and have the meaning of *choice indicators.*

Thursday, July 15 — The Method of Postulates, PROFESSOR HUNTINGTON.[1]

The method of postulates may be briefly described as a recently developed method for clarifying and simplifying the processes of argumentation. Although it was first developed

[2]My thanks are due to Professor E. V. Huntington for several valuable suggestions regarding the symbolic presentation of the logical base.

[1]This lecture was originally delivered at Brunswick, Maine, April 13, 1937, as part of the Institute of Philosophy held from April 6 to 16 at Bowdoin College. It will be published in full in *Philosophy of Science.*

Note to Table on p. 68:

*If any of these postulates is admitted, they are assumed to hold also for the other signs, just as the similar relations $>$, $<$, and $=$ in ordinary algebra. If local determinateness and transitivity hold, the elements K_2 form a partly ordered set when combined by $|$ with any element of K_1. If interlocal determinateness and transitivity hold, the simple elements of K_3 form a partly ordered set; and similarly for the collectional elements of K_3 if collectional determinateness and transitivity hold.

within the field of mathematics, the method itself has a range of applicability which includes not only the physical sciences, but the social sciences as well. In particular, some attempt to apply the postulational method to the science of economics would appear to be especially useful and timely at the present stage of development in that science.

The first inkling of the postulational method dates back to Euclid's *Elements of Geometry*, although the real significance of the method was not appreciated by the Greeks. The earliest influential example of the method was the discovery of non-euclidean geometry by Bolyai and Lobatschewski (about 1830). The next landmark is the publication of George Boole's so-called *Laws of Thought* in 1854. The systematic development of postulational technique began with Peano and his school in Italy about 1890, and ten years later Hilbert's *Foundations of Geometry* attracted wide attention to the method. The next landmark is the publication of *Principia Mathematica* by A. N. Whitehead and Bertrand Russell in 1910, leading to widespread interest in the postulation foundations of logic. (For a complete bibliography of this field, see the current issue of the new *Journal of Symbolic Logic.*) Within the last few years so many papers have been published in the general field of postulate theory that it would be invidious to attempt to select any short list to be mentioned in this place.

The postulational method is primarily a method of classification of observed systems according to their logical structure. Every observed system must be recognized as belonging to some definite "universe of discourse," and as dealing with definite types of entities and relations within that universe. For example, Boolean algebra is a system dealing with a class of elements, a binary rule of combination, a unary operator, and a dyadic relation. An economic system recently presented by Dr. Frisch deals with three classes of elements, two subclasses, four binary operators, one unary operator, and two dyadic relations. In any case the entities and relations must be regarded as abstract symbols which possess no properties except those which are explicitly set forth in agreed-upon "postulates." From any consistent set of "postulates," we may then proceed to deduce "theorems" by purely mathematical reasoning. Neither the postulates nor the theorems have any quality of truth or falsity. All that we can say is that any system which has the properties set down in the postulates will also have, automatically, all the properties stated in the theorems.

Having thus classified, in abstract symbols, various systems belonging to some interesting universe, the next step (and the more difficult one) is to ascertain, by empirical observation, whether some given concrete system belongs to one or other of these classes. If it does, the further study of the concrete system is greatly simplified by our knowledge of the abstract properties of the whole class of systems to which it belongs.

The abstract postulational method is especially useful in rendering clear and precise the exact meanings of terms which are introduced by definition, for the purpose of abbreviation. (The choice of definitions in any scientific theory is a matter for the selective ingenuity of the author of the theory.)

Technical details which may be mentioned are: the proofs of independence (or nonredundancy) of the postulates; the possibilities of finding a "categorical" set of postulates in any given field; the possibility of reducing the number of undefined concepts in any universe; etc.

The method of postulates has been largely responsible for the astounding success achieved in mathematics and the other exact sciences. In the opinion of many forward-looking thinkers, the progress of civilization may be said to depend directly on the degree to which this same method is applied in economics and the other social sciences.

Thursday, July 15 — An Exact Theory of Social Relations and Groups, KARL MENGER, Professor of Mathematics, University of Notre Dame.

The sociological theory presented in this paper is related to the research of econometricians not in its results but in its spirit.[1] The spirit of many other sociological theories is very different. In some German standard works on sociology we used to find statements beginning with the words "thus," "hence," "therefore," which are not at all implied by what was said before. The unsatisfactory form of these books seems to come from two reasons: the lack of logical training of the authors but also the very nature of the topic. For statements on the content of rules, the nature of the good and just, the meaning of institutions, etc. (which form the topic of these theories) can only very rarely be logically connected with each other.

Aside from these *"intentional"* theories the author has tried to develop an *"extensional"* theory *dealing with the relations between human individuals and groups which result from the fact that the individuals and groups are partly similar and partly different from each other in their properties as well as in their tastes and attitudes.* The method of this theory consists in starting from simple assumptions as to these properties and attitudes (e.g., that a group of men is divided into two subgroups, that each member has the same attitude towards each two members who belong to the same group, etc.). Gradually more complicated assumptions are introduced (e.g., that the total group is divided into three subgroups, that the attitude of a man towards another depends not only on the properties of the other but also on the attitudes of the other, etc.). In each case conclusions are rigorously deduced from the assumptions. A statement begins with the word "therefore" if and only if it is implied by the assumptions.

Although a theory of this type is not part of what is usually called economics it may eventually have applications to the theory of economic policy. The latter theory has consisted so far of advice: This measure ought to be adopted, that measure ought to be avoided. Authors with better logical

[1]Cf. the author's paper, "An Exact Theory of Social Relations and Groups," to be published soon. Also his book, *Moral, Wille und Weltgestaltung, Grundzüge einer Logik der Sitten,* Wien, Springer, 1934; and the lecture, "Einige neuere Fortschritte in der exakten Behandlung sozialwissenschaftlicher Probleme," in the book, *Neuere Fortschritte in den exakten Wissenschaften,* Wien, Deuticke, 1936.

training used to announce their advice in the more careful form: If the aim A is to be attained, then this means ought to be adopted and that means ought to be avoided. In this latter form the advice has, as a matter of fact, the form of a prediction: Adoption of this means is followed by attaining the aim A, adoption of that means is followed by not attaining the aim A. As to the aim A, we find authors who take it for granted naturally, others who claim to deduce it, others who simply state what it implies. But all authors so far have studied *one common aim.* It is, however, an empirical fact that different men have different aims.

Our above-mentioned sociological theory gives a survey of possible forms of organization of a group, and also includes the case where different subgroups of the group follow different aims $A_1, A_2, \cdots$. A survey of this kind may lead to hints concerning new forms of organization. The theory does not claim to give advice as to which form of organization ought to be adopted, for it fully realizes that the actual formation of the world is left to voluntary decisions of individuals and of groups. But the suggestion of new possibilities may be of practical use, since it enlarges the field in which we may choose.

Friday, July 16 — The *Fonctionnelle* Nature of Utility, PROFESSOR DAVIS.

The concept of utility has always been an attractive, but somewhat elusive one in economics. The search for a measure of utility may be compared with the search of the physicists for potentials in the theory of thermodynamics. In that branch of science the differential relationship: $dE = dQ + dW$, where E is the internal energy, Q is heat, and W work, presented an important question: Which of these quantities is a function and which is a functional? It was one of the supreme discoveries of science that the quantity dQ/T, the differential of entropy, is exact and hence that its integral is a true potential function. Both Q and W are functionals, but E is again a potential function of the variables of state.

It is interesting to observe that a similar question occurred to V. Pareto as early as 1906 as the result of a suggestion due to Vito Volterra, the father of the modern theory of functionals. In his *Manuel d'économie politique,* Paris, 1909, citing an earlier paper in the *Giornale degli Economisti,* Pareto defines as an index of ophelimity (utility) the integral

$$I = \int A dx + B dy + \cdots + N dt,$$

where $x, y, \cdots, t$ are commodities and $A, B, \cdots, N$ functions of these variables. He then says (p. 556):

> If one then considers a commodity path, which departing from a point $x, y, \ldots, t$ returns again to this point, one says that he has traversed a *closed cycle* provided he returns to this point with the same index of ophelimity with which he left it. This case corresponds to indifference in the order of consumption.
>
> One will say that he has traversed an *open cycle,* if he returns to the point of departure with an index of ophelimity different from that with which he departed. This case corresponds to that in which the order of consumption influences the pleasure which it occasions.

Several examples of the use of the concept of utility occur in economics. The first of these is the classical definition by Bernoulli of the marginal utility of money, which he introduced in order to explain the St. Petersburg paradox. The theory of equilibrium between the marginal utilities of a set of commodities, their prices, and the marginal utility of money appears in the recent budgeting studies of Allen and Bowley. The proof [see Bibliography: Allen (7)] that the true index of the change in the cost of living lies between the index numbers of Laspeyres and Paasche, is an example of the usefulness of the utility concept. The attempt of Ragnar Frisch to make a statistical measurement of marginal utility through his concept of money flexibility may also be cited.

In pursuing the isomorphism with thermodynamics, which seems to be indicated by inherent aspects of the economic problem, the following set of postulates is set up:

1. That there exists a differential of expenditure

$$dE = P dQ,$$

where P is a price index and Q a measure of the quantity of goods purchased.

2. That there is a differential of income, I,

$$dI = dS + dE,$$

where S denotes savings.

3. That there is an equation of state

$$PQ = W,$$

where W is the total value of goods purchased.

4. That there exists a money utility function, U, the differential of which is

$$dU = F(W)\,dI,$$

where $F(W)$ is a function which satisfies the condition

$$\lim_{W=\infty} F(W) = 0.$$

5. That S plays the rôle of a potential function, that is

$$S = \int_C (dI - dE) = 0,$$

where C is a closed path in the variables of state.

6. That money utility is a potential function, that is,

$$U = \int_C F(W)\,dI = 0$$

where C is a closed path in the variables of state.

Several results inherent in this system of postulates were developed. Thus if the coefficients A, B, C, and D are defined by the differentials

$$di = AdP + BdW, \quad Q \text{ constant};$$

$$dI = CdQ + DdW, \quad P \text{ constant};$$

it may be shown that as a consequence of the postulates the following relationship holds between C and D:

$$C - D = -1/\eta(W),$$

where $\eta(W) = d(\log F)/d(\log W)$ is the marginal elasticity of money. This relation is within the reach of statistical verification.

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Friday, July 16—Utilities and Probabilities in Human Choice, J. MARSCHAK, University of Oxford.

Many problems discussed in treatises on the Logic of Probability involve human decisions: cf. Venn, Borel, Keynes, Jeffreys; and, earlier, Bernoulli's "moral expectation," a mixture of the probability and utility concept. The purpose of the paper is to throw light on such problems by introducing *estimated probabilities of future events* into the Theory of Choice familiar to economists. Such a generalization of the Choice Theory is fundamental for any economic analysis of investing, gambling, speculating, insurance, and entrepreneurship.

The problem is approached in successive stages, or rational patterns, at all of which two kinds of data are necessary to explain rationally the individual's decision: (a) his preferences, or tastes, described in a list of all possible combinations of r "things" x, with information added as to which combination is preferred to which — a family of $(r—1)$ -

dimensional indifference surfaces $f(x_1, \cdots, x_r) = c$; (b) his opportunities or resources, i.e., a list of all combinations accessible to the individual (by production or exchange) — an r-dimensional region bounded by an opportunity surface $g(x_1, \cdots, x_r) = 0$. The number of "things" considered becomes larger as we proceed from stage to stage, but the formal problem is always expressed as "$f =$ max., subject to $g = 0$," and is determinate.

Stage 1 can be described as the consumer's situation, stages 2-3 as the situation of a safe investor, stages 4-5 idealize the situation in games of chance and in insurance, stages 6-7 are rational patterns of entrepreneurship. All these stages 1-7 explain the individual's equilibrium when his tastes and the opportunities as he sees them in a given point of time are given. The change of these data in time gives rise to stage 8.

Stage 1. "Consumption." Choice between combinations of n *final goods* $y_1, \cdots, y_n$, accessible to the individual. f and g are functions of the y's.

Stage 2. "Short safe investment." Choice between combinations of n final goods $y_1, \cdots, y_n$ at t points of *time*. f and g are functions of $y_{11}, \cdots, y_{nt}$.

Stage 3. "Long safe investment." Choice between combinations of m *assets* $a_1, \cdots, a_m$ at t points of time (time series of balance sheets, or "plan") accessible to the individual and producing jointly n final goods $y_1, \cdots, y_n$ at t points of time. As before f is a function of $y_{11}, \cdots, y_{nt}$, but g is a function of the assets $a_{11}, \cdots, a_{mt}$. In addition, however, the individual supposes to know *nt production functions* of the form $y_{ik} = h_{ik}(a_{11}, \cdots, a_{mt})$. By substituting these expressions we transform $f(y_{11}, \cdots, y_{nt})$ into, say, $F(a_{11}, \cdots, a_{mt})$ and are thus able, at this and all the following stages, to express both tastes and opportunities in terms of assets, viz., $F(a_{11}, \cdots) = c$, $g(a_{11}, \cdots) = 0$.

Stage 4. "Games of Chance." Choice between combinations (plans) of m *known probability distributions* of m assets in t points of time, each plan producing n known probability distributions of the n final goods at t points of time. Each probability distribution may be described by u parameters. We have then: $F(a_{11}{}^1, \cdots, a_{mt}{}^u) = \max$; $g(a_{11}{}^1, \cdots, a_{mt}{}^u) = 0$.

Example: $u = 2$ parameters, viz., the mean (mathematical expectation) and the range ("risk"). E.g., in roulette, put either (1) \$1 on red, or (2) \$100 on red, or (3) \$1 on a figure. Mathematical expectation is 0 in all cases, but the range is respectively: \$2, \$200, \$36. (The necessity of thus taking account of some second distribution parameter as a measure of risk was neglected by Bernoulli but mentioned more and more explicitly by Marshall, I. Fisher, J. Hicks, and K. Menger. The tastes (indifference line in the plane "gain, risk") and the opportunities (in the same plane; they are mainly given by the man's property), will decide the choice.

Stage 5. "Insurance." Same as stage 4 but the simultaneous distribution p of the variates (e.g., described by the means, standard deviations, and correlations) is taken into account; example: the "pooling of risks" point of view, so important for insurers and investment trusts. Using functional notation,

$$F[p(a_{11}, \cdots, a_{nt})] = \text{max.}, \quad g[p(a_{11}, \cdots, a_{nt})] = 0.$$

Stage 6. "Statistically-minded enterprise." Here it is not assumed any more that the individual is supposed to know the parameters of the relevant distribution. He has to estimate them. This is the main distinction between games of chance and enterprise (F. H. Knight). Different individuals will estimate with narrower or larger "confidence limits" in Neyman's sense: the more precise the estimate (whether right or wrong) the more "enterprising" is the man. With 2 confidence limits α_i' and α_i'' for each of v parameters $\alpha_1, \cdots, \alpha_v$ of the simultaneous distribution we obtain: $F(\alpha_1', \cdots, \alpha_v'') = \text{max.}$, $g(\alpha_1', \cdots, \alpha_v'') = 0$. Or, introducing R. A. Fisher's likelihood functions l, we have, in functional notation:

$$F[l_1(\alpha_1), \cdots, l_v(\alpha_v)] = \text{max.}, \quad g[l_1(\alpha_1), \cdots, l_v(\alpha_v)] = 0.$$

Stage 7. "Enterprise." Besides sampling, the entrepreneur uses any other source of evidence to make the estimates. By multiplying, with Bayes, the a priori probability of each small value range of a parameter by the likelihood of this value range, we obtain weighted likelihood functions as the objects of the individual's choice (cf. Fry).

Stage 8. "Good and Bad Luck." The quantities of goods do not depend, in the preceding stages, on whether the expectations have proved to be right or wrong. The relative scarcity of "vegetarians" (stage 1), lenders (stages 2-3), gamblers (stage 4-5), entrepreneurs (stage 6-7) on the market influences the exchange ratios in their favour, thus creating "rewards" for, respectively, "meat abstinence," patience, audacity, enterprising — irrespective of the actual validity of expectations. Each individual maximizes his $F(a_1', \cdots, a_v'')$ according to his estimates; only to learn, at the next point of time, that his estimates were more or less wrong and that he missed the maximum. The better were the estimates the smaller is his "loss due to malinvestment" — a dynamic or "windfall" loss quite independent of the reward for entrepreneurship.

If utilities (degrees of desire) and/or probabilities (degrees of belief) are regarded as ordinal numbers only, arbitrary monotonically increasing functions of the utility functions F and of the probability functions p must be used; this does not affect the solutions.

Monday, July 19—The Analysis of Variance for Two or More Variables, S. S. WILKS, Assistant Professor of Mathematics, Princeton University.

For the past ten or fifteen years the method of the analysis of variance, due to R. A. Fisher, has been a widely used device for making significance tests in certain types of exploratory and experimental work, particularly in some of the biological sciences. The method is essentially a highly specialized form of the multiple and partial correlation technique used in conjunction with probability theory. The specialization has been made in the direction of placing greater attention on the selection of factors $F_1, F_2, \cdots, F_k$ and the corresponding independent variables $x_1, x_2, \cdots, x_k$, and on the choice of linear relationships among the regression coefficients b_1,

$b_2, \cdots, b_k$ in the regression equation

$$(1) \qquad Y = a + b_1x_1 + b_2x_2 + \cdots + b_kx_k.$$

The expression for Y is the mean value of a dependent variable y for given values of the x's. In many of the problems to which the analysis of variance has been applied the independent variables have been chosen so that they take on the values 0 or 1, 0 being assigned when a corresponding factor is absent and 1 when it is present.

The main assumptions involved in the analysis of variance are: (a) that the unknown constants $a, b_1, \cdots, b_k$ enter into equation (1) linearly; (b) that the residuals

$$\varepsilon = (y - a - b_1x_1 - \cdots - b_kx_k)$$

be normally distributed with a given variance σ^2 for all configurations of values of the x's which arise. The actual application of probability is made in connection with testing the hypothesis that a given set of the b's are zero. In principle it is as follows: In testing the hypothesis that $b_{r+1}, \cdots, b_k$ are zero from the information furnished by a sample of observations on n individuals, we calculate the minimum of the sum of squares $\sum\varepsilon_i^2$ for variations of the a and all b's; denote the minimum by S'. The minimum is then calculated with $b_{r+1}, \cdots, b_k$ all set equal to zero; denote this minimum by S''. Then assuming $b_{r+1}, \cdots, b_k$ all to be zero and that there are no linear restrictions among the b's, S' and $S'' - S'$ are independently distributed sums of squares for estimating σ^2 with $n - k - l$ and $k - r$ degrees of freedom respectively. Fisher's z test can then be applied to these two estimates for testing the hypothesis that $b_{r+1}, \cdots, b_k$ are zero. The constituents for making the test can be set up in the familiar form as shown in Table I.

TABLE I

Variance due to	Degrees of Freedom	Sum of Squares	Mean Sum of Squares
Factors $F_{r+1}, \cdots, F_k$	$k-r$	$S''-S'$	$m_1 = (S''-S')/(k-r)$
Error	$n-k-1$	S'	$m_2 = S'/(n-k-1)$
Total	$n-r-1$	S''	$z = (1/2)\log(m_1/m_2)$

In case there are linear restrictions among the b's, and the x's take only the values 0 and 1, as in the various layouts used in agronomy, the same procedure is applied, but S' and S'' are usually simple sums of squares.

The analysis of variance has been confined, in its application, to problems in which there is only one dependent variable y. There are situations in which it may be useful to consider several interrelated dependent variables simultaneously in conjunction with the given independent variables $x_1, \cdots, x_k$. Consider the case of two dependent variables y and y', and let the regression equations be

$$(2) \qquad \begin{aligned} Y &= a + b_1 x_1 + \cdots + b_k x_k, \\ Y' &= a' + b_1' x_1 + \cdots + b_k' x_k. \end{aligned}$$

The hypothesis to be tested is that y and y' are jointly independent of $x_{r+1}, \cdots, x_k$; in other words that $b_{r+1}, \cdots, b_k$ and $b_{r+1}', \cdots, b_k'$ are all zero. For a sample of n individuals we can find the minima of the sums of squares

$$\sum (y_i - Y_i)^2 \quad \text{and} \quad \sum (y_i' - Y_i')^2$$

corresponding to those in Table I. Let the minima of the first sum of squares be S_{11}' and S_{11}'' and those of the second sum of squares be S_{22}' and S_{22}''. There will be corresponding product terms from $\sum (y_i - Y_i)(y_i' - Y_i')$ where Y and Y' are to be substituted from those obtained by minimizing the sums of squares under the two conditions respectively. Denote these product terms by S_{12}' and S_{12}''. The constituents are shown in Table II.

TABLE II

Variance due to	Degrees of Freedom	Sum of Squares (y)	Sum of Squares (y')	Products
Factors $F_{r+1}, \cdots, F_k$	$k-r$	$S_{11}''-S_{11}'$	$S_{22}''-S_{22}'$	$S_{12}''-S_{12}'$
Error	$n-k-1$	S_{11}'	S_{22}'	S_{12}'
Total	$n-r-1$	S_{11}''	S_{22}''	S_{12}''

The criterion for testing the hypothesis that y and y' are independent of the factors $F_{r+1}, \cdots, F_k$ as measured by $x_{r+1}, \cdots,$ x_k is the ratio of determinants

$$(3) \qquad U = \frac{\begin{vmatrix} S_{11}' & S_{12}' \\ S_{12}' & S_{22}' \end{vmatrix}}{\begin{vmatrix} S_{11}'' & S_{12}'' \\ S_{12}'' & S_{22}'' \end{vmatrix}},$$

a quantity having values between 0 and 1. The smaller the value of U, the more highly significant the discrepancy between the observations and this hypothesis of independence. Under the assumption of independence and the normality of residuals, U has the distribution

$$(4) \qquad f(U)\,dU = K(\sqrt{U})^{n-k-4}(1-\sqrt{U})^{k-r-1}\,dU,$$

where K is a constant, from which critical values of U for any desired significance level can be found from the *Tables of the Incomplete Beta-Function.*

The validity of the use of U can be established by means of the principle of maximum likelihood, and also geometrically. The geometrical grounds for the criterion are similar to those used by Frisch in his notions of *scatterance.* U can be expressed in various other forms which are more convenient for computation. In the case of one dependent variable U reduces to a simple function of Fisher's z.

If there are more than two independent variables, say t, then U will be the ratio of two determinants of sums of squares and second-order moments of order t. The exact distribution is more complicated than in the case $t = 2$, but for large samples $-n \log U$ will be approximately distributed like χ^2 with $t(k-r)$ degrees of freedom, thus making significance tests fairly simple in large samples.

Monday, July 19—Index-Number Bias, W. V. LOVITT, Professor of Mathematics, Colorado College.

One requisite of a good index number is that it shall satisfy the time-reversal test, which requires that the product of the index number for any given year on a base year by the index number for the base year on the given year shall be unity. Thus, if an article sells in 1929 for sixty cents and in 1937 for forty cents the index number of price with 1929 for base is 2/3; the index number with 1937 for base is 3/2; the product of 2/3 and 3/2 is unity. For one article, with prices unweighted, this test is always satisfied.

If this product is greater than unity, the index number is said to have an upward bias. If this product is less than unity, the index number is said to have a downward bias.

Many formulas have been devised for computing an index number. It is of interest to know whether an index number, computed from one of the formulas, is too large or too small. It is also of interest to compare index numbers computed from two different formulas. Will an index number computed from formula A always be larger (smaller) than the corresponding index number computed from formula B?

For a number of the more often used index numbers, the author has answered the above questions.

For all index numbers considered in the present paper no permanent-type bias exists. To prove this it is sufficient to exhibit sets of values such that for one set the index number obtained has a downward bias and for another set the index number obtained has an upward bias.

Let us represent an index number computed from

a simple arithmetic average of relatives by A,
a simple harmonic average of relatives by H,
a simple geometric average of relatives by G,
the median of relatives by Md.

The four following methods of weighting by value are in current use:

I. Each weight equals base-year price times base-year quantity (p_0q_0);

II. Each weight equals base-year price times given-year quantity (p_0q_1);

III. Each weight equals given-year price times base-year quantity (p_1q_0);

IV. Each weight equals given-year price times given-year quantity (p_1q_1).

Let the corresponding weighted index numbers be represented by A_i, H_i, G_i; $i = 1, 2, 3, 4$.

We show that H_i, G_i have no permanent bias.

We have made the following comparisons and find that if unequal the inequality is not permanently directed:
G_1 vs. G_4; G_1 vs. G_2; G_2 vs. G_3; G_3 vs. G_4; A_i vs. Md; H_i vs. Md; H_i vs. H_j; A_1 vs. H_1; A_1 vs. G_4; A_2 vs. G_2; H_1 vs. G_2; H_2 vs. G_1; H_3 vs. G_2; H_3 vs. G_2; H_3 vs. G_4; H_4 vs. G_1; H_4 vs. G_3.

Tuesday, July 20—The Influence of Income and Interest Rate on Savings, DR. MARSCHAK.

P. Douglas has summarized graphically the wide variety of contradicting hypotheses set a priori by various writers on the influence of interest rate on national savings. Similarly, Mr. Keynes' "psychological law" that the marginal propensity to save increases with income has been occasionally disputed. A correlation analysis of the relevant time series cannot yield safe results unless we either assume that the relationship studied remained unchanged during the time under observation, or that it changed according to some known time shape.

It is, however, possible to estimate the relationship between individual savings and incomes for a given point of time; and to establish some theoretical relationships between the "elasticity of individual savings with respect to incomes" (s_i) and the "elasticity of individual savings with respect to interest rate" (s_ρ), and thus to evaluate the latter elasticity as well; it is further possible, with the existing statistical data, to use these estimates for the evaluation of the corresponding "national" elasticities.

Write (for an individual): s = savings; i = income; ϱ = interest rate (accumulated continuously); t = accumulation time; w = wage income; c = capital; $m = c\varrho/i$; $s_i = d \log s/d \log i$, (ϱ = const.); $s_\rho = d \log s/d \log \varrho$, ($i$ = const.); $s_\rho' = d \log s/d \log \varrho$, ($i = c\varrho + w$). (1)

For Germany, 1927-28, $s_i = 1.52$, as found from Wagemann's figures (based on original income-tax material) of capital changes in various income groups. For U.S.A., 1929, the perusal of the family-budget estimates of the Brookings Institution (*America's Capacity to Consume*) yielded $s_i = 1.235$; $s_i > 1$ confirms Keynes' "psychological law."

Writing $\log s = \varphi(\log \varrho, \log i)$ and differentiating completely with respect to $\log \varrho$, we get $s_\rho' = s_\rho + ms_i$. Further, writing $f = se^{\rho t}$ for the value of savings in t years, and $f_\rho = d \log f / d \log \varrho$, it follows that

$$s_\rho = f_\rho - \varrho t, \quad s_\rho' = f_\rho - \varrho t + ms_i'. \tag{2}$$

The negative term $-\varrho t$ may be called Marshall effect, and the positive ms_i', Carver effect. There remains f_ρ (Böhm-Bawerk effect). Write for the "elasticity of substitution between present and future goods" $\sigma = d \log \frac{f}{i-s} / d \log e^{-\rho t}$, the differentials being taken along the indifference line defined by the condition $e^{-\rho t} df + d(i - s) = 0$. It will be noticed that, in the definition of σ, $f/(i - s)$ is the ratio of the amount of future goods to the amount of present goods purchased by the individual, and $e^{-\rho t}$, the ratio of their respective prices. Then, by the Hicks-Allen theorem (*Economica*, 1934), the elasticity of demand for future goods $= d \log f / d \log e^{-\rho t} = f_\rho / \varrho t = s_i s/i + \sigma(i-s)/i$. (The paper contained an easy proof of the Hicks-Allen theorem for two goods.) From the last equation we find f_ρ, and, inserting it into (2), we get

$$s_\rho' = ms_i - \varrho t[1 - s_i s/i - \sigma(i-s)/i]. \tag{3}$$

Now by definition (1) $s = a\varrho^{s_\rho}$ where a is a constant. As m and s are known for the various income groups, and the income distribution within the nation is also known, it is possible to integrate $s = a\varrho^{s_\rho}$ over the whole range of incomes. Let national savings $S =$

$$a \int_0^\infty e^{s_\rho} N(i)\, di,$$

where $N(i)$ is the distribution density of incomes. Then $d \log S / d \log \varrho$, the "national elasticity of savings with respect to interest rate," does not depend on a and can be calculated from

(3) provided σ is known within limits, or can be neglected — a probable assumption. Of course, the result does not enable us to evaluate the important dynamic relationship between savings and the rate of income change in time, or the income of the preceding year.

Tuesday, July 20 — A Problem of Laplace, T. H. RAWLES, Dean of Freshmen, Colorado College.

Laplace[1] proposed the following problem; "Let us consider now two urns A and B each holding N balls, and let us suppose that in the total number of balls, $2N$, there are as many white as black. Let us imagine that someone draws at the same time a ball from each urn, and then puts in each urn the ball drawn from the other. Let us suppose that this operation is repeated any number of times, r, each time shaking the urns to mix the balls well; and let us seek the probability that after r operations there will be x white balls in urn A." In his treatment Laplace approximated difference expressions by first- and second-order derivatives and obtained in that manner the second-order partial differential equation which yielded his solution.

Molina[2] has recently called attention to several applications of this problem, notably, to one from Lotka.[3] It occurred to the author of this note that an operational technique which he used in a problem in biological theory would be useful in this, and that proved to be the case.[4]

We shall call $s_r = \sum a_{rn} x_n$ the state of the system after r operations, the coefficients a_{rn} indicating the probability that there are n white balls at this state. The operator

$$\Delta = \sum \left(\frac{n^2}{N^2} x_{n-1} + \frac{2n(N-n)}{N^2} x_n + \frac{(N-n)^2}{N^2} x_{n+1} \right) \frac{\partial}{\partial x_n}$$

[1]Laplace, *Theorie Analytique des Probabilités*, 1847, pp. 314-332.
[2]Molina, *Scientific Monthly*, July, 1937, pp. 55-57.
[3]Lotka, *Elements of Physical Biology*, 1925.
[4]Rawles, *Human Biology*, Vol. 8, No. 1, pp. 126-132.

serves to change each state into the succeeding one.

The infinite sum $S(e) = s_0 + s_1 e + s_2 e^2 \cdots$ satisfies the partial differential equation

$$s_0 = S - e\Delta S,$$

whose solution may be found in the form

$$S = \sum A_n(e) x_n.$$

We obtain a system of linear equations in the A_n, the determinant of which may be factored as follows,

$$D = \Pi \left\{ 1 - \frac{[(N-p)^2 - p]e}{N^2} \right\},$$

the factors being obtained by the method described in detail elsewhere.[5] Employing partial fractions, we find

$$A_n(e) = \sum_{p=0}^{N} \frac{C_{pn}}{1 - \frac{[(N-p)^2-p]e}{N^2}},$$

and

$$a_{rn} = \sum C_{pn} \left(\frac{(N-p)^2 - p}{N^2} \right)^r.$$

The expression

$$\left(\frac{(N-p)^2 - p}{N^2} \right)^r$$

approaches zero as r approaches ∞, except when $p = 0$, in which case it is always 1. It is from this term, then, that we determine the probability of the various possible states when r becomes large. In other words the ultimate probability of having n white balls in urn A is given by

$$C_{0n} = \frac{(N!)^4}{(2N)![n!(N-n)!]^2}.$$

This result is independent of the original state of the system and, as one might expect, it is identical with the distribution which would be obtained if the $2N$ balls were dealt at random into the two urns.

[5]Rawles, *loc. cit.*, pp. 128-129.

Finally, if we put $n = N/2 + s$ and make use of Stirling's formula, we find

$$F(s) = \frac{2}{\sqrt{N\pi}} e^{-4s^2/N}$$

as the distribution function when N is large. This corresponds to one of the results obtained by Laplace.

Wednesday, July 21 — The Maximization of Utility Over Time,[1] GERHARD TINTNER, Research Fellow, Cowles Commission (now Assistant Professor of Economics and Mathematics, Iowa State College).

This problem is studied under two aspects: 1) The discontinuous case, where consumption takes place at n discontinuous points in time. 2) The continuous case, where consumption is continuous over an interval of time.

Under the assumption that only three goods, x, y, z, are consumed we define in the discontinuous case a utility function F which depends on the quantities consumed of every good at the points in time $1 \cdots n$. Let p_j, q_j, r_j, be the prices of x, y, z expected for the point in time j, I_j the expected income, s_j the saving, E_j the expenditure and i_j the rate of interest expected. We then have to maximize a function G:

$$G = F + \lambda_1[I_1 - E_1 - s_1] + \lambda_2[I_2 + s_1(1 + i_1) - E_2 - s_2] + \cdots + \lambda_n[I_n + s_{n-1}(1 + i_{n-1}) - E_n)],$$

where the λ are Lagrange multipliers and the subsidiary conditions result from relations between income, saving, and expenditure. From the solutions we deduce first that the λ_j, marginal utilities of money expected for the point in time j, are equal to the marginal utility of money expected for the point in time 1 discounted at the expected rates of interest. Further-

[1]The full paper has been accepted for publication in *Econometrica*.

more, the sum of the expected incomes and the expected money value of expenditure at every point in time, discounted at the expected rates of interest, must be equal. The solution can be put into the following form:

$$\frac{\partial F}{\partial x_j}\frac{1}{p_j}=\frac{\partial F}{\partial y_j}\frac{1}{q_j}=\frac{\partial F}{\partial z_j}\frac{1}{r_j}=\frac{\lambda_1}{(1+i_1)(1+i_2)\cdots(1+i_{j-1})}\,.$$

The weighted marginal utilities (i.e., the marginal utilities divided by the prices) of the quantities of the commodities which the individual expects to consume at the point in time j must be equal to the discounted marginal utility of money at the point 1.

In the continuous case utility is defined as a functional f of the totality of the values of the quantities of the commodities x, y, and z consumed between the points in time 0 and n. The problem is again to find the maximum of a functional g, defined as

$$g=f[x\overset{n}{\underset{0}{(t)}},y\overset{n}{\underset{0}{(t)}},z\overset{n}{\underset{0}{(t)}}]-\Lambda\int_0^n\{p(t)x(t)$$

$$+q(t)y(t)+r(t)z(t)\}\,e^{-\int_0^t\rho(s)ds}\,dt,$$

where $\varrho(s)$ is the expected force of interest, and Λ again a Lagrange multiplier. If we designate by $f'_x[x,y,z;t]$ the functional derivative of the utility functional with respect to x at the point in time t and similarly for y and z, the equilibrium conditions can be written:

$$\frac{f'_x[x,y,z;t]}{p(t)}=\frac{f'_y[x,y,z;t]}{q(t)}=\frac{f'_z[x,y,z;t]}{r(t)}=\Lambda\,e^{-\int_0^t\rho(s)ds}$$

Or at every point in time t the weighted functional marginal utility of every commodity consumed ought to be equal to the marginal utility of money at the point 0 discounted by the expected rates of interest.

It is also possible to use the concepts of marginal rates of substitution introduced by Hicks and Allen in their celebrated articles in *Economica,* 1934. In the discontinuous and in the continuous case we can substitute any index function with positive derivative for the utility function or functional since the solution does not involve the marginal utilities or functional marginal utilities themselves but only their ratios. Hence it is possible to formulate the solutions in terms of marginal rates of substitution and functional marginal rates of substitution defined as these ratios.

Thursday, July 22—The Application of the Analysis of Variance with Three or More Criteria of Classification to the Study of Time Series, HARRY PELLE HARTKEMEIER, Associate Professor of Business Statistics, University of Missouri.

Monetary theorists have discussed the interrelationship of the various factors affecting the supply and use of member-bank reserve funds. This interrelationship affects the seasonal variation in one of the factors, bills discounted, as can be shown easily and advantageously by the analysis of variance.

BILLS DISCOUNTED FOR MEMBER BANKS, 1922-1931. MILLIONS OF DOLLARS

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Federal Reserve Districts	11	1,522,936.77	138,449
Years	9	455,212.12	50,579
Months	11	18,632.45	1,694
Interactions			
Districts and years	99	484,172.87	4,891
Districts and months	121	42,195.92	349
Months and Years	99	168,060.22	1,698
Remainder (Triple interaction)	1089	253,997.19	233
	1439	2,945,207.54	

The first six mean squares are significantly larger than triple interaction, which is the best available measure of experimental error. The presence of seasonal variation is indicated by the mean square 1,694. As there is not much evidence of secular trend in a graphic presentation of the data, the mean square 50,579 is due to changes in the business cycle. At least one, and probably several, of the twelve Federal Reserve Districts are different from the rest with respect to the amount of bills discounted, as is evident from the large mean square 138,449. The three interactions indicate that the differences between districts do not remain the same from year to year or from month to month, and that the seasonal variation changes from district to district and from year to year. The fact that the interaction between districts and years is much larger than the interaction between districts and months indicates that the business cycle causes more of a change than seasonal variation in the differences between Federal Reserve Districts with respect to bills discounted. The business cycle causes the districts to be different by larger amounts than does the seasonal variation. The interactions indicate also that if we desire to adjust the data for seasonal variation we should not use the same index of seasonal variation for all districts or for all years. It is necessary to measure seasonal variation by using a method which permits changes to occur in the seasonal index for different years. It is possible that the seasonal variation may be changing in absolute amounts but is still constant so far as percentage changes are concerned. In order to investigate this possibility the analysis was repeated using logarithms of bills discounted instead of the original data. Similar conclusions were reached.

The application of the analysis of variance to data on the rate of interest earned by Federal Reserve banks on total earning assets yields the information that the rate of earnings is not the same in all Federal Reserve Districts, that there is a significant seasonal variation in the rate of earnings, and that the rate of earnings is decidedly affected by the business cycle. Furthermore, although the interaction between districts and years is large (indicating that the business cycle causes a significant change in the district differences) the interaction between districts and months is not significant (indicating that the district differences do not change within the year, or, stated differently, the seasonal variation is the same for all districts). Valuable information may be obtained by applying the analysis of variance to rates charged customers, department store sales, etc., using three or more criteria of classification. Lack of space prevents adequate discussion but an analysis of the trade-in value of used cars with four criteria of classification is given below.

ALLOWANCES ON USED 1934 MODEL CARS. PERCENTAGE OF F.O.B. PRICES

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Makes (Chev., Ford, Ply.)	2	107.59	53.80
Body Styles (Coupe, Coach, Sedan)	2	201.76	100.88
Grades (Stand., DeLuxe)	1	144.39	144.39
Months (from 4/19/35 on)	11	11,735.07	1,066.82
First Order Interactions			
Makes and Body Styles	4	32.74	8.18
Makes and Grades	2	10.92	5.46
Makes and Months	22	74.70	3.40
Body Styles and Grades	2	10.34	5.17
Body Styles and Months	22	13.13	.60
Grades and Months	11	4.53	.41
Higher Order Interactions	136	63.73	.47
	215	12,398.90	

Friday, July 23—Price-of-Living Comparisons Between Different Countries, PROFESSOR FRISCH.

Using the theoretical apparatus developed in the three lectures on choice-field theory, it was shown how a price-of-living index with a rational meaning could be constructed even between markets that were structurally different in the sense that they did not have the same *kinds* of commodities and services.

Friday, July 23 — The Mean External Appreciation of Money and a New Definition of So-called External Prices, FELICE VINCI, Professor in the University of Bologna.

(Because Professor Vinci was prevented from carrying out his plans to be present, this paper was read by Professor Harold T. Davis.)

The development of econometrics depends on the number and precision of the statistical concepts used as the foundation. Among the concepts which need greater precision are the related ones of *the external appreciation of money* and *the variation of external prices.*

For that purpose let us consider n states, and the mean rates of exchange, $c_{2,1}, c_{3,1}, \cdots$, of the States $2, 3, \cdots, n$ with respect to State 1, expressed in the money of State 1 and with reference to a given interval of time.

Considering the transactions made during the time designated, for example, a year, we may write the equation:

$$(1) \qquad m_{1,2\ldots n} = c_{2,1} m_{2,1} + c_{3,1}\, m_{3,1} + \cdots + c_{n,1} m_{n,1},$$

where

$$m_{1,2\ldots n} = \sum (pq)_{1,2\ldots n} + \sum E_{1,2\ldots n}$$

= value of goods and services exported by State 1 + money items exported by State 1;

$$m_{i,1} = \sum (pq)_{i,1} + \sum E_{i,1} \qquad (i = 2, \cdots, n)$$

= value of goods and services imported by State 1 + money items imported by State 1.

It is clear that we can write as many relations of the form (1) as there are states under consideration. These being n in number, it follows that the number of rates of exchange is equal to $n(n-1)$; but, as is known, Cournot has shown that, by admitting the action of arbitrage, these rates of exchange are reduced to $n-1$, and they prove to be determined by the system of equations of form (1), because the number of such equations is reduced to $n-1$, in view of the fact that one of them is derived from the others.

Denoting by $r_{i,1}$ the legal parities of the moneys of countries i with respect to country 1, and by $k_{i,1}$ the exchange rates relative to parity, and by $k_{23\ldots n,1}$ the arithmetical mean of the $k_{i,1}$'s, calculated with weights equal to $r_{i,1}\, m_{i,1}$, we may write equation (1) as follows:

$$(2) \qquad m_{1,2\ldots n} = k_{2\ldots n,1} \sum_{2}^{n} r_{i,1}\, m_{i,1} \qquad \text{or}$$

$$(3) \qquad k_{2\ldots n,1} = \frac{m_{1,2\ldots n}}{\sum_{2}^{n} r_{i,1}\, m_{i,1}}.$$

This $k_{2\ldots n,1}$, computed from the balance-of-trade and exchange-rate data, is precisely a measure of the external appreciation of money for which we have been searching.

Similar formulas may also be used when it is desired to consider transactions with only a limited number of states, s, instead of with all n states. These formulas may prove specially useful in considering monetary disturbances, and they lead to a general definition of the concept of the variation of external prices. It is known that especially in a state which is suffering from monetary devaluation or revaluation, the levelling of the exchange of the national money is not bound up with the variation of any groups of internal prices. These prices, if obliged to increase or diminish greatly in order to reach a new position of equilibrium, do not all display the same sensitivity, and their respective positions change especially because of the resistances, the contractual constraints, the monetary losses and gains, and the consequent variations in tastes, demands, and costs; there are groups of prices with maximum sensitivity, groups with slightly less sensitivity, and so on.

The internal prices of a state, expressed in the money of another state, based on the relative rate of exchange, have been called (by the term introduced by W. Lexis in 1893) external prices of the first state with respect to the second. Where P_1 is the index number of the price of a good or group of goods with respect to a base year, and $C_{2,1}$ is the index number of the rate of exchange, we must consider at first that the external price may be expressed by

$$p_1/c_{2,1} = p_1/k_{2,1}\, r_{2,1}.$$

If $r_{2,1}$ remains fixed, the indices of $c_{2,1}$ coincide with those of $K_{2,1}$ of the relative exchange rates $k_{2,1}$, and we obtain

$$P_1/C_{2,1} = P_1/K_{2,1}.$$

With the use of (3) or the corresponding formula for a limited number of states, we can make such calculations of external prices not only with regard to one state, but with regard to all or any number of states.

In general, the relation between the index numbers of the prices of a good or of a group of goods and the corresponding index numbers computed from the mean ratio to parity of the exchange rates of the money of a state with respect to the other states with which it may have trade operations will be called by us variation of the external prices of that good or that group of goods in that state.

Suggestions for the Quantitative Study of the Business Cycle,[1] JOSEPH A. SCHUMPETER, Professor of Economics, Harvard University.

(NOTE: Three lectures on the above subject were scheduled for July 14, 15, and 16, 1937; but Professor Schumpeter at the last moment was unable to be present. The following is an abstract which he had prepared in advance.)

Clearing the Ground (compare Tinbergen's Survey I and II).

(1) When we say that "cycles" are present in economic time series, we merely mean the facts (a) that time series or their time-derivatives are never monotonic, (b) that their "fluctuations" are related to each other. There is no reason to assume that the cycles in this sense are all of one kind, or that cycles of any given kind should display constant periods. In those cases, however, in which the periodogram analysis

[1]This title is the same as that chosen by Professor Tinbergen for his excellent survey in *Econometrica*, Vol. 3, July, 1935, pp. 241-308. It has been thought convenient to arrange topics in the order in which they are dealt with by Tinbergen, and to start from Tinbergen's treatment.

has been successful (Crum, Greenstein) actual fluctuations must fairly correlate with a pure sine curve. Prima facie evidence justifies the expectation that most causes give rise to *several* "fluctuations" or "vibrations."

(2) The time-shape of a factor which produces cycles need not be cyclical itself. Such a factor may be an "impulse," or simply a property of the system. If it is an impulse of the noncyclical kind, then the cycles are the form in which the system responds to disturbance, and the analogy with a resonator suggests itself ("waves of adaptation"). The most familiar example is afforded by the "animal cycles" (hogs, sheep, cattle). Unless we exclude all impulses from our variables and include them among data, our system is not "causal" in the sense of Birkhoff and Lewis, that is to say, it is not possible to describe its evolution in time uniquely in terms of initial conditions. Even so far as this is possible the system may not have any static solutions. The value of the concept of equilibrium as a tool of business-cycle analysis is however not much impaired by this possibility.

(3) Cycles in the system as a whole (general cycles) can, of course, arise from particular causes, such as harvests. The variable that sets into motion a general cycle in social aggregates (total deposits, total output, price level, and so on) need not itself be an aggregate. Even if it is (alteration of gold content of the dollar would, for instance, involve a change in an aggregate) it practically never directly acts on other aggregates, but only through, say, prices and quantities of individual commodities. Hence statements about relations in time between aggregate quantities ("macrodynamic theories") never go beyond the surface and are likely to be either empty as regards diagnosis or misleading as regards prognosis. Quantitative study should be, much more than it actually is, directed towards the details of the industrial processes that lie behind those aggregates.

Facts, Schemata, Theories (compare Tinbergen's Survey III, IV, V).

(1) Most of the facts we agree on are facts of aggregative behavior (see I.3). Difference of opinion arises, how-

ever, as soon as these facts are being treated or even expressed. The outstanding example of the former type of difference is trend analysis. The outstanding example of the latter type is the manner in which different investigators deal with exceptions. Nothing is more plausible, for instance, than that general unemployment percentage inverted should vary with the rate of change of price level. In England, 1871-1914, it does so in 22 instances and fails to do so in 16. The question what such a finding means immediately leads into "theory."

(2) Schemata are descriptions of certain pieces of the cyclical mechanism; in the most ambitious instances they claim to depict the whole of it. One class may be exemplified by the schemata of lagged adaptation (mentioned in I.2) or of reaction depending not only on the values of some independent variables but also on their time-derivatives. Another class is represented by a development which flows from the old quantity theory of money which can easily be "dynamized" in such a way as to issue into the Keynes-Amoroso-Vinci equations (see the latter's article in *Econometrica,* Vol. 2, 1934, pp. 125-139).

These schemata are purely formal and compatible with almost any views about the nature and causes of business cycles. But their authors often add an interpretation which goes beyond those mechanisms and then are likely to believe that statistical facts which verify the schema also verify that interpretation. To clear up this matter it will be useful to analyze the Kalecki schema and to ask what it really asserts about the cyclical process.

(3) Interpretations of mechanisms or *causes* we call Theories. They can be grouped under the headings: External-Factor Theories, Monetary Theories, Others. Since any analysis must take account of all the facts to which the various theories appeal, common ground should extend very far. If it does not, this can be due only:

(a) To unwarranted assertions about fact. For instance, the speaker believes that the current views about the effects of saving are not only wrong in logic but also in fact, since they imply gross overstatement of the actual yearly saving.

(b) To faulty argument. A theory may be right in all the assertions it makes about the facts from which it starts and yet fail to establish the desired conclusion. Some oversaving theories again afford an instance.

(c) To difference in emphasis and particularly in evaluation of causal significance. The "Hayek effect" for instance is very real. Its presence can be well established historically. Yet it fails as a theory of the cycle.

Questions of forecasting and policy (compare Tinbergen's Survey VI).

Although neither forecasts nor policies are among the primary objects of these lectures, questions relating to both may serve to clarify and develop a few additional points.

(1) Forecasting from a series of states of the economic organism and forecasting what will actually happen are, of course, entirely different things. So are forecasts about future events as such and forecasts about the exact timing of these events. Forecasts of the former type are possible and interesting, although of very limited value to the businessman or short-time speculator.

(2) The first question as to policy which business-cycle analysis can be expected to answer is what sense there is in trying to counteract cyclical variations at all. Agreement can be hoped for only with respect to certain features or phases of those variations, beyond which difference of opinion is likely to persist.

But another group of questions is more hopeful: we can strictly remain in the realm of scientific reasoning if we confine ourselves to an investigation of the effects to be expected from any given policy. As an example the influence on cyclical situations of various wage policies will be discussed.

ABSTRACTS OF PUBLIC LECTURES

Wednesday, July 7 — Some Lessons of Recent Money Policy in the United States, JAMES HARVEY ROGERS, Sterling Professor of Political Economy, Yale University.

To describe the recent currency experiences of the United States is to recount one of the strangest and most dramatic experiences in monetary history. While occasional depreciation of the world's chief units of value apparently has been the rule rather than the exception in the past ages, the motivating influence in every earlier recorded experience seems to have been the financial needs of the sovereign (and more recently of the government) rather than the economic well-being of the country. Until 1933 history had failed to record a single instance of a great country's resorting to currency depreciation for the avowed and sole purpose of reviving internal prosperity.

On March 4, 1933, the day of the inauguration of President Roosevelt, all the commercial banks in the country were closed, as a result of the drastic credit contraction. On March 5 a temporary embargo was placed on the export of gold. On March 9 this temporary embargo was extended, and on April 19 the President ordered an embargo on all exports of gold except that earmarked for foreign countries. Thus did the United States leave the gold standard.

Unlike that in Great Britain this action was taken deliberately and purposely. The depreciation of the pound sterling in terms of which many leading international markets were conducted had had an extremely depressing effect on the prices of many of our leading exports — notably agricultural products — and one of the purposes of the depreciation of the dollar was to correct that situation. The disparity of farm prices had led to farm foreclosures at an appalling rate, and widespread revolt among farm owners made its appearance.

With the departure from the gold standard, in the United

States as in Great Britain, three outstanding domestic advantages appeared:

1. Prices of basic raw materials with markets largely outside the United States rose.

2. In foreign trade the United States gained an advantage over countries with undepreciated currencies and recovered its position with respect to the sterling group which had taken a similar advantage when they earlier left gold.

3. Complete monetary freedom, making possible long-continued low interest rates, was assured. It was this reduction in interest rates, coupled with a revival of financing at these rates, which, in the opinion of the writer, marked the beginning of recovery in the United States.

The same lesson is shown by the experience of other countries: recovery has followed conservative depreciation of the currency, while economic conditions in the gold-bloc countries declined. But while the domestic advantages of depreciation have been proved by experience to be real and effective, the international disadvantages have been great, because of the decline in international trade brought about by restrictions intended to combat depreciation in other countries.

With the devaluation of many currencies the monetary value of the world's gold stocks has been greatly increased, and the production and dehoarding of gold have been stimulated. This has brought serious problems for the future.

The basic money of the country is supplied from the gold stock, Treasury currency, and the total earning assets of the Federal Reserve banks. This basic money is utilized chiefly for circulation, for member-bank reserves, and for the Stabilization Fund and the deposit balances of the Federal Treasury at the Federal Reserve banks.

From a balance sheet, prepared on the above basis, of the supply and utilization of basic money month by month a quantitative study of monetary phenomena is greatly simplified.

The currency-supporting capacity of the gold stock was greatly increased by devaluation and by the subsequent inflow of gold. Its effect, however, was counteracted to some extent by tieing up two-thirds of the gold profit in the Stabilization Fund and using much of the rest to retire National bank notes, The effect of subsequent gold inflows has been offset by doubling the reserve requirements of the member banks, and, in recent months, by purchasing gold with Treasury bills and sterilizing it. By all these devices member-bank reserves have been reduced to somewhat less than a billion dollars while approximately seven billions have been thrown out of use.

Certain by-products of currency depreciation, completely unanticipated, already begin to appear as most important of all the effects. Attempts in some countries to defend the gold standard at all costs stimulated the trend toward economic nationalism and state capitalism. The depreciation of the world's major currencies aggravated these vicious tendencies but likewise threw into the limelight the hopeless economic inferiority of certain countries most actively sponsoring them.

Thursday, July 15 — Probability and the Telephone Plant, THORNTON C. FRY, Bell Telephone Laboratories, Inc.

Wherever the demand for service arises from a large number of individuals acting more or less independently of one another, wide fluctuations are bound to occur in the number of demands. This is true of the number of customers in a barber shop or at a theater ticket window; of the power load on the generators of an electric company; or of the work load of an attendant in a spinning mill when the spindles stop from time to time because of thread breakage. It is also true of the number of telephone calls which must be handled simultaneously in a telephone exchange.

As all these fluctuations are accidental in character, they raise many questions that fall naturally into the field of the theory of probability. We would therefore expect that probability theory could be applied to the efficient management of any such enterprise; and, in particular, that it would play an important part in the design and operation of telephone exchanges. It is a curious paradox that so far as telephony is concerned, this statement is both false and true.

So far as the day-to-day operation of the plant is concerned, it is false; for not only are the purely random fluctuations of demand seriously modified by the changing personal habits of the subscribers and by the normal growth of the exchange, but also the routine observations which are necessary to assure the successful functioning of the exchange automatically give answers to the very questions which the theory of probability might otherwise be called upon to solve.

But the telephone art is not a static art. Instead, it is continually being subjected to scrutiny and development in the interests of greater economy and higher standards of service. For example, new kinds of switching equipment for use in dial offices and new methods of using standard equipment are constantly being suggested. Some of these suggestions are good, others are worthless; and it is only by appraising them accurately that the telephone plant of the future can be brought to the highest possible efficiency. To appraise them, however, it is necessary to answer many questions more or less similar to those met in day-to-day operation, but lying outside the range of past experience. It is in appraising the merits of these brain-children, of which so many are born and so few survive, that probability plays a really important role.

This is true for two reasons: First, an empirical study is frequently not feasible because its cost would be prohibitive. Second, even where empirical studies with actual traffic are feasible, their results are often difficult of interpretation because the various systems which are to be compared cannot be tried under identical traffic conditions; just as experiments to determine the relative yield of two kinds of wheat are frequently inconclusive because they cannot both be grown on the same plot of ground. In other words, "controlled" empirical studies are difficult to arrange.

In a theoretical approach, however, neither of these difficulties arises; for a great deal of mathematical brain-power can be bought for the cost of even a small experimental exchange, and of course identical traffic conditions may be assumed in as many alternative studies as may be desired.

Monday, July 19 — The Relation of High Profits to High Wages, MR. SNYDER.

It is now well known that the United States have become by far the richest nation, with the widest dissemination of general comfort and well-being that ever existed. The average wages, real wages in purchasing power, were in 1930, as computed by the International Labor Office, at least one-half again as much as those of the next leading nation, Great Britain, near to twice the average real wages of Germany, and between two and three times those of Italy.

Actual wages in this country are at least ten times as high as the average wage in the better class of factories in India and perhaps also of China. They seem at least five or six times those of Japan. Why this extraordinary primacy of the United States? What created all this wealth and well-being?

We now have solid statistical evidence that the extraordinary growth of the United States in manufacturing, mining, and the like (not agriculture) has been closely parallel to the increase of capital invested. In turn this capital investment seems in the last hundred years or more to have grown almost parallel to the amount of horsepower employed in our industries, excluding, of course, the farm and now the automobile. This quantity of horsepower in turn seems to measure closely the average amount of machinery employed in industry as a whole.

All this means that the increase in the amount of machinery or horsepower employed in industry *per worker* has risen almost exactly with the amount of capital invested per worker; and, in turn, the wages of the workers appear to have risen at a corresponding rate.

In other words, we now have a clear and definite picture of the process or mechanism by which our high wages have been created. They are due solely to the increased product per worker which has in turn steadily risen in the last one hundred years in close proportion to the amount of machinery employed, and to the amount of capital invested in industry. These facts are incontestable and the further fact seems evident that the only way that the general rate of wages can be increased is through this increase of product per worker.

This increase, in turn, is not due to any fabled gain in the "efficiency" of the workers. There is little evidence that skilled workers are today any more skillful than they were a hundred years ago. The increase in product seems due solely to invention, discovery, the creation of new processes, and the realization of them through largely automatic machinery. This in turn has required a vast expenditure of capital. The capital supply then is an ineluctable factor in high wages and the gain in our general well-being.

Whence now comes this capital employed in industry and especially in manufacturing, of all kinds, wherein has been our greatest industrial gain? The rather surprising answer is that it has come, in by far the larger part, from the industries themselves; and even each industry to a large degree has generated its own capital. The supposed role of "the savings of the nation," or of "the people," seems to play a quite subordinate and almost unimportant part. And this is especially true of new industries, a very vital point. For it is almost solely through the new industries that the increase in real wages comes. In the older industries, long established and often showing little more than a population rate of growth, there is little incentive or capacity to raise the general wage level because their profits are relatively low and tend to be stationary.

Now it is almost solely through the profits of industry that the new capital, and especially the new capital for new industries, is derived, either through dividends or from the attendant capital gains. In turn it is the new industries which are expanding rapidly, which make, therefore, larger profits, for the larger supply of capital funds for expansion and improvements, and, it may be added, for technical research. (The commercial corporations of the United States are today expending more than three-quarters of a million dollars per day in scientific investigations.) It is these new industries which create the new demands for labor and from their high profits they are able to bid higher for the most capable workingmen; and this is almost the sole source of the rise in real wages in the United States. This increase is automatic and for all practical purposes the influence of legislation, or of the labor unions, or the altruistic sentiments of employers, or the like, has been zero.

We have now a complete and detailed picture of the whole process by which the well-being of this country, since three-quarters or more of the population constitute "the workers," has been achieved. We may dismiss the myths. These are the facts.

Now if the rate of industrial progress and, therefore, of the rise of wages is finally determined by the capital supply, then it follows that the limitation of profits in any fashion is inimical to the welfare of the country and opposed to its best interests. For the rest, the amount of these profits even at their highest is negligible. The net profits of industry, even excluding farming, have in the last century rarely exceeded from 4 to 6 per cent, and have averaged around something like 5 per cent of the national income. Yet this is the vital and spermatic element in our prosperity, the seed corn of all industrial improvement. Beware lest we destroy it.